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Solution of the Three Forces Problem in the Case of Two Forces Being Mutually Orthogonal

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The present paper delivers some new numerical and exact solutions of the three forces problem that is one of fundamental problems of Michell's truss theory. The present problem is to find the lightest fully stressed truss transmitting three self-equilibrated co-planar forces. In this study, we limit our considerations to the case of two forces being mutually orthogonal. The aim of this paper is to classify possible layouts of optimal trusses depending on the position of the applied lateral point load (the positions of the other two forces are fixed, which however does not restrict the scope of the study). The exact analytical solutions are obtained with a great help of numerical solutions that enable proper prediction of the optimal layouts.

Key words: topology optimization, Michell's trusses, three forces problem.

1. Introduction

The three forces problem (3FP), till now unsolved in general, is one of the most important tasks of Michell's truss theory. The problem is formulated as follows: *find the lightest fully stressed truss transmitting three self-equilibrated co-planar forces*, assuming the feasible domain is the whole plane. A significant class of solutions of 3FP was derived by Chan [1] and used by Sokół and Lewiński [2–4] and Sokół and Rozvany [5] to obtain the optimal trusses with two point loads between supports. Some geometrical aspects of the optimal solution of the 3FP for discretized Michell's trusses were investigated by Mazurek [6]. Recently, the present authors obtained some new exact solutions of simply supported Michell's trusses generated by a lateral point load and constrained in a unit square domain [7]. In the present paper, we expand this class of solutions by extending the feasible design domain and the position of the point load to half or even full plane. However, as before, we assume that two of the three forces are orthogonal, so the problem can be stated as it is shown in Fig. 1. Note that although the supports can be replaced by the appropriate reaction forces, it is convenient to retain the supports to eliminate rigid body motions and to obtain a uniquely determined displacement field.

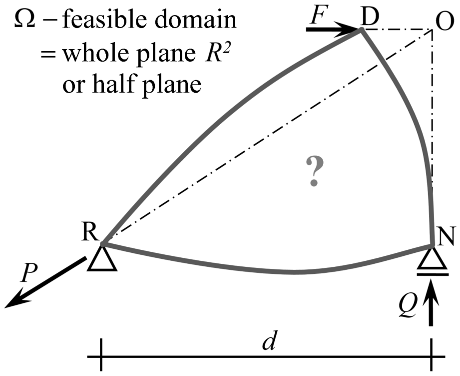

Fig. 1. The three forces problem in the case of two forces being mutually orthogonal.

2. Numerical prediction of the optimal layout

An exact analytical solution of any Michell's problem is very hard to obtain due to the necessity of solving a hyperbolic differential system; therefore, it requires a good prediction of the proper layout that usually has to be divided into appropriate regions. This difficult task can be performed in a reliable and effective way using the *adaptive ground structure method*. This is not the main topic of the paper and will not be given in detail here (a description of the latest, most powerful version of this method and related references can be found in [8]). Nevertheless, it should be pointed out that this method allows the use of very dense ground structures, which reduces the discretization error, and thus makes it possible to obtain very accurate approximations of the optimal solutions to be constructed.

The problem investigated in this paper (see Fig. 1) consists of constructing the two-parameter family of solutions depending on the position of the applied lateral force (i.e., x_{D} and y_{D} coordinates of point D). Upon performing the numerical tests for different positions of load, we can recognize and classify the possible optimal layouts, which then can be confirmed by exact analytical formulae. Selected numerical solutions for fixed value of $y_{\mathrm{D}} = 2d$ and different values of x_{D} are presented in Fig. 2.

