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Continuum Model of Orthotropic Tensegrity Plate-Like Structures with Self-Stress Included

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A continuum model of the orthotropic tensegrity plate-like structures with self-stress included is proposed within the six-parameter flat shell theory. This approach allows to simplify calculations, that is, it is not necessary to describe the whole complex tensegrity cable-strut structure with the use of computational methods. Average displacements, strains and internal forces for orthotropic tensegrity plate-like structures can be determined within the model. The closed form solutions for selected tensegrity plate strips and simply supported rectangular plate with sinusoidal load are presented in the paper. Tensegrity modules, which are based on the four-strut expanded octahedron modules with additional connecting cables are proposed as the examples. Self-stress and some geometrical parameters are introduced for parametric analysis.

Key words: tensegrity plate-like structure, six-parameter flat shell theory.

1. Introduction

Tensegrities are defined as cable-strut structures consisting of isolated compressed elements inside a continuous net of tensioned members [1, 2]. The concept of tensegrity concerns specific trusses structures, with a node configuration that ensures occurrence of infinitesimal mechanisms balanced with self-stress states. Tensegrity structures are complex regarding both geometry and mechanical properties. The tensegrity concept has found wide applications within architecture and civil engineering [3]. In order to estimate their actual properties and identify features of the structure as a whole, a continuum model is considered.

The present paper focuses on the application of a continuum theory based on the three-dimensional tensegrity plate-like structure. As a result, the two-dimensional plate theory is built to describe mechanics of the space tensegrity structure. The first step of the proposed modelling is the selection of an orthotropic repetitive segment, which is taken out from the tensegrity plate-like structure. Then, the selected representative segment undergoes numerical homogenization [4, 5]. By comparing the elastic strain energy from the FEM truss formulation to the energy of a solid, a continuum model of the segment is obtained. The homogeneous segments are afterwards joined together to create the three-dimensional orthotropic continuum, which includes the effect of self-stress [6]. After applying the assumptions of plate theory to the moderately thick plates and integration over the thickness, a two-dimensional plate model is obtained for both membrane and bending deformations. The model includes the effect of self-stress initially applied to the tensegrity structure.

2. Six-parameter plate theory

Mathematical model of the tensegrity plate is a linearized form of the six-parameter shell theory [7] with the curvature tensor $b_{\alpha\beta} = 0$ $(\alpha, \beta = 1, 2)$. A detailed discussion of the theory can be found in [8]. The rectangular flat shell of thickness h in a global coordinate system x_1, x_2, x_3 is considered. The middle plane is taken as the reference surface. A displacement field is described by three linear displacements u_α, w and three rotations ϕ_α, ψ. The following equations are to be valid:

- geometrical relationships:

$$\begin{aligned} &\gamma_{\alpha\beta} = u_{\alpha,\beta} - \epsilon_{\alpha\beta}\psi, \qquad \gamma_{\alpha 3} = \phi_\alpha + w_{,\alpha}, \\ &\gamma_{33} = \psi, \qquad \kappa_{\alpha\beta} = \phi_{\alpha,\beta}, \qquad \kappa_{\alpha 3} = \psi_{,\alpha}, \end{aligned} \tag{2.1}$$

  where $\gamma_{\alpha\beta}$, $\gamma_{\alpha 3}$, γ_{33} $\kappa_{\alpha\beta}$, $\kappa_{\alpha 3}$ are the strain components and $\epsilon_{\alpha\beta}$ is the Ricci symbol,

- constitutive relationships:

$$\begin{aligned} &N_{\alpha\beta} = B^0_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}, \qquad N_{\alpha 3} = k^2 B^0_{\alpha 3\beta 3}\gamma_{\beta 3}, \\ &M_{\alpha\beta} = \frac{h^2}{12} B^0_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}, \qquad M_{\alpha 3} = \frac{h^2}{12} l^2 B^0_{\alpha 3\beta 3}\kappa_{\beta 3}, \end{aligned} \tag{2.2}$$

  where $N_{\alpha\beta}$, $N_{\alpha 3}$, $M_{\alpha\beta}$, $M_{\alpha 3}$ are the internal forces and k^2, l^2 are correction factors,

- equilibrium equations:

$$N_{\alpha\beta,\alpha} + f_\beta = 0, \qquad N_{\alpha 3,\alpha} + f_3 = 0,$$
$$M_{\alpha\beta,\alpha} - N_{\beta 3} + m_\beta = 0, \qquad M_{\alpha 3,\alpha} + \epsilon_{\alpha\beta} N_{\alpha\beta} + m_3 = 0, \tag{2.3}$$

where f_β, f_3, m_β, m_3 are the external loads. The constitutive equations for orthotropic tensegrity plate-like structures are proposed below.

3. Constitutive relations for orthotropic tensegrity plate-like structure

The orthotropic tensegrity plate-like structure (Fig. 1a) is selected to present the constitutive relations. The example is a system of dully connected repetitive expanded four-strut octahedron modules [1, 2] with additional cables (Figs. 1b, c, d) used to ensure stability of the structure. It is assumed that a single tensegrity module is included in the cube equal to the thickness of the plate. To obtain a fully orthotropic system the distances between three orthogonal pairs of struts are defined as follows [4]:

$$\frac{x}{X} = 0.65, \qquad \frac{y}{Y} = 0.30, \qquad \frac{z}{Z} = 0.56. \tag{3.1}$$

This system consists of struts, regular cables and connecting cables, which are described by stiffness: $(EA)_{\text{strut}}$, $(EA)_{\text{cable}}$ and $(EA)_{\text{connection}}$, respectively.

