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Research Paper

A Reliable Numerical Algorithm for the Fractional Klein-Gordon Equation

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The key purpose of the present work is to introduce a numerical algorithm for the solution of the fractional Klein-Gordon equation (FKGE). The numerical algorithm is based on the applications of the operational matrices of the Legendre scaling functions. The main advantage of the numerical algorithm is that it reduces the FKGE into Sylvester form of algebraic equations which significantly simplify the problem. Numerical results derived by using suggested numerical scheme are compared with the exact solution. The results show that the suggested algorithm is very user friendly for solving FKGE and accurate.

Key words: fractional Klein-Gordon equation; Legendre scaling functions; operational matrices.

1. Introduction

The standard Klein-Gordon equation (KGE) is written as

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial y^2} + w = g(y,t), \qquad y \geq 0, \quad t \geq 0, \tag{1.1}$$

where u indicates an unknown function in variables y and t, $g(y,t)$ stands for the source term. Fractional extension of differential equation is similarly useful with great advantages because of the non-local nature of fractional derivatives [1–12]. For handling the initial and boundary conditions of our physical problem we will replace integer order derivative to fractional order derivative. The fraction in derivative suggests a modulation or weighting of system memory. A broad literature of models having space fractional derivative can be found in [13–17]. Therefore in this article, we will consider more general form of KGE by changing integer order space derivative by Liouville-Caputo derivative of fractional order in the following manner:

$$\frac{\partial^2 w(y,t)}{\partial t^2} - \frac{\partial^\alpha w(y,t)}{\partial y^\alpha} + w(y,t) = g(y,t), \qquad 1 < \alpha \leq 2, \tag{1.2}$$

having the initial conditions:

$$w(y,0) = g_1(y), \qquad \frac{\partial w(y,0)}{\partial t} = g_2(y), \qquad \text{for} \qquad 0 \leq y, \quad t \leq 1, \tag{1.3}$$

and boundary conditions:

$$w(0,t) = h_1(t), \qquad w(1,t) = h_2(t). \tag{1.4}$$

In view of great importance of KGE in science, especially in quantum field theory, plasma, optical fibers and dispersive wave-phenomena many authors have studied it by using various analytical and numerical schemes [18–27] with their own shortcomings and limitations. The operational matrix method [28–38] was also applied to solve problems in fractional calculus.

In the present paper, we are using a computational technique which is based on the operational matrices of Legendre scaling functions. In this method, first we take finite dimensional approximation of unknown function. Further, making use of operational matrices in the FKGE, we find a system of algebraic equations in Sylvester form whose solution yields approximate solution for the FKGE. To show the utility and accuracy of the suggested approach we have compared the obtained results with exact solutions and numerical results by some existing methods.

2. Preliminaries

Here we give some theoretical foundation of fractional order differentiation and integration:

Definition 2.1. The Riemann-Liouville (RL) fractional integral operator is presented as

$$I^{\alpha}\sigma(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1}\sigma(u)\,\mathrm{d}u, \qquad \alpha > 0, \quad x > 0,$$

$$I^{0}\sigma(x) = \sigma(x).$$

Definition 2.2. The Liouville-Caputo fractional derivative of order β is expressed as

$$D^{\beta}\sigma(x) = I^{m-\beta}D^{m}\sigma(x) = \frac{1}{\Gamma(m-\beta)} \int_0^x (x-u)^{m-\beta-1}\frac{\mathrm{d}^m}{\mathrm{d}u^m}\sigma(u)\,\mathrm{d}u,$$

$$m-1 < \beta < m, \qquad x > 0.$$

Lemma 2.1. If $p-1 < \alpha \le p$, $p \in N$, and $\sigma \in L^2[0,1]$ then $D^{\alpha}I^{\alpha}\sigma(y) = \sigma(y)$ and

$$I^{\alpha}\,D^{\alpha}\sigma(y) = \sigma(y) - \sum_{k=0}^{p-1}\sigma^{(k)}(0^{+})\frac{y^k}{k!}, \qquad y > 0.$$

Proof. Please see [39].

The Legendre scaling functions $\{\psi_j(y)\}$ in one dimension are expressed in the following manner

$$\psi_j(y) = \begin{cases} \sqrt{(2j+1)}P_j(2y-1), & \text{for} \quad 0 \le y < 1. \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $P_j(y)$ is standing for Legendre polynomials of order j on the interval $[-1,1]$.

