ENGINEERING TRANSACTIONS • *Engng. Trans.* • **63**, 3, 251–272, 2015
Polish Academy of Sciences • Institute of Fundamental Technological Research (IPPT PAN)
National Engineering School of Metz (ENIM) • Poznan University of Technology

Radial Vibrations of a Thick-Walled Spherical Reservoir Forced by an Internal Surge-Pressure in a Compressible Elastic Medium

Edward WŁODARCZYK, Bartosz FIKUS

Faculty of Mechatronics and Aerospace
Military University of Technology
Sylwestra Kaliskiego 2, 00–908 Warszawa, Poland
e-mail: bfikus@wat.edu.pl

The authors investigated radial vibrations of a metal thick-walled spherical reservoir forced by an internal surge-pressure. The reservoir is located in a compressible elastic medium. In this paper, the medium's compressibility is represented by the Poisson's ratio ν. Analytical closed-form formulae determining the dynamic state of mechanical parameters in the reservoir wall have been derived. These formulae were obtained for the surge pressure $p(t) = p_0 = \text{const}$. From analysis of these formulae it follows that the Poisson's ratio ν, substantially influences variations of the parameters of reservoir wall in space and time. All parameters intensively decrease in space along with an increase of the Lagrangian coordinate r. On the contrary, these parameters oscillate *versus* time around their static values. These oscillations decay in the course of time. We can mark out two ranges of parameter ν values in which vibrations of the parameters are "damped" (there is no energy loss due to internal friction, energy is transferred from reservoir to further layers of the medium) at a different rate. Thus, Poisson's ratio in the range below about 0.4 causes intensive decay of parameter oscillations and reduces reservoir dynamics to static state in no time. On the other hand, in the range $0.4 < \nu < 0.5$, the "damping" of parameter vibrations of the reservoir wall is very low. In the limiting case when $\nu = 0.5$ (incompressible medium) "damping" vanishes and the parameters harmonically oscillate around their static values. In the range $0.4 < \nu < 0.5$, insignificant increase of Poisson's ratio causes a considerable increase of the parameter vibration amplitude and decrease of vibration "damping".

Key words: dynamics of continuous media, vibrations of engineering systems, divergent damping, spherical reservoir.

1. Introduction

The issues of the dynamics of a spherical cavity and of the thick-walled spherical reservoir, surrounded by compressible medium and loaded with an internal surge pressure (explosion), are important with respect to theoretical knowledge and useful applications. The comprehensive analysis of these issues is presented by Hopkins in [1]. Recently, several papers [2–6] have been published within this scope.

Two problems have been theoretically investigated in the above mentioned papers:

1. Dynamic expansion of spherical cavity and distributions of mechanical parameters of an expanding spherical stress wave in infinite compressible elastic medium [2–5], due to surge pressure applied to the cavity surface. Among other things, the anomalous influence of the compressibility of the elastic medium on vibration of the cavity surface has been revealed. The compressibility of elastic medium is presented in the papers by the Poisson's ratio ν.
2. Dynamics of a thick-walled spherical reservoir located in vacuum and loaded with internal surge pressure [6]. Material of the reservoir is incompressible. The reservoir vibrates radially with angular frequency and behaves like one degree of freedom system. Spherical sections of the reservoir wall vibrate with constant amplitudes whose values are determined by Lagrangian coordinate r. Its value decreases with increasing of the coordinate r, due to divergence of the displacement of the wall elements.

Coupling the above mentioned two problems into one initial- boundary value problem gives rise to a new type damping of a vibration of the reservoir wall. This damping is due to transfer of energy from vibrating reservoir wall by an expanding stress wave propagating into infinite compressible medium that surrounds the reservoir. The wave is generated by motion of the reservoir wall due to an internal surge pressure.

From the perspective of technical applications, significant problem is the theoretical estimation of the damping coefficient of the reservoir vibration in infinite, compressible medium. This problem has been solved in the analytical form for the linear elastic medium in this paper.

2. Formulation of the problem

Let us consider the radial vibration of a thick- walled metal spherical reservoir, which is in an isotropic linear elastic medium that is infinite. The reservoir was loaded with the internal surge pressure $p(t)$. For the convenience of the reader of analysis of the problem under investigation, successive mechanical quantities will be defined directly as we progress in the current study. Let r_0 and r_1 denote the internal and external radii of the reservoir. The problem has been solved in the spherical system of Lagrangian coordinates r, φ, θ. Taking into account spherical symmetry, the problem can be assumed as a spatially one-dimensional boundary value problem. Therefore, the states of stress and strain in the materials of the reservoir and medium can be represented by the following components: σ_r – radial stress, $\sigma_\varphi = \sigma_\theta$ – tangential stresses, ε_r – radial strain and $\varepsilon_\varphi = \varepsilon_\theta$ – tangential strains.

The rest of components of the stress and strain tensors are equal to zero in this coordinate system.

The problem has been solved according to the linear elasticity theory. Therefore, the following relations can be written [7, 8]:

$$\varepsilon_r(r,t) = \frac{\partial u(r,t)}{\partial r}, \qquad \varepsilon_\varphi(r,t) = \varepsilon_\theta(r,t) = \frac{u(r,t)}{r}, \tag{2.1}$$

$$\sigma_r(r,t) - \sigma_\varphi(r,t) = 2\mu\left[\varepsilon_r(r,t) - \varepsilon_\varphi(r,t)\right] = 2\mu\left[\frac{\partial u(r,t)}{\partial r} - \frac{u(r,t)}{r}\right], \tag{2.2}$$

$$\mu = \frac{E}{2(1+\nu)}, \tag{2.3}$$

where symbols E, μ and ν denote Young's modulus, Lame's constant (shear modulus) and Poisson's ratio, respectively, u denotes the radial displacement of infinitesimal elements of the reservoir or medium, while r denotes their Lagrangian coordinate. In the following considerations the reservoir parameters have been denoted by index z.

For the sake of small strains (2.1), value of the pressure $p(t)$ is limited.

We assumed that reservoir material is incompressible, i.e., $\nu_z = 0.5$ and its density $\rho_z = \text{const}$. These assumptions have been discussed analytically in [2] and investigated experimentally in [9–13]. The error caused by these simplifications is of the order of a fraction of a percent. At these assumptions, dynamics of the reservoir wall, deformed within the scope of small strains can be determined by means of equations:

$$\frac{\partial u_z}{\partial r} + 2\frac{u_z}{r} = 0, \tag{2.4}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + 2\frac{\sigma_{rz} - \sigma_{\varphi z}}{r} = \rho_z \frac{\partial^2 u_z}{\partial t^2}. \tag{2.5}$$

In turn, the equation of motion of a linear elastic medium takes the form [2]:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - \frac{2u}{r^2} = \frac{1}{c_e^2}\frac{\partial^2 u}{\partial t^2}, \tag{2.6}$$

where

$$c_e^2 = \frac{1-\nu}{(1+\nu)(1-2\nu)}c_0^2, \qquad c_0^2 = \frac{E}{\rho}, \tag{2.7}$$

ρ – density of medium.

Components of the stress in elastic medium are determined by formulae:

$$\sigma_r = (2\mu + \lambda)\frac{\partial u}{\partial r} + 2\lambda\frac{u}{r}, \tag{2.8}$$

$$\sigma_\varphi = \sigma_\theta = \lambda\frac{\partial u}{\partial r} + 2\,(\mu + \lambda)\,\frac{u}{r}, \tag{2.9}$$

where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{2.10}$$

The linearized system of equations (2.4), (2.5) and (2.6) has been solved for the following boundary conditions:

$$\sigma_{rz}(r_0, t) = -p(t) \qquad \text{for} \qquad r = r_0, \tag{2.11}$$

$$u_z(r_1, t) = u(r_1, t) \qquad \text{for} \qquad r = r_1, \tag{2.12}$$

$$\sigma_{rz}(r_1, t) = \sigma_r(r_1, t) \qquad \text{for} \qquad r = r_1, \tag{2.13}$$

$$u(r_1 + c_e t, t) \equiv 0 \qquad \text{for} \qquad r = r_1 + c_e t, \tag{2.14}$$

$$\sigma_r(r = \infty, t) \equiv 0 \qquad \text{for} \qquad r = \infty, \tag{2.15}$$

where $r = r_1 + c_e t$ is the trajectory of the front of a stress wave propagating on the outside of the reservoir into the medium (Fig. 1).