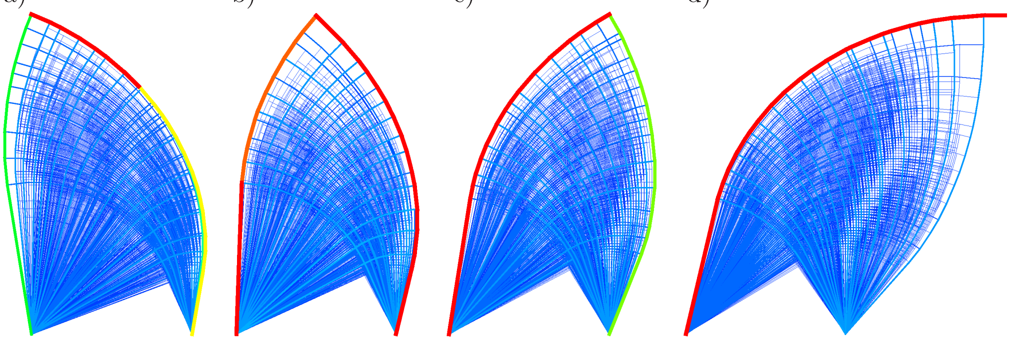

FIG. 2. Numerical solutions of the3FP for half-plane domain and $y_{\mathrm{D}} = 2d$: a) $x_{\mathrm{D}} = 0$, b) $x_{\mathrm{D}} = d/2$, c) $x_{\mathrm{D}} = d$, d) $x_D = 2d$.

3. Adjusting the optimal solution by exact analytical formulae

One of the most important classes of solution of the 3FP is presented in Fig. 3. The optimal layout consists of two circular fans RAB and NAC and the region ABDC with a Hencky net of mutually orthogonal lines (see also [1, 2]). This region is parametrized by the curvilinear system (λ, μ):

$$\tag{3.1}\begin{aligned}\frac{x(\lambda,\mu)}{d} &= \sin\gamma_2\, k_1(\lambda,\mu) - \cos\gamma_2\, k_2(\lambda,\mu),\\ \frac{y(\lambda,\mu)}{d} &= -\sin\gamma_2\, h_1(\lambda,\mu) + \cos\gamma_2\, h_2(\lambda,\mu),\end{aligned}$$

where the functions k_1, k_2, h_1, h_2 are defined as follows:

$$\tag{3.2}\begin{aligned}k_1(\lambda,\mu) &= -\sin(\gamma_2+\mu-\lambda)\, F_2(\lambda,\mu) + \cos(\gamma_2+\mu-\lambda)\, F_1(\lambda,\mu),\\ k_2(\lambda,\mu) &= \sin(\gamma_2+\mu-\lambda)\, F_1(\lambda,\mu) - \cos(\gamma_2+\mu-\lambda)\, F_0(\lambda,\mu),\\ h_1(\lambda,\mu) &= -\cos(\gamma_2+\mu-\lambda)\, F_2(\lambda,\mu) - \sin(\gamma_2+\mu-\lambda)\, F_1(\lambda,\mu),\\ h_2(\lambda,\mu) &= \cos(\gamma_2+\mu-\lambda)\, F_1(\lambda,\mu) + \sin(\gamma_2+\mu-\lambda)\, F_0(\lambda,\mu)\end{aligned}$$

and F_n (λ, μ) are Lommel's functions (see LEWIŃSKI *et al.* [9]).

The coordinates of point D (point of application of force F) are given by

$$\tag{3.3}\begin{aligned}x_{\mathrm{D}} &= x(\theta_1,\theta_2) = d\,[\sin\gamma_2\, k_1(\theta_1,\theta_2) - \cos\gamma_2\, k_2(\theta_1,\theta_2)],\\ y_{\mathrm{D}} &= y(\theta_1,\theta_2) = d\,[-\sin\gamma_2\, h_1(\theta_1,\theta_2) + \cos\gamma_2\, h_2(\theta_1,\theta_2)].\end{aligned}$$

Note that the layout in Fig. 3 is defined by three angles θ_1, θ_2 and γ_2. Of course, these angles depend on the position of point D and they must satisfy (3.3). Thus, we need one additional equation to determine all three angles