The two dimensional Legendre scaling function ψ_{j_1,j_2} are defined as

$$\psi_{j_1,j_2}(y,t) = \psi_{j_1}(y)\,\psi_{j_2}(t), \qquad j_1,\, j_2 \in N = \{1,\,2,\,3,...\}\,,$$

where $\psi_{j_1}(y)$ and $\psi_{j_2}(y)$ are one dimensional Legendre scaling functions as defined in Eq. (2.1).

(ψ_{j_1,j_2}) form a complete orthonormal basis with the following property:

$$\int_0^1\int_0^1 \psi_{i_1,i_2}(y,t)\psi_{j_1,j_2}(y,t)\,\mathrm{d}y\,\mathrm{d}t = \begin{cases} 1, & i_1 = j_1 \quad \text{and} \quad i_2 = j_2, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, a function $h(y,t) \in L^2([0,1]\times[0,1])$, can be approximated in the following manner

$$h(y,t) \cong \sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2} c_{j_1,j_2}\psi_{j_1,j_2}(y,t) = C^{\mathrm{T}}\mu_{n_1,n_2}(y,t), \tag{2.2}$$

where

$$C = [c_{0,0},...,c_{0,n_2},...,c_{n_1,1},...,c_{n_1,n_2}]^{\mathrm{T}},$$

$$\mu_{n_1,n_2}(y,t) = [\psi_{0,0}(y,t),...,\psi_{0,n_2}(y,t),...,\psi_{n_1,1}(y,t),...,\psi_{n_1,n_2}(y,t)]^{\mathrm{T}}.$$

The coefficients c_{j_1,j_2} in the expansions of $h(y,t)$ are presented as

$$c_{j_1,j_2} = \int_0^1\int_0^1 h(y,t)\,\psi_{j_1,j_2}(y,t)\,\mathrm{d}y\,\mathrm{d}t. \tag{2.3}$$

On employing matrix notation Eq. (2.2) can be presented as,

$$h(y,t) \cong \mu_{n_1}^{\mathrm{T}}(y)\,C\mu_{n_2}(t), \tag{2.4}$$

where

$$\mu_{n_1}(y) = [\psi_0(y),...,\psi_{n_1}(y)]^{\mathrm{T}}, \qquad \mu_{n_2}(t) = [\psi_0(t),...,\psi_{n_2}(t)]^{\mathrm{T}}$$

and

$$C = (c_{j_1,j_2})_{(n_1+1)\times(n_2+1)}\,.$$

Theorem 2.1. If Legendre scaling vector $\mu_n(y) = [\psi_0(y),...,\psi_n(y)]^{\mathrm{T}}$, and consider $\alpha > 0$, then

$$I^{\alpha}\psi_i(y) = I^{(\alpha)}\mu_n(y). \tag{2.5}$$

In the above $I^{(\alpha)} = (\varsigma(c,d))$, is $(n+1)\times(n+1)$ operational matrix of fractional integral of order α and its (c,d)-th entry is written as

$$\varsigma(c,d) = (2c+1)^{1/2}(2d+1)^{1/2}\sum_{k=0}^{c}\sum_{l=0}^{d}(-1)^{c+d+k+l}$$
$$\cdot\,\frac{(c+k)!(d+l)!}{(c-k)!(d-l)!(k)!(l!)^2(\alpha+k+l+1)\Gamma(\alpha+k+1)}, \qquad 0 \le c, \quad d \le n.$$

Proof. Please see [40, 41].

Theorem 2.2. If Legendre scaling vector $\mu_n(y) = [\psi_0(y), ..., \psi_n(y)]^{\mathrm{T}}$, and consider $\beta > 0$, then

$$D^{\beta}\psi_i(y) = D^{(\beta)}\mu_n(y), \tag{2.6}$$

where $D^{(\beta)} = (\tau(c,d))$ is $(n+1)\times(n+1)$ operational matrix of Liouville-Caputo fractional derivative of order β and its (c,d)-th entry is given by

$$\tau(c,d) = (2c+1)^{1/2}(2d+1)^{1/2}\sum_{k=\lceil\beta\rceil}^{c}\sum_{l=0}^{d}(-1)^{c+d+k+l} \cdot \frac{(c+k)!(d+l)!}{(c-k)!(d-l)!(k)!(l!)^2(k+l+1-\beta)\Gamma(k+1-\beta)}.$$

Proof. Please see [40, 41].