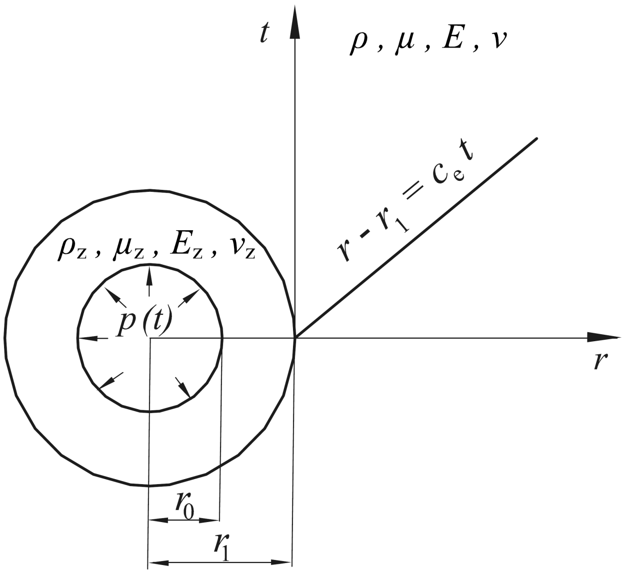


Fig. 1. Scheme of the under investigation initial-boundary value problem.

The physical scheme of the under investigation initial-boundary value problem is shown in Fig. 1.

Such a formulated problem, completed by initial conditions, has been analytically solved in the next section of this paper.

3. Analytical solution of the problem

The general solution of Eq. (2.4) has the following form:

$$u_z(r,t) = \frac{C(t)}{r^2}, \tag{3.1}$$

where $C(t)$ denotes a continuous and twice-differentiable time function, which satisfies the condition $C(0) = 0$, because $u_z(r,0) \equiv 0$.

Subsequently, from relationships (2.2) and (3.1) it follows that

$$\sigma_{rz}(r,t) - \sigma_{\varphi z}(r,t) = -6\mu_z \frac{C(t)}{r^3} = -6\mu_z \frac{u_z}{r}. \tag{3.2}$$

Upon substitution of expressions (3.1) and (3.2) into Eq. (2.5) and integration in respect to r, as well as using condition (2.11), the following expression has been obtained:

$$\sigma_{rz}(r,t) = 4\mu_z \left(\frac{1}{r_0^3} - \frac{1}{r^3}\right) C(t) + \rho_z \left(\frac{1}{r_0} - \frac{1}{r}\right) \ddot{C}(t) - p(t), \tag{3.3}$$

where $\ddot{C}(t) = d^2C/dt^2$.

Furthermore, the relationships (3.2) and (3.3) yield

$$\sigma_{\varphi z}(r,t) = 2\mu_z \left(\frac{2}{r_0^3} + \frac{1}{r^3}\right) C(t) + \rho_z \left(\frac{1}{r_0} - \frac{1}{r}\right) \ddot{C}(t) - p(t). \tag{3.4}$$

The general solution of Eq. (2.6) fulfilling boundary condition (2.15) $\sigma_r(\infty,t) \equiv 0$ at infinity, can be written in the form [2, 7, 8]:

$$u(r,t) = \frac{\varphi'[r - (r_1 + c_e t)]}{r} - \frac{\varphi[r - (r_1 + c_e t)]}{r^2}, \tag{3.5}$$

where the symbol φ' denotes the derivative of function φ with respect to its argument.

In agreement with relationships (2.8), (2.9) and (3.5) the components of the stresses in the medium can be written as

$$\sigma_r(r,t) = (2\mu+\lambda)\frac{\varphi''}{r} - 4\mu\frac{\varphi'}{r^2} + 4\mu\frac{\varphi}{r^3}, \tag{3.6}$$

$$\sigma_\varphi(r,t) = \lambda\frac{\varphi''}{r} + 2\mu\frac{\varphi'}{r^2} - 2\mu\frac{\varphi}{r^3}, \tag{3.7}$$

$$\sigma_\varphi(r,t) - \sigma_r(r,t) = -2\mu\frac{\varphi''}{r} + 6\mu\frac{\varphi'}{r^2} - 6\mu\frac{\varphi}{r^3}. \tag{3.8}$$

Substitution of solutions (3.1) and (3.5) into condition (2.12) gives:

$$C(t) = r_1\varphi'(-c_e t) - \varphi(-c_e t). \tag{3.9}$$

In turn, differentiation of expression (3.9) with respect to time yields:

$$\begin{aligned} \dot{C}(t) &= -r_1 c_e \varphi''(-c_e t) + c_e \varphi'(-c_e t), \\ \ddot{C}(t) &= r_1 c_e^2 \varphi'''(-c_e t) - c_e^2 \varphi''(-c_e t). \end{aligned} \tag{3.10}$$

Upon substitution of expressions (3.9) and $(3.10)_2$ as well as $\mu_z = E_z/2(1+\nu_z) = E_z/3$ into (3.3) and transformations, we have

$$\begin{aligned} \sigma_{rz}(r,t) = \rho_z c_e^2 \left(\frac{r}{r_0} - 1\right)\varphi'''(-c_e t) - \rho_z c_e^2 \left(\frac{r}{r_0} - \frac{1}{r}\right)\varphi''(-c_e t) \\ + \frac{4}{3}E_z r\left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\varphi'(-c_e t) - \frac{4}{3}E_z\left(\frac{1}{r_0^3} - \frac{1}{r^3}\right)\varphi(-c_e t) - p(t). \end{aligned} \tag{3.11}$$

Furthermore, the relationships (2.8), (2.10) and (3.6) yield:

$$\sigma_r(r,t) = \frac{1}{r}\rho c_e^2 \varphi''(-c_e t) - \frac{2E}{1+\nu}\frac{1}{r^2}\varphi'(-c_e t) + \frac{2E}{1+\nu}\frac{1}{r^3}\varphi(-c_e t). \tag{3.12}$$

Subsequently, from condition (2.13), and expressions (3.11) as well as (3.12), we obtain:

$$\varphi'''(-c_e t) - \frac{\dfrac{r_1}{r_0} - 1 + \dfrac{\rho}{\rho_z}}{r_1\left(\dfrac{r_1}{r_0} - 1\right)}\varphi''(-c_e t) + \frac{(1+\nu)\,(1-2\nu)}{(1-\nu)} \frac{\dfrac{4}{3}\dfrac{E_z}{E}\left[\left(\dfrac{r_1}{r_0}\right)^3 - 1\right] + \dfrac{2}{1+\nu}}{r_1^2\left(\dfrac{r_1}{r_0} - 1\right)} \frac{\rho}{\rho_z}\varphi'(-c_e t) - \frac{(1+\nu)(1-2\nu)}{(1-\nu)} \frac{\dfrac{4}{3}\dfrac{E_z}{E}\left[\left(\dfrac{r_1}{r_0}\right)^3 - 1\right] + \dfrac{2}{1+\nu}}{r_1^2\left(\dfrac{r_1}{r_0} - 1\right)} \frac{\rho}{\rho_z}\varphi(-c_e t) = \frac{(1+\nu)(1-2\nu)}{1-\nu}\frac{r_0}{r_1 - r_0}\frac{\rho}{\rho_z}\frac{p(t)}{E}. \tag{3.13}$$

It is necessary to complete this equation with values of initial conditions, namely $\varphi(0)$, $\varphi'(0)$, $\varphi''(0)$.