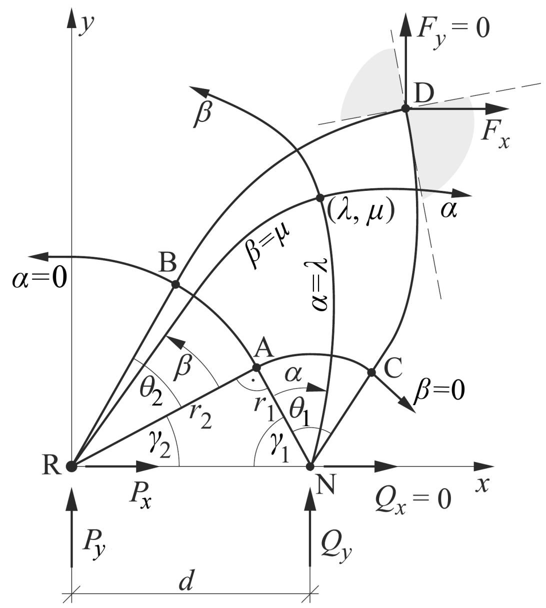

FIG. 3. Parameterization of the fans RBA and NAC and the fibrous domain ABDC.

in a unique way. This equation was derived by CHAN [1] and then confirmed by the kinematic method in [2]. For the problem studied in the present paper we initially assume that $F_y = 0$ and $Q_x = 0$, so the Chan formula can now be written in a very simplified form:

$$\cos\gamma_2\, h_1(\theta_1, \theta_2) + \sin\gamma_2\, h_2(\theta_1, \theta_2) = 0. \tag{3.4}$$

Equations (3.3) and (3.4) form a set of three transcendental equations in three unknowns θ_1, θ_2 and γ_2, which, in general, can be solved only numerically. Nevertheless, it can be solved with any arbitrary precision, so the solutions obtained in this way can be regarded as the exact analytical ones.

According to the kinematic method (which corresponds to the dual formulation of the primal form of least weight structure) the volume of the optimal truss in Fig. 3 can be expressed by

$$V = \frac{F\,\overline{w}_x^{\mathrm{D}}}{\sigma_P}, \tag{3.5}$$

where $\overline{w}_x^{\mathrm{D}}$ is the horizontal component of 'adjoint displacement' of point D defined as (see also Eqs. (6), (7) and (23) in [3])

$$\overline{w}_x^{\mathrm{D}} = u^{\mathrm{D}}\cos(\gamma_2 + \theta_2 - \theta_1) - v^{\mathrm{D}}\sin(\gamma_2 + \theta_2 - \theta_1) - y_{\mathrm{D}}(1 - \sin 2\gamma_2), \tag{3.6}$$

where u^{D} and v^{D} denote the displacements of point D in local curvilinear system and are given by (see also Eq. (8) in [3])

$$
\begin{aligned}
u^{\mathrm{D}} &= 2r_1\theta_1 G_0(\theta_1,\theta_2) + r_2\left[G_0(\theta_1,\theta_2) + 2\theta_2 G_1(\theta_1,\theta_2)\right] \\
&\qquad + (r_2 - r_1)\left[F_1(\theta_1,\theta_2) - F_2(\theta_1,\theta_2)\right], \\
v^{\mathrm{D}} &= -2r_1\theta_2 G_1(\theta_1,\theta_2) - r_2(1+2\theta_2)G_0(\theta_1,\theta_2) \\
&\qquad + (r_2 - r_1)\left[F_1(\theta_1,\theta_2) + F_2(\theta_1,\theta_2)\right].
\end{aligned}
\tag{3.7}
$$

To save space we give only the formula for the horizontal displacement of point D, which is necessary for the calculation of the optimal volume (3.5). The detailed derivation of the displacement field in all regions can be found in [2].

A more detailed study of the system of Eqs. (3.3) and (3.4) clearly limits its validity to the case: $\theta_2 + \gamma_2 - \pi/2 \leqslant \theta_1 \leqslant \theta_2 + \gamma_2$, which corresponds to the force $F = \pm F_x$ lying in the greyed regions marked in Fig. 3. If the force F is applied somewhat further to the right of support N, then θ_1 becomes equal to $\theta_2 + \gamma_2$ and the straight horizontal tie appears to transmit this force to the rest of the structure RBDCNAR (see Fig. 2d). A similar observation can be made for the force F applied to the left of the support R (see Fig. 4a). Note, however, that this type of layouts is valid only for a half-plane domain. If we let the truss lay on the whole plane, the solution will be different with a slightly smaller volume, see Fig. 4b.