3. Method of solution

In the present part, we describe algorithm to obtain the approximate solution of the Eq. (1.2) with initial condition Eq. (1.3) by taking $n_1 = n_2 = n$, for any approximations. Let

$$\frac{\partial^2 w(y,t)}{\partial t^2} = \mu_n^{\mathrm{T}}(y)C\mu_n(t). \tag{3.1}$$

Integrating Eq. (3.1) twice with respect to t and using Lemma 2.1, we have

$$w(y,t) = \mu_n^{\mathrm{T}}(y)CI^{(2)}\mu_n(t) + tg_2(y) + g_1(y), \tag{3.2}$$

where $I^{(2)}$ can be calculated using Eq. (2.5). Let

$$tg_2(y) + g_1(y) \approx \mu_n^{\mathrm{T}}(y)A\mu_n(t). \tag{3.3}$$

From Eqs (3.2) and (3.3), we can write

$$w(y,t) = \mu_n^{\mathrm{T}}(y)CI^{(2)}\mu_n(t) + \mu_n^{\mathrm{T}}(y)A\mu_n(t). \tag{3.4}$$

Taking differentiation of order α with respect to y on the both side of Eq. (3.4), we arrive at the subsequent result

$$\frac{\partial^{\alpha} w(y,t)}{\partial y^{\alpha}} = \mu_n^{\mathrm{T}}(y)D^{(\alpha),\mathrm{T}}CI^{(2)}\mu_n(t) + \mu_n^{\mathrm{T}}(y)D^{(\alpha),\mathrm{T}}A\mu_n(t), \tag{3.5}$$

where $D^{(\alpha)}$ can be calculated using Eq. (2.6). Further, approximating the inhomogeneous term as

$$g(y,t) \approx \mu_n^{\mathrm{T}}(y)G\mu_n(t). \tag{3.6}$$

Using Eqs (3.1), (3.4), (3.5) and (3.6) in Eq. (1.2), we get

$$\mu_n^{\mathrm{T}}(y)C\mu_n(t) - \mu_n^{\mathrm{T}}(y)D^{(\alpha),\mathrm{T}}CI^{(2)}\mu_n(t) - \mu_n^{\mathrm{T}}(y)D^{(\alpha),\mathrm{T}}A\mu_n(t) + \mu_n^{\mathrm{T}}(y)CI^{(2)}\mu_n(t) + \mu_n^{\mathrm{T}}(y)A\mu_n(t) = \mu_n^{\mathrm{T}}(y)G\mu_n(t).$$

Equation (3.7) can be written as

$$\left(D^{(\alpha),\mathrm{T}} - I^d\right)C - C\left(I^{(2)}\right)^{-1} = \left(A - F - D^{(\alpha),\mathrm{T}}A\right)\left(I^{(2)}\right)^{-1}. \tag{3.7}$$

Equation (3.8) is known a Sylvester equation which can be solved very easily to determine the unknown matrix C. Making use of the value of C in Eq. (3.4), we can obtain an approximate solution for FKGE.

4. Numerical experiments and discussion

Example 1. We consider the subsequent FKGE [26]

$$\frac{\partial^2 w(y,t)}{\partial t^2} - \frac{\partial^\alpha w(y,t)}{\partial y^\alpha} = g(y,t), \qquad 1 < \alpha \le 2, \tag{4.1}$$

with initial conditions:

$$w(y,0) = y^\alpha(1-y), \qquad \frac{\partial w(y,0)}{\partial t} = y^\alpha(y-1), \qquad \text{for} \quad 0 \le y, \quad t \le 1, \tag{4.2}$$

and boundary conditions:

$$w(0,t) = 0, \qquad w(1,t) = 0, \tag{4.3}$$

with source function $g(y,t) = y^\alpha(1-y)\exp(-t) - [\Gamma(\alpha+1) - \Gamma(\alpha+2)y]\exp(-t)$. The exact solution of FKGE (4.1) is $w(y,t) = y^\alpha(1-y)\exp(-t)$.

In Fig. 1, we have shown the behaviour of approximate solution for integer order KGE. In Fig. 2, we have plotted absolute errors by our proposed method for integer order KGE at $n = 5$.

From Fig. 2, it observed that our numerical results show excellent agreement with the exact solution for $\alpha = 2$. In Fig. 3 we have plotted approximate and exact solution for different values of α.

From Fig. 3 it is clear that our numerical results have a great agreement with exact results for the fractional order involved in KGE. In Figs 4 and 5, we have plotted absolute errors by our proposed method for different values of $\alpha = 1.5$, 1.6, 1.7, 1.8 and 1.9 at $t = 0.1$ and 0.5, respectively.