Substitution of relationship (3.5) into condition (2.14) gives:

$$\frac{\varphi'(0)}{r_1 + c_e t} - \frac{\varphi(0)}{(r_1 + c_e t)^2} \equiv 0. \tag{3.14}$$

From identity (3.14) it follows that

$$\varphi(0) = 0 \quad \text{and} \quad \varphi'(0) = 0. \tag{3.15}$$

In turn, from relationships (3.1) and $(3.10)_1$, we have:

$$\begin{aligned} v_z(r_1, 0) &= \left.\frac{\partial u_z(r_1, t)}{\partial t}\right|_{t=0} = \frac{\dot{C}(0)}{r_1^2} = 0, \\ \dot{C}(0) &= r_1^2 v_z(r_1, 0) = r_1^2\left[-r_1 c_e \varphi''(0) + c_e \varphi'(0)\right] = 0. \end{aligned} \tag{3.16}$$

From expressions $(3.15)_2$ and (3.16) it follows that

$$\varphi''(0) = 0. \tag{3.17}$$

As seen, the under investigation problem has been referred to as ordinary differential equation of the third order (3.13) for function $\varphi(-c_e t)$, that along with its derivatives $\varphi'(-c_e t)$, $\varphi''(-c_e t)$ and $\varphi'''(-c_e t)$ uniquely determines all problem parameters. Therefore, we are going to solve Eq. (3.13) for conditions (3.15) and (3.17) in the following considerations.

In order to simplify notations of successive expressions, the dimensionless quantities have been introduced in the following considerations:

$$
\begin{gathered}
\xi = \frac{r}{r_0}, \qquad \eta = \frac{c_0 t}{r_0}, \qquad \beta = \frac{r_1}{r_0}, \\
\eta_0 = \frac{x_0}{r_1} = -\frac{c_e}{\beta c_0}\eta, \qquad g = \frac{\rho}{\rho_z}, \qquad k = \frac{\mu}{\mu_z}, \\
j = \frac{E}{E_z}, \qquad P = \frac{p_0}{E}, \qquad S_{rz} = \frac{\sigma_{rz}}{p_0}, \\
S_{\varphi z} = \frac{\sigma_{\varphi z}}{p_0}, \qquad S_z = \frac{\sigma_{\varphi z} - \sigma_{rz}}{p_0}, \qquad U_z = \frac{u_z}{r_0}, \qquad (3.18)\\
S_r = \frac{\sigma_r}{p_0}, \qquad S_\varphi = \frac{\sigma_\varphi}{p_0}, \qquad S = \frac{\sigma_\varphi - \sigma_r}{p_0}, \\
\varphi(x_0) = r_1^3 \psi(\eta_0), \qquad \varphi'(x_0) = r_1^2 \psi'(\eta_0), \qquad \varphi''(x_0) = r_1 \psi''(\eta_0), \\
\varphi'''(x_0) = \psi'''(\eta_0),
\end{gathered}
$$

where

$$
x_0 = -c_e t, \qquad c_e = \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}}\, c_0, \qquad c_0 = \sqrt{\frac{E}{\rho}}.
$$

Upon using of quantities (3.18), Eq. (3.13) can be written in a simple form, namely:

$$
a\psi'''(\eta_0) + b\psi''(\eta_0) + c\psi'(\eta_0) + d\psi(\eta_0) = ep\left(-\frac{x_0}{c_e}\right) \Big/ E \tag{3.19}
$$

with initial conditions:

$$
\psi(0) = \psi'(0) = \psi''(0) = 0, \tag{3.20}
$$

where

$$a = \beta - 1, \qquad b = -(\beta - 1 + g),$$

(3.21)

$$c = -d = \frac{2g}{j}\frac{(1+\nu)(1-2\nu)}{(1-\nu)(1+\nu_z)}(\beta^3 - 1 + k), \qquad e = \frac{(1+\nu)(1-2\nu)}{1-\nu}g.$$

Note the fact, that for $\beta = 1$, i.e. when the reservoir vanishes and remains only a medium with cavity factor $a = 0$ and Eq. (3.19), upon using (3.18) and (3.21) can be transformed to the form

$$\varphi''(x_0) - 2h\varphi'(x_0) + \frac{2h}{r_0}\varphi(x_0) = -\frac{(1+\nu)(1-2\nu)}{1-\nu} r_0 p\left(-\frac{x_0}{c_e}\right) \Big/ E,$$

where

$$h = \frac{1-2\nu}{1-\nu}\frac{1}{r_0} \geq 0, \qquad x_0 = -c_e t.$$

This is the equation of the problem presented in [2], which is numbered by (3.4) on page 467. Thus the solution of the problem given in [2] is with particular case of the under consideration issue in this paper

The particular solution of the following homogenous equation

$$a\psi'''(\eta_0) + b\psi''(\eta_0) + c\psi'(\eta_0) + d\psi(\eta_0) = 0 \tag{3.22}$$

can be written as:

$$\psi_p(\eta_0) = \exp(z\eta_0). \tag{3.23}$$

Upon substitution (3.23) into (3.22) we obtain:

$$az^3 + bz^2 + cz + d = 0. \tag{3.24}$$

The discriminant of Eq. (3.24) is negative, i.e.

$$\Delta = -4ac^3 + b^2c^2 + 18abcd - 27a^2d^2 - 4b^3d < 0.$$

Accordingly the roots of Eq. (3.24) are

(3.25)

$$z_1 = \frac{1}{3}\sqrt[3]{\frac{9abc - 27a^2d - 2b^3 + 3a\sqrt{-3\Delta}}{2a^3}} - \left(\frac{3ac - b^2}{3\,a^2\sqrt[3]{\dfrac{9abc - 27a^2d - 2b^3 + 3a\sqrt{-3\Delta}}{2a^3}}}\right) - \frac{b}{3a},$$

$$z_2 = x_1 + ix_2, \qquad z_3 = x_1 - ix_2,$$

where

$$x_1 = \frac{1}{6}\left(-A + \frac{3ac - b^2}{Aa^2}\right) - \frac{b}{3a},$$

$$x_2 = \frac{\sqrt{3}}{6}\left(A + \frac{3ac - b^2}{Aa^2}\right) - \frac{b}{3a},$$

$$A = \sqrt[3]{\frac{9abc - 27a^2d - 2b^3 + 3a\sqrt{-3\Delta}}{2a^3}}.$$

As seen, the root z_1 is real number; on the contrary the roots z_2 and z_3 are conjugate complex numbers that can be expressed by trigonometric functions.

Bearing in mind the above-mentioned roots, the general solution of Eq. (3.22) can be written by means of relationship:

$$\psi_1(\eta_0) = C_1 \exp(z_1\eta_0) + \exp(x_1\eta_0)\left[C_2 \sin(x_2\eta_0) + C_3 \cos(x_2\eta_0)\right]. \tag{3.26}$$

In order to estimate maximal dynamical parameters of the reservoir it was assumed that pressure of detonation products is surge-pressure and it equals $p(t) = p_{\max} = p = \text{const}$. For this loading, the particular solution of nonhomogeneous equation (3.19) can be expressed by formula:

$$\psi_2 = \frac{e}{d}P \tag{3.27}$$

and general solution of this equation can be written with the help of the following sum:

$$\begin{aligned} \psi(\eta_0) &= \psi_1(\eta_0) + \psi_2 \\ &= C_1 \exp(z_1\eta_0) + \exp(x_1\eta_0)\left[C_2 \sin(x_2\eta_0) + C_3 \cos(x_2\eta_0)\right] + \frac{e}{d}P. \end{aligned} \tag{3.28}$$