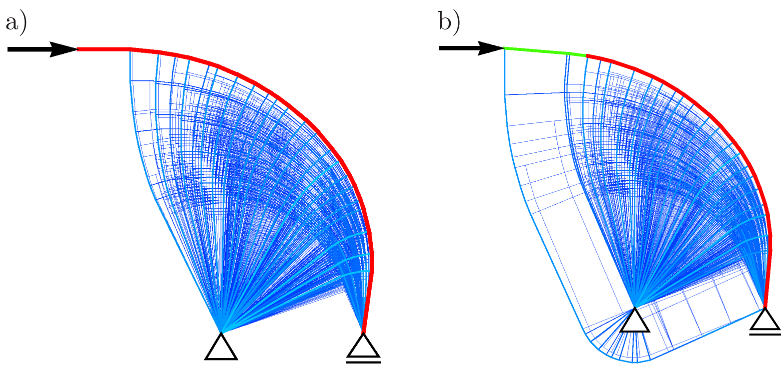

FIG. 4. Comparison of solutions of the 3FP for $x_{\mathrm{D}} = -d$, $y_{\mathrm{D}} = 2d$: a) half-plane domain, b) full-plane domain.

The other types of layouts recognized by the numerical method [8] can be observed if the position of the force F is closer to the horizontal line connecting the supports, see Fig. 5. Here, the layout is composed only of one circular fan and a Hencky net. This layout can be obtained as a special degenerated case of a more general layout shown in Fig. 3 with one of radii r_1 or r_2 equal to

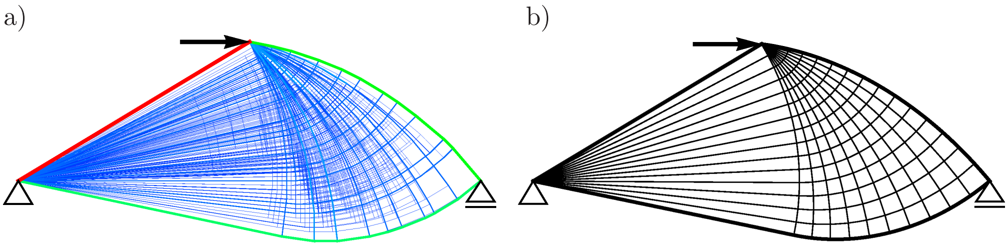

Fig. 5. Exemplary solution of the 3FP for $x_\mathrm{D} = 0.5d$, $y_\mathrm{D} = 0.3d$: a) numerical and b) analytical result.

zero. A more detailed study indicates that this type of layouts can occur only for $x_\mathrm{D}^2 + y_\mathrm{D}^2 \leqslant d^2$, $x_\mathrm{D} \geqslant g(y_\mathrm{D})$ and $y_\mathrm{D} \geqslant h(x_\mathrm{D})$, where $g(\cdot)$ and $h(\cdot)$ are some smooth decreasing functions defined in [7] (see Fig. 7 in [7] and related discussion in this paper for details). If the force F is shifted to the left, so that $x_\mathrm{D} < g(y_\mathrm{D})$ (thus also for negative x_D) then both fans appear to be visible with $r_1, r_2 > 0$, but in contrary to the layouts presented in Figs. 2 and 4 the circular fans start from points R and D.

At the end, it should be pointed out that the solutions presented in the present paper do not cover all possible types of layouts. If the force is very close to or far from the line connecting supports (i.e., $0 \leqslant y_\mathrm{D} < h(x_\mathrm{D})$ or $y_\mathrm{D} > \widetilde{h}(x_\mathrm{D})$), then totally different and new kind of layouts can appear (see examples presented in Fig. 9 in Sokół and Lewiński [7], see also He and Gilbert [10] for the modified 3FP).

4. Final remarks

Among Michell's problems still unsolved, the three forces problem is probably the most important. The solutions to this problem are the basic solutions from which general solutions to Michell problems can be obtained. If the three forces problem is extended to the spatial case of the four forces problem, it may deliver patterns for new designs, especially those that can be produced by additive manufacturing with the use of metallic components, see Smith *et al.* [11].

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