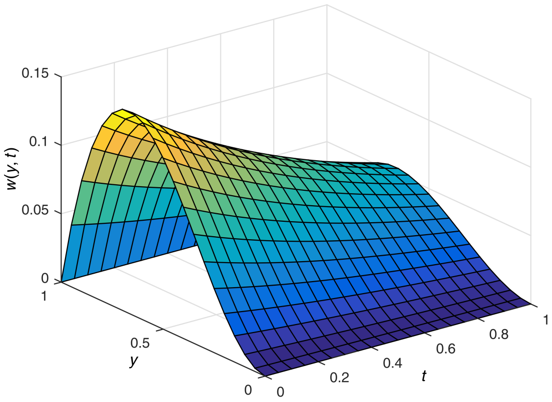

FIG. 1. Approximate solution at $\alpha = 2$, Example 1.

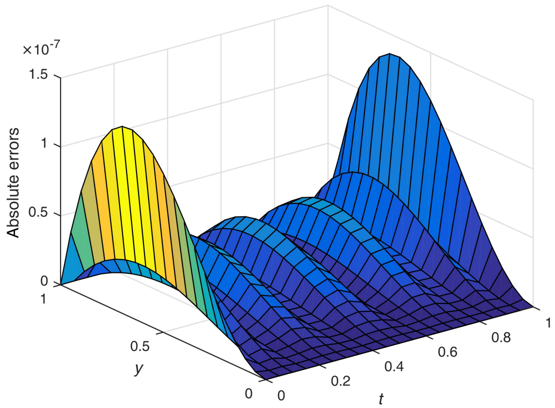

FIG. 2. Absolute errors at $\alpha = 2$, Example 1.

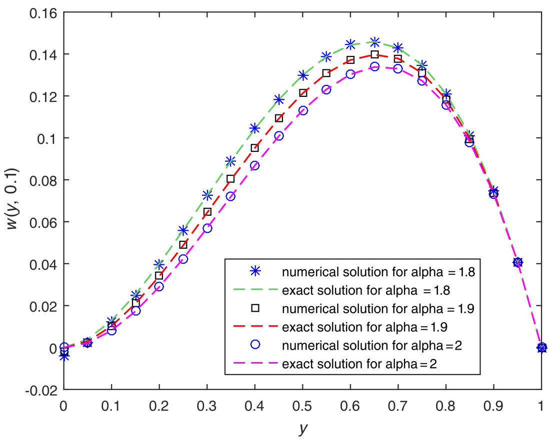

FIG. 3. Behaviour of exact and approximate solution for different values of $\alpha = 1.8$, 1.9 and 2 at $t = 0.1$, Example 1.

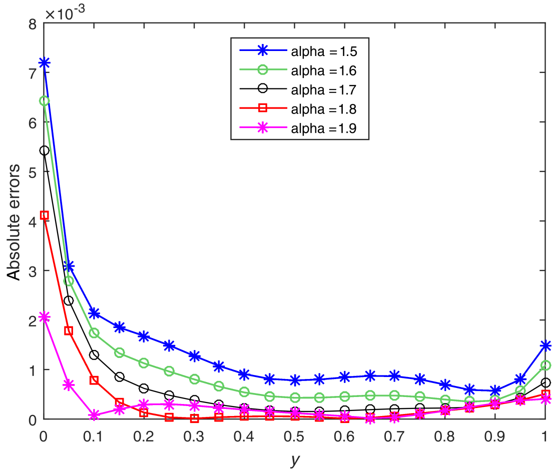

FIG. 4. Absolute errors for different values of $\alpha = 1.5$, 1.6, 1.7, 1.8 and 1.9 at $t = 0.1$, Example 1.

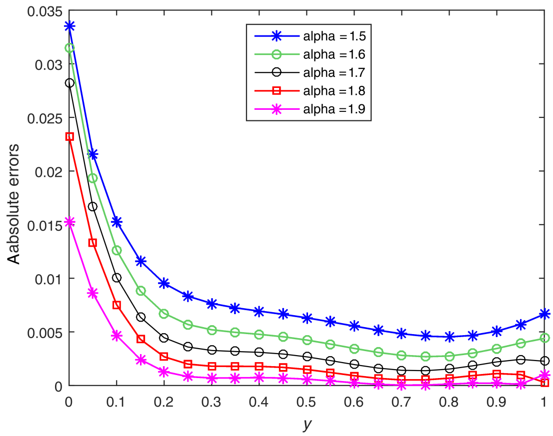

FIG. 5. Absolute errors for different values of $\alpha = 1.5$, 1.6, 1.7, 1.8 and 1.9 at $t = 0.5$, Example 1.