Substitution of the function $\psi(\eta_0)$ and its derivatives into the initial conditions (3.20) gives three equations:

$$C_1 + C_3 = -\frac{eP}{d},$$

$$z_1C_1 + x_2C_2 + x_1C_3 = 0,$$

$$z_1^2C_1 + 2x_1x_2C_2 + \left(x_1^2 - x_2^2\right)C_3 = 0$$

and upon solution of these equations, we obtain:

$$
\begin{aligned}
\frac{C_1}{P} &= \frac{-\dfrac{e}{d}\left[x_2(x_1^2 - x_2^2) - 2x_1^2 x_2\right]}{x_2(x_1^2 - x_2^2) - 2x_1^2 x_2 + 2x_1 x_2 z_1 - z_1^2 x_2},\\
\frac{C_2}{P} &= \frac{\dfrac{e}{d}\left[z_1(x_1^2 - x_2^2) - z_1^2 x_1\right]}{x_2(x_1^2 - x_2^2) - 2x_1^2 x_2 + 2x_1 x_2 z_1 - z_1^2 x_2},\\
\frac{C_3}{P} &= \frac{-\dfrac{e}{d}\left[2z_1 x_1 x_2 - z_1^2 x_2\right]}{x_2(x_1^2 - x_2^2) - 2x_1^2 x_2 + 2x_1 x_2 z_1 - z_1^2 x_2}.
\end{aligned}
\tag{3.29}
$$

Thus an explicit function $\psi(\eta_0)$ has been determined, which together with its derivatives uniquely defines all dynamical parameters of the reservoir.

4. The parameters characterizing reservoir dynamics after explosion

By means of relationships derived in Secs. 2 and 3, after using function $\psi(\eta_0)$ and its derivatives, the dimensionless parameters characterizing reservoir dynamics can be written as:

$$
\begin{aligned}
\frac{U_z(\xi,\eta)}{P} = \frac{\beta^3}{\xi^2}\Bigg[\frac{C_1}{P}(z_1 - 1)\exp\left(-\frac{c_e z_1}{c_0\beta}\eta\right) &+ \frac{A_1(\eta)}{P}\sin\left(\frac{c_e x_2}{c_0\beta}\eta\right)\\
&+ \frac{B_1(\eta)}{P}\cos\left(\frac{c_e x_2}{c_0\beta}\eta\right) - \frac{e}{d}\Bigg],
\end{aligned}
\tag{4.1}
$$

$$
\varepsilon_{rz}(\xi,\eta) = -\frac{2}{\xi}U_z(\xi,\eta), \qquad \varepsilon_{\varphi z}(\xi,\eta) = \frac{1}{\xi}U_z(\xi,\eta),
\tag{4.2}
$$

$$
\begin{aligned}
S_{rz}(\xi,\eta) = -1 + \left(1 - \frac{1}{\xi}\right)\Bigg\{\frac{C_1}{P}\beta(z_1 - 1)\Bigg[&\frac{2\beta^2}{(1+\nu_z)j}\left(1 + \frac{1}{\xi} + \frac{1}{\xi^2}\right)\\
&+ \frac{(1-\nu)z_1^2}{(1+\nu)(1-2\nu)g}\Bigg]\exp\left(-\frac{c_e z_1}{c_0\beta}\eta\right)\\
+ \beta\exp\left(-\frac{c_e x_1}{c_0\beta}\eta\right)\Bigg[\frac{A_2(\xi)}{P}\sin\left(\frac{c_e x_2}{c_0\beta}\eta\right) &+ \frac{B_2(\xi)}{P}\cos\left(\frac{c_e x_2}{c_0\beta}\eta\right)\Bigg]\\
&- \frac{2\beta^3}{(1+\nu_z)j}\left(1 + \frac{1}{\xi} + \frac{1}{\xi^2}\right)\frac{e}{d}\Bigg\},
\end{aligned}
\tag{4.3}
$$

$$S_{\varphi z}(\xi,\eta) = S_{rz}(\xi,\eta) + \frac{2}{P}\frac{U_z}{\xi}, \tag{4.4}$$

$$S_z(\xi,\eta) = S_{\varphi z}(\xi,\eta) - S_{rz}(\xi,\eta) = \frac{2}{P}\frac{U_z}{\xi}, \tag{4.5}$$

where

$$\begin{aligned}
\frac{A_1(\eta)}{P} &= -\exp\left(-\frac{c_e x_1}{c_0\beta}\eta\right)\left((1-x_1)\frac{C_2}{P} + x_2\frac{C_3}{P}\right),\\
\frac{B_1(\eta)}{P} &= -\exp\left(-\frac{c_e x_1}{c_0\beta}\eta\right)\left(x_2\frac{C_2}{P} + (x_1-1)\frac{C_3}{P}\right),\\
\frac{A_2(\xi)}{P} &= \frac{2\beta^2}{(1+\nu_z)j}\left[(1-x_1)\frac{C_2}{P} + x_2\frac{C_3}{P}\right]\\
&\qquad\cdot\left(1+\frac{1}{\xi}+\frac{1}{\xi^2}\right) - \frac{1-\nu}{(1+\nu)(1-2\nu)g}\\
&\qquad\cdot\left[(x_1^3 - 3x_1x_2^2 - x_1^2 + x_2^2)\frac{C_2}{P} + (x_2^3 - 3x_1^2x_2 + 2x_1x_2)\frac{C_3}{P}\right],\\
\frac{B_2(\xi)}{P} &= \frac{2\beta^2}{(1+\nu_z)j}\left[x_2\frac{C_2}{P} + (x_1-1)\frac{C_3}{P}\right]\\
&\qquad\cdot\left(1+\frac{1}{\xi}+\frac{1}{\xi^2}\right) + \frac{1-\nu}{(1+\nu)(1-2\nu)g}\\
&\qquad\cdot\left[(3x_1^2x_2 - x_2^3 - 2x_1x_2)\frac{C_2}{P} + (x_1^3 - 3x_1x_2^2 - x_1^2 + x_2^2)\frac{C_3}{P}\right],
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&\frac{c_e}{c_0} = \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}}, \qquad \xi = \frac{r}{r_0}, \qquad \eta = \frac{c_0 t}{r_0},\\
&\eta_0 = \frac{x_0}{r_1} = -\frac{c_e}{\beta c_0}\eta, \qquad c_0 = \sqrt{\frac{E}{\rho}},
\end{aligned} \tag{4.7}$$

S_z denotes reduced stress in reservoir wall.

It should be noted, that in accordance with Eq. (4.1), the value of U_z/P is independent of the pressure value, which is the result of C_1/P, C_2/P and C_3/P being independent of the pressure level (according to Eqs. (3.29)).

The static values of the mechanical parameters of the reservoir wall can be determined by substituting infinite value $\eta = \infty$ into the above mentioned expressions. Then we have:

$$\lim_{\eta\to\infty} \exp\left(-\frac{c_e}{c_0}\frac{x_1}{\beta}\eta\right) = 0,$$

$$\lim_{\eta\to\infty} \exp\frac{A_1(\eta)}{P} = 0,$$

$$\lim_{\eta\to\infty} \exp\frac{B_1(\eta)}{P} = 0$$

and the parameters presented in Eqs. (4.1)–(4.5) take on the following forms:

$$\left.\frac{U_z(\xi)}{P}\right|_s = -\frac{e}{d}\frac{\beta^3}{\xi^2}, \tag{4.8}$$

$$\varepsilon_{rz}(\xi)|_s = 2\frac{e}{d}\left(\frac{\beta}{\xi}\right)^3, \qquad \varepsilon_{\varphi z}(\xi)|_s = -\frac{e}{d}\left(\frac{\beta}{\xi}\right)^3, \tag{4.9}$$

$$S_{rz}(\xi)|_s = -\frac{2\beta^3}{(1+\nu_z)j}\left(1-\frac{1}{\xi^3}\right)\frac{e}{d} - 1, \tag{4.10}$$

$$S_{\varphi z}(\xi)|_s = -2\beta^3\frac{e}{d}\left[\frac{1}{\xi^2} + \frac{1}{(\nu_z+1)j}\left(1-\frac{1}{\xi^3}\right)\right] - 1, \tag{4.11}$$