After the procedure described in the previous section (see also [4, 5] for details) non zero coefficients of the elasticity tensor of the tensegrity plate-like structure are as follows:

$$\begin{gathered} B^0_{1111} = \frac{2EA}{h}\delta_{11}, \qquad B^0_{2222} = \frac{2EA}{h}\delta_{22}, \\ B^0_{2323} = \frac{EA}{h}\delta_{23}, \qquad B^0_{1313} = \frac{EA}{h}\delta_{13}, \\ B^0_{1122} = 2B^0_{1212} = 2B^0_{1221} = \frac{EA}{h}\delta_{12}, \end{gathered} \tag{3.2}$$

where

$$\delta_{11} = 1 + 1.52325n + 0.13125m + 0.129225\sigma, \tag{3.3}$$

$$\delta_{22} = 1 + 1.35912n + 0.35m + 0.137028\sigma, \tag{3.4}$$

$$\delta_{23} = 1.51283n - 0.168813\sigma, \tag{3.5}$$

$$\delta_{13} = 1.26604n - 0.153207\sigma, \tag{3.6}$$

$$\delta_{12} = 0.845615n - 0.105243\sigma. \tag{3.7}$$

The parameters (3.3)–(3.7) and, in consequence, displacements, strains and internal forces of the tensegrity plate, depend on the coefficients n and m that describes proportions of member properties (Young's modulus and cross-sections) and on the level of self-stress σ (assumed even for each module and multiplied by the force S):

$$n = \frac{(EA)_{\text{cable}}}{(EA)_{\text{strut}}}, \qquad m = \frac{(EA)_{\text{connection}}}{(EA)_{\text{strut}}},$$
$$(EA)_{\text{strut}} = EA, \qquad \sigma = \frac{S}{EA}. \tag{3.8}$$

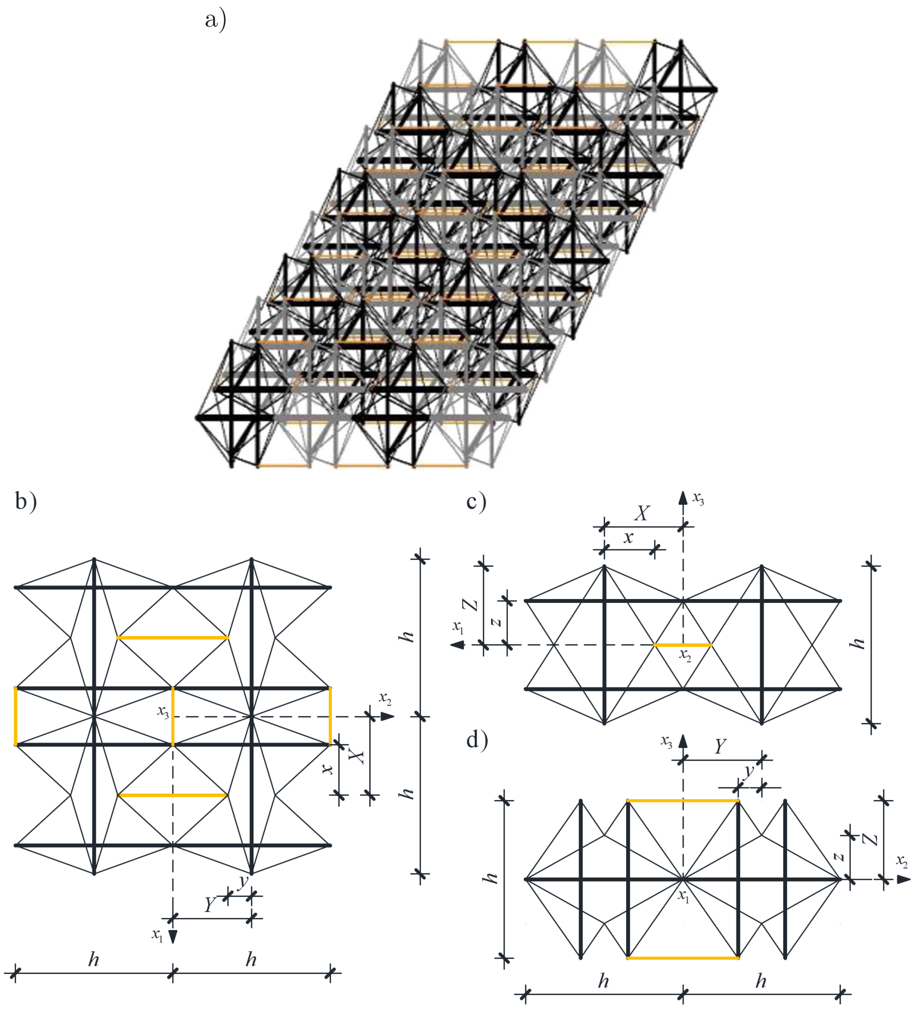

FIG. 1. a) Tensegrity plate, b), c), d) views of four modules of tensegrity based on the expanded octahedron with additional connecting cables (yellow colour).

4. Examples

The maximum displacements for three instructive tensegrity plate examples, as a function of the coefficients n, m and σ, are presented in the