From Figs 4 and 5, it observed that our numerical results show excellent agreement with the exact solution for $\alpha = 1.5$, 1.6, 1.7, 1.8 and 1.9. In Table 1, we have compared the numerical results from our proposed scheme and method in [26]. From Table 1, it is observed that our technique is more accurate in comparison with finite difference scheme as given in [26].

Example 2. We consider the following FKGE [27]

$$\frac{\partial^2 w(y,t)}{\partial t^2} - \frac{\partial^\alpha w(y,t)}{\partial y^\alpha} = w(y,t), \qquad 1 < \alpha \le 2, \tag{4.4}$$

Table 1. Comparison of absolute errors by our proposed method and method in [26] at $t = 1$ and $n = 3, 5$.

y	$n = 3$		$n = 5$	
	Present method	Method in [26]	Present method	Method in [26]
0.0	2.0700e–08	1.1234e–03	8.1115e–12	8.1245e–04
0.1	3.2932e–06	2.7894e–03	8.2932e–09	6.8754e–04
0.2	1.1752e–05	4.4561e–03	2.9504e–08	4.9541e–04
0.3	2.3152e–05	1.7418e–03	5.8092e–08	2.4875e–04
0.4	3.5286e–05	7.8527e–03	8.8523e–08	8.5154e–04
0.5	4.5951e–05	5.9634e–03	1.1527e–07	4.0092e–04
0.6	5.2940e–05	6.8527e–03	1.3279e–07	6.1457e–04
0.7	5.4049e–05	3.1237e–03	1.3556e–07	6.9541e–04
0.8	4.7072e–05	1.7595e–03	1.1804e–07	7.1478e–04
0.9	2.9804e–05	3.0030e–03	7.4704e–08	2.0854e–04
1.0	4.1401e–08	0.0129e–03	1.6230e–11	0.0034e–04

with initial conditions:

$$w(y,0) = 1 + \sin(y), \qquad \frac{\partial w(y,0)}{\partial t} = 0, \quad \text{for} \quad 0 \leq y, \quad t \leq 1, \tag{4.5}$$

and boundary conditions:

$$w(0,t) = \cosh(t), \qquad w(1,t) = \sin(1) + \cosh(t). \tag{4.6}$$

The exact solution of this equation for integer order is $w(y,t) = \sin(y) + \cosh(t)$. In Fig. 6, we have shown the behaviour of approximate solution for

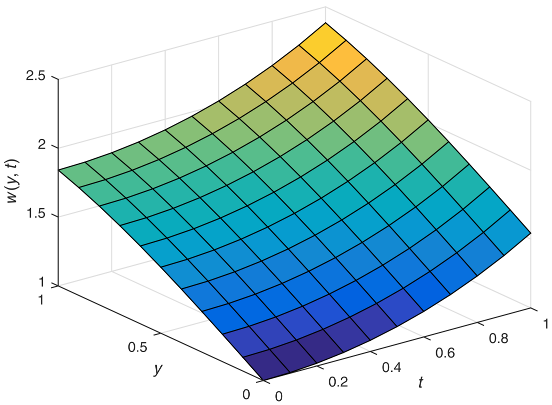

FIG. 6. Approximate solution at $\alpha = 2$, Example 2.

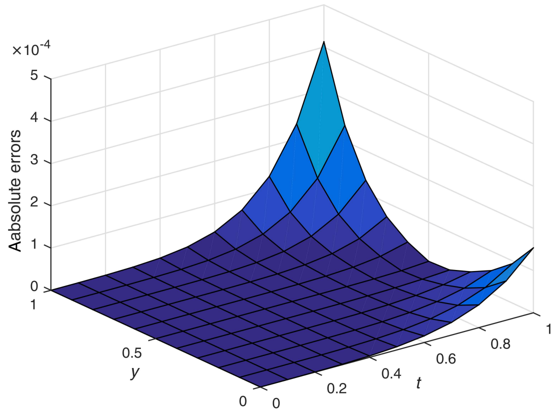

FIG. 7. Absolute errors at $\alpha = 2$, Example 2.

integer order KGE. In Fig. 7, we have plotted absolute errors by our proposed method for integer order KGE at $n = 8$.