$$S_z(\xi)|_s = -2\frac{e}{d}\frac{\beta^3}{\xi^3}, \tag{4.12}$$

where

$$\frac{e}{d} = \frac{-3(1+\nu)}{4(\beta^3-1)(1+\nu)+6}. \tag{4.13}$$

Dynamical parameters of the spherical reservoir in a vacuum, i.e. with no medium, loaded with the internal surge-pressure, according to [6] can be written as

$$\frac{U_z^a(\xi,\eta)}{P_z} = \frac{3}{4}\frac{\beta^3}{\beta^3-1}\frac{1}{\xi^2}\left(1-\cos\sqrt{\frac{4}{3}\left(1+\frac{\beta+1}{\beta^2}\right)}\eta\right), \qquad P_z = \frac{p_0}{E_z}, \tag{4.14}$$

$$\varepsilon_{\varphi z}^a(\xi,\eta) = -\frac{1}{2}\varepsilon_{rz}^a(\xi,\eta) = \frac{1}{\xi}U_z^a(\xi,\eta), \tag{4.15}$$

$$S_{rz}^a(\xi,\eta) = -\frac{1}{\beta^3-1}\left[\left(\frac{\beta}{\xi}\right)^3 - 1\right] + A(\xi)\cos\sqrt{\frac{4}{3}\left(1+\frac{\beta+1}{\beta^2}\right)}\eta, \tag{4.16}$$

$$A(\xi) = \frac{1}{\beta^3 - 1}\left[\left(\frac{\beta}{\xi}\right)^3 - \left(\beta^2 + \beta + 1\right)\left(\frac{\beta}{\xi} - 1\right) - 1\right],$$

(4.17)

$$S^a_{\varphi z}(\xi,\eta) = S^a_{rz}(\xi,\eta) + \frac{2}{P_z}\frac{U^a_z(\xi,\eta)}{\xi},$$

$$S^a_z(\xi,\eta) = S^a_{\varphi z}(\xi,\eta) - S^a_{rz}(\xi,\eta) = \frac{2}{P_z}\frac{U^a_z(\xi,\eta)}{\xi}, \tag{4.18}$$

where the additional index a denotes the parameters for the reservoir wall without the surrounding medium.

5. Quantitative analysis of dynamical parameters of the reservoir

We assume that the densities and Young's moduli of materials of the reservoir and medium are identical, i.e. $g = (\rho/\rho_z) = 1$ and $j = (E/E_z) = 1$. Then quantitative analysis of all dimensionless parameters of the reservoir (3.16) can be carried out in terms of dimensionless independent variables ξ and η.

Exemplary motions of the relative displacement of the reservoir internal surface ($\xi = 1$), $U_z(1,\eta)/P$, *versus* dimensionless time $\eta = ct/r_0$, for a few values of Poisson's ratio ν and $\beta = 2$, are shown in Fig. 2a. Parameter ν, similarly as bulk modulus characterizes compressibility of the medium in this paper. As it can be seen, the parameter ν substantially influences the behaviour of the function $U_z(1,\eta)/P$ in terms of η.

We can determine two ranges of ν values in which vibration of the reservoir internal surface is "damped" (there is no energy loss due to internal friction in reservoir wall, but energy is transferred from vibrating reservoir to further layers of medium) with a different degree. Thus, decrease of the parameter ν below the value of about 0.4 causes intense decaying of the reservoir internal surface vibration. In this range of ν values the displacement of the internal surface of the reservoir approaches its static value, i.e. $(U_z/P)|_s = 12(1+\nu)/[14(1+\nu)+3]$, after a few cycles of vibrations (Fig. 2a). On the other hand in the range $0.4 < \nu < 0.5$, i.e. in quasi-incompressible media the vibration "damping" is very low. In the limiting case, when $\nu = 0.5$, i.e. in the incompressible medium "damping" vanishes and the reservoir surface harmonically vibrates around its static position $(U_z/P)|_s = 0.75$, with the constant amplitude $A_u = 0.75$ (Fig. 2a).

The increase of the thickness of the reservoir wall $\beta - 1$ also reduces damping of its vibration (see Figs. 2a and 2b). This damping is due to increase of the reservoir mass.

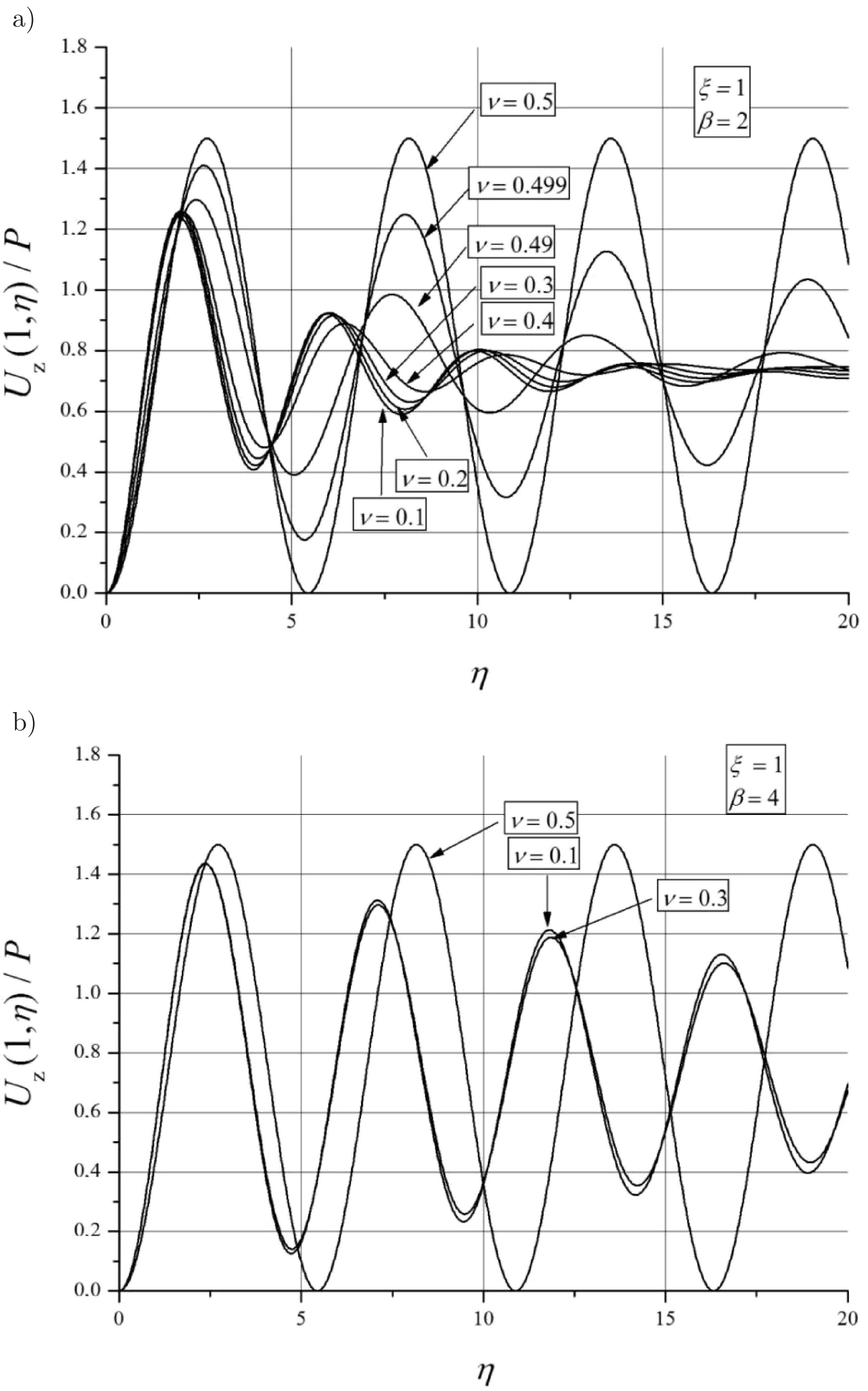


FIG. 2. Calculated relative displacement of the internal surface of the reservoir $U_z(1,\eta)/P$ *versus* η for selected values of the Poisson's ratio ν and the thickness of the reservoir wall $\beta-1$.

In the other limiting case, when $\beta = 1$, i.e., where there is only cavity in medium with no reservoir, the motions of the relative displacement of the cavity surface $U_z(1,\eta)/P$ *versus* $\eta = ct/r$, for a few values of ν, are shown in Fig. 3. The same results for vibrations of the spherical cavity surface in elastic medium, loaded with surge – pressure, has been published in [2] (see Fig. 2 – p. 470).

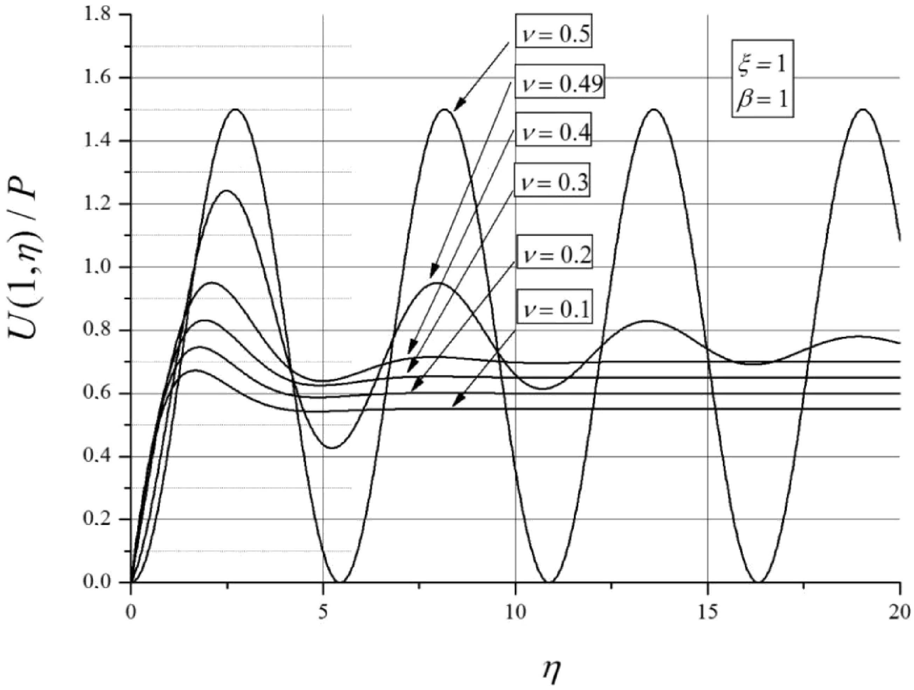


FIG. 3. Calculated relative displacement of the cavity surface $U(1,\eta)/P$ with no reservoir ($\beta = 1$) *versus* η for selected values of the Poisson's ratio ν.

The relative displacement of the boundary surfaces of the reservoir wall (internal $\xi = 1$ and external $\xi = 2$) and of the surface of its middle spherical section $\xi = 1.5$ *versus* dimensionless time η for $\nu = 0.3$ and $\beta = 2$ are depicted in Fig. 4 (solid line). Vibrations of these surfaces intensively decay and approach their static values, i.e. $(U_z/P)|_s = 0.73585/\xi^2$ after a few cycles of vibrations.

If the reservoir is in a vacuum (with no medium), then according to (4.14) the same surfaces of the reservoir wall harmonically vibrate around their static positions $(U_z^a/P)|_s = (6/7)/\xi^2$ with constant amplitudes $A_u^a = (6/7)/\xi^2$. This case is depicted in Fig. 4 by dashed lines. As it can be seen, the elastic compressible medium intensively takes up the energy of the vibrating reservoir and very effectively reduces its dynamics to static state in range of several periods of reservoir vibrations.

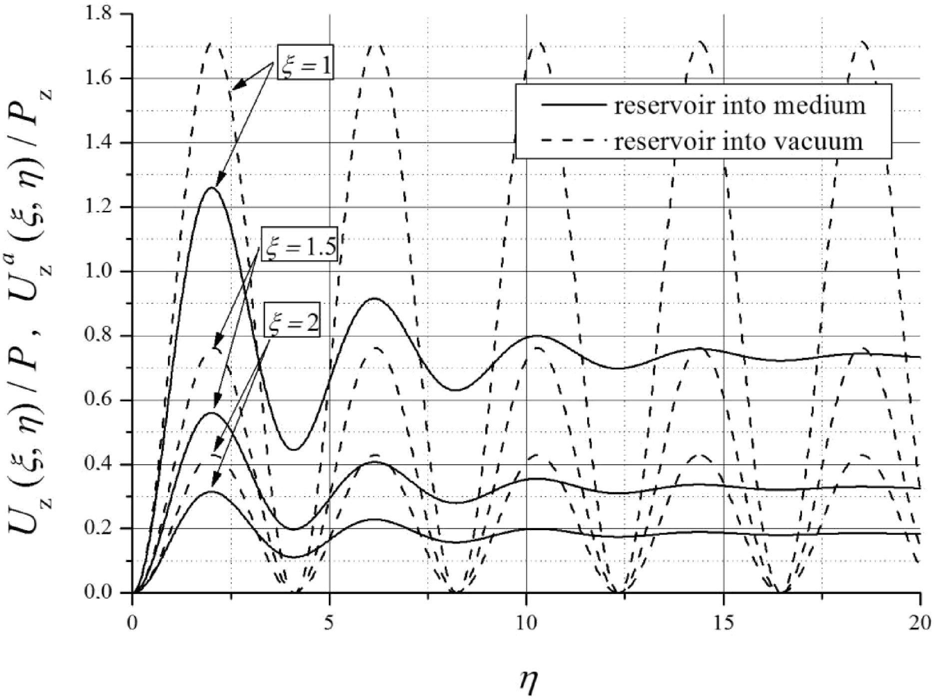


FIG. 4. Calculated relative displacements $U_z(\xi,\eta)/P$ and $U_z^a(\xi,\eta)/P$ of the selected reservoir surfaces: $\xi = 1$, $\xi = 1.5$ and $\xi = 2$ for $\beta = 2$ and $\nu = 0.3$ *versus* η.

Figures 5a and 5b show distributions of maximal values of the functions $S_{rz}(\eta_a,\xi) = S_{rz\,\max}(\xi)$ and $S_{rz}^a(\eta_a,\xi) = S_{rz\,\max}^a(\xi)$ in terms of ξ for $\nu = 0.3$ as well $\beta = 2$ (Fig. 5a) and $\beta = 4$ (Fig. 5b), where parameter η_a denotes dimensionless time $\eta = ct_a/r$, where t_a denotes time of reaching of first maximal values by above – mentioned functions in given sections ξ of the reservoir wall during its vibration. Graphs of the static distributions of the radial stress $S_{rz}(\xi)|_s$ along the thickness of the reservoir wall are also depicted in these figures.

From the graphs presented in Figs. 5a and 5b it follows that the thickness of the reservoir wall, represented by difference $\beta-1$, has substantial influence on distributions of the functions $S_{rz\,\max}(\xi)$, $S_{rz\,\max}^a(\xi)$ and $S_{rz}(\xi)|_s$ *versus* ξ. Above all, increase of the parameter β brings about decaying of the "damping" of the reservoir vibration, due to the medium, and at $\beta = 4$ is $S_{rz\,\max}(\xi) \approx S_{rz\,\max}^a(\xi)$. In addition, in graphs presented in Fig. 5b it can be seen that the functions $S_{rz\,\max}(\xi)$ and $S_{rz\,\max}^a(\xi)$ can change their sign from negative to positive (tension) on the internal reservoir wall. The relationships $S_{rz\,\max}(4) \approx S_{rz\,\max}^a(4) \approx S_z(4)|_s \approx 0$ occurs on the boundary surface with the medium.