From Fig. 7, it observed that our numerical results show nice agreement with the exact solution for $\alpha = 2$. In Fig. 8, we have shown the behaviour of approximate solution for distinct values α at $t = 0.1$.

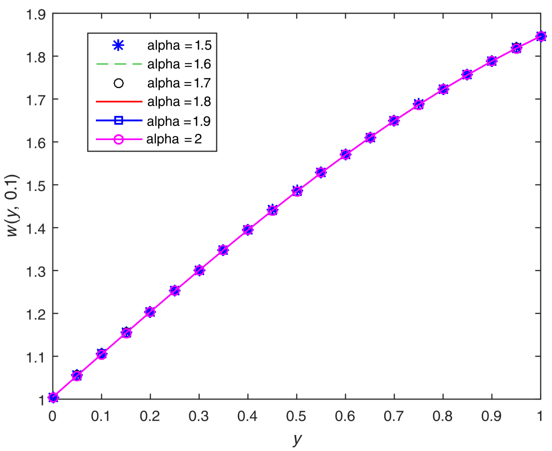

FIG. 8. Behaviour of approximate solution for different values of $\alpha = 1.5, 1.6, 1.7, 1.8$, 1.9, and 2 at $t = 0.1$, Example 2.

In Fig. 9, we have shown the nature of approximate solution for various values α at $t = 1$. From Figs 8 and 9, it is to be noticed that approximate solution changes continuously from fractional order to integer order solution. In Table 2, we have listed the numerical outcomes from our suggested technique at two different values of n. From Table 2 it is noticed that our scheme is accurate.

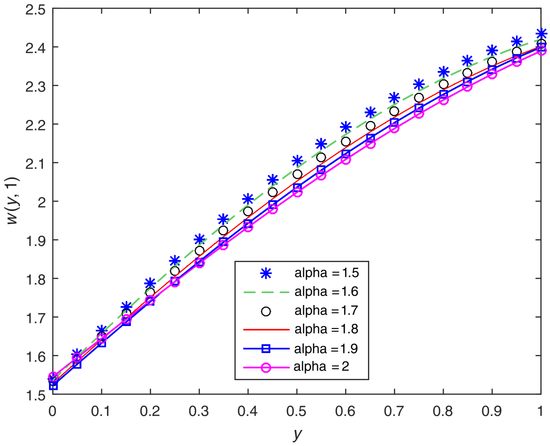

Fig. 9. Behaviour of approximate solution for different values of $\alpha = 1.5, 1.6, 1.7, 1.8, 1.9$ and 2 at $t = 1$, Example 2.

Table 2. Comparison of absolute errors by our proposed method at two different values of $n = 5$, 8.

y	$n = 5$	$n = 8$
0.0	2.1183e–03	1.5372e–04
0.1	1.4260e–03	9.3541e–05
0.2	9.3639e–04	5.4153e–05
0.3	6.2133e–04	2.8306e–05
0.4	4.6593e–04	9.7867e–06
0.5	4.7204e–04	7.4249e–06
0.6	6.6241e–04	3.0138e–05
0.7	1.0850e–03	6.7116e–05
0.8	1.8180e–03	1.3049e–04
0.9	2.9753e–03	2.3753e–04
1.0	4.7125e–03	4.1277e–04

5. Concluding remarks

The solutions are derived by solving Sylvester's form of equation so it is very simple and user friendly for computational purposes. The numerical results obtained for FKGE are shown in graphical and tubular form. From numerical results we can see that the solution varies continuously for various values of α. For $\alpha = 2$ the numerical solution for the classical KGE are obtained. The numerical results are obtained for left hand side Liouville-Caputo fractional derivative. It should be noticed that physical interpretation of the model is at this moment unclear due to the fact that in the fractional operator half axes is assumed only. The key advantage of this study is to find an approximate solution of FKGE where the exact analytical solutions are not easy to derive. The outcomes of the present study are very helpful for the scientists and engineers working in the mathematical modelling of natural phenomena. We carried out our technique in the domain $[0, 1] \times [0, 1]$. The proposed scheme can be utilized in any bounded domain $[-k, k] \times [0, k]$, by scaling Legendre functions very carefully. In a nutshell we can say that with the aid of this scheme we can examine FKGE for quantum field theory, plasma, optical fibers and dispersive wave-phenomena. For future study we can employ operational matrices of various orthonormal polynomials to achieve higher accuracy.

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