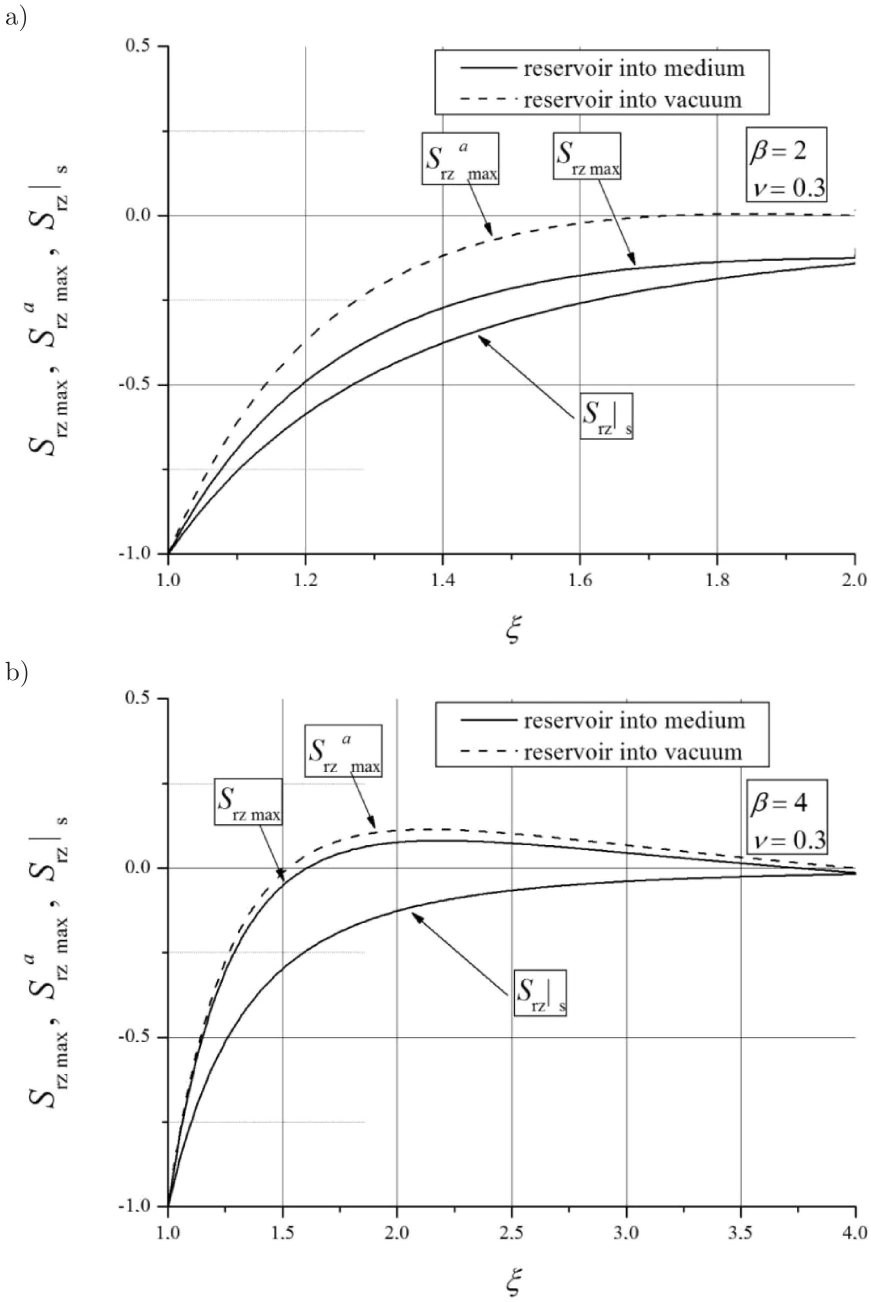
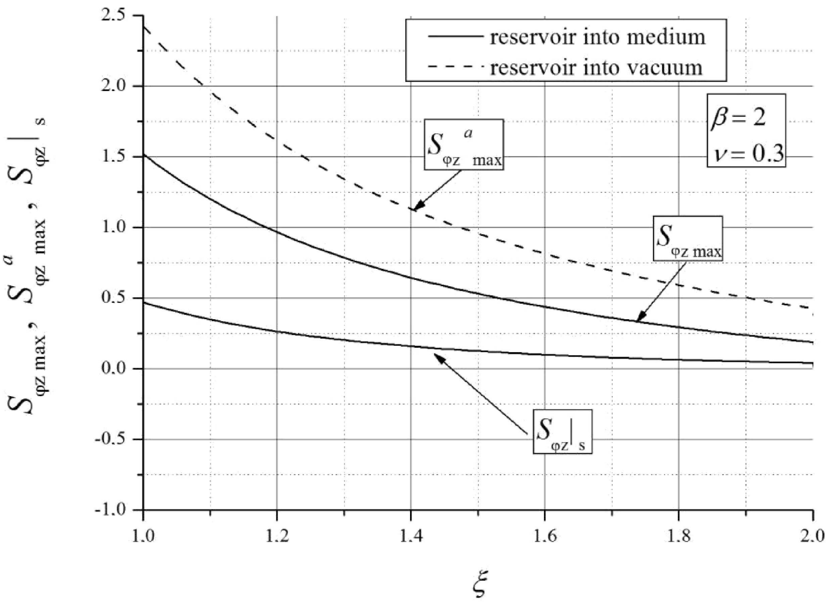


FIG. 5. Calculated distributions of the functions $\mathrm{S}_{rz\ \max}(\xi)$, $S^a_{rz\ \max}(\xi)$, $S_{rz}(\xi)|_s$ *versus* ξ.

Figures 6 and 7 show that functions $S_{\varphi z\max}(\xi)$, $S^a_{\varphi z\max}(\xi)$, $S_{\varphi z}(\xi)|_s$, $S_{z\max}(\xi)$, $S^a_{z\max}(\xi)$, $S_z(\xi)|_s$ are positive in whole reservoir wall independently from the a)



b)

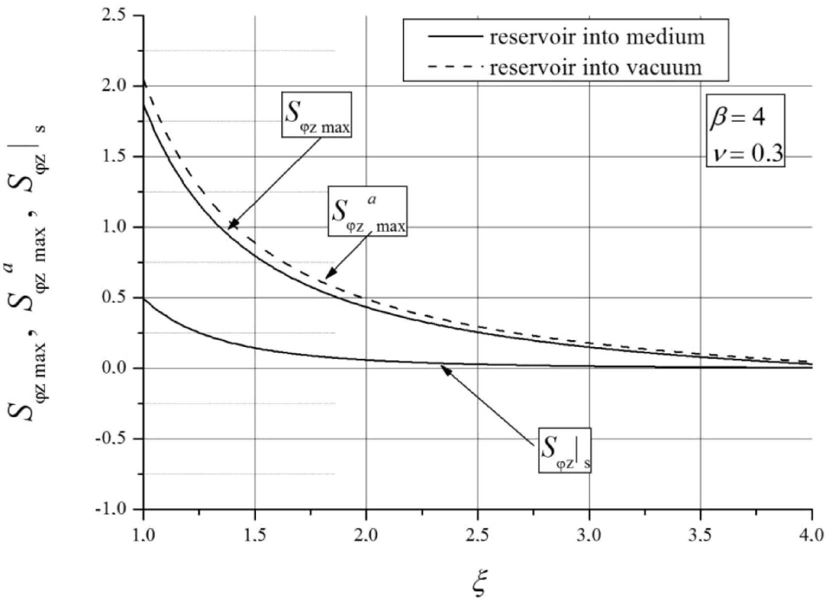


FIG. 6. Calculated distributions of the functions $S_{\varphi z\,\max}(\xi)$, $S^a_{\varphi z\,\max}(\xi)$, $S_{\varphi z}(\xi)|_s$ *versus* ξ.

value of β. Besides, influence of thickness of the reservoir wall on distributions of the above mentioned functions *versus* ξ is similar as in case of the radial functions (Fig. 5).

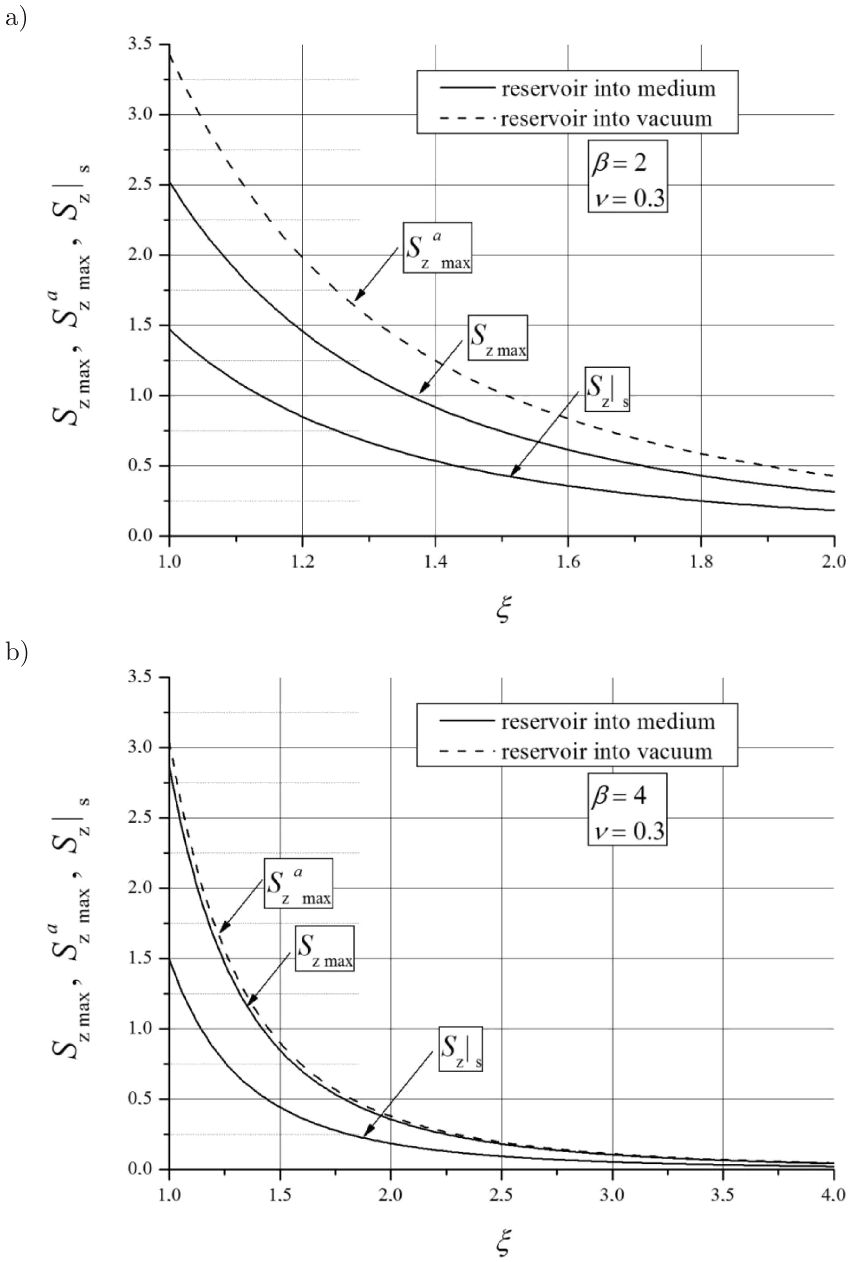


FIG. 7. Calculated distributions of the functions $S_{z\,\max}(\xi)$, $S^a_{z\,\max}(\xi)$, $S_z(\xi)|_s$ *versus* ξ.

6. Final conclusion

The main conclusions derived from the above presented theoretical investigations may be briefly summarized as follows:

- Analytical solution of the radial vibration problem of the thick-walled spherical reservoir in compressible linear elastic medium has been solved in the closed form. The vibration is forced by an internal surge-pressure. In this paper, the medium's compressibility is represented by the Poisson's ratio ν.
- The Poisson's ratio ν substantially influences vibrations of the parameters in spherical sections of the reservoir wall in the course of time.
- All parameters intensively and monotonously decrease in space along with an increase of the Lagrangian coordinate r due to divergence of the displacement of the wall elements.
- These parameters oscillate *versus* time around their static values. The oscillations decay in the course of time for $\nu < 0.5$. During the "damping" of the parameter vibration there is no energy loss due to internal friction in the reservoir wall, but energy is transferred from the vibrating reservoir to further layers of the infinite medium.
- The compressibility of the medium of Poisson's ratio in the range below about 0.4 causes intense decay of parameters' oscillations and reduces reservoir dynamics to static state in the range of several vibration's periods.
- On the contrary, in the range $0.4 < \nu < 0.5$, the "damping" of parameter vibrations of the reservoir wall is very low, and when $\nu = 0.5$ (incompressible medium) "damping" vanishes and the parameters harmonically oscillate around their static values.
- The reservoir located in vacuum vibrates radially with an angular frequency and behaves like one degree of freedom system. Its spherical sections vibrate with constant amplitudes whose values are determined by coordinate r.
- Results of the presented analysis can be applied during engineering design of reservoirs located in elastic media.
- Increase of the thickness of the reservoir wall $\beta-1$ brings about decaying of the "damping" of the reservoir vibration, due to the medium.
- According to the authors' best knowledge, the results presented in this paper have not been presented so far in available literature.

References

1. Hopkins H.G., *Dynamic expansion of spherical cavities in metals*, Progress in Solid Mechanics, **1**, 83–164, 1960.
2. Włodarczyk E., Zielenkiewicz M., *Influence of elastic material compressibility on parameters of an expanding spherical stress wave*, Shock Waves, **18**, 465–473, 2009.

3. Włodarczyk E., Zielenkiewicz M., *Influence of elastic material compressibility on parameters of the expanding spherical stress wave. I. Analytical solution to the problem*, Journal of Theoretical and Applied Mechanics, **47**, 1, 127–141, 2009.

4. Włodarczyk E., Zielenkiewicz M., *Analysis of the parameters of a spherical stress wave expanding in a linear isotropic elastic medium*, Journal of Theoretical and Applied Mechanics, **47**, 4, 761–778, 2009.

5. Włodarczyk E., Zielenkiewicz M., *On the dynamic coefficient of load generating an expanding spherical stress wave in elastic medium*, Journal of Theoretical and Applied Mechanics, **49**, 2, 457–475, 2011.

6. Włodarczyk E., Zielenkiewicz M., *Dynamics of a thickwalled spherical casing loaded with a time depending internal pressure*, Journal of Theoretical and Applied Mechanics, **46**, 1, 21–40, 2008.

7. Achenbach J.D., *Wave propagation in elastic solids*, North Holland Publ. Co., American Elsevier, Amsterdam – New York, 1975.

8. Wasley R.J., *Stress wave propagation in solids, an introduction*, Marcel Dekker, New York, 1973.

9. Gurney R.W., *The initial velocities of fragments from bombs, shells and granades*, BRL Rep. 405 (September), 1943.

10. Gurney R.W., *Fragmentation of bombs, shells and granades*, BRL Report 635 (March), 1947.

11. Duvall G.E., *Shock waves in the study of solids*, Appl. Mech. Reviews, **15**, 11, 1962.

12. Baym F.A., Orlenko Ł.Ł. Stanukowych K.P., Chelychev W.P., Tchehter B.I., *The physics of explosion* [in Russian], Nauka, Moskva, 1975.

13. Rinehart J.S., Pearson J., *Behavior of metals under impulsive loads*, ASM, Cleveland, Ohio, 1954.

Received July 21, 2014; accepted December 18, 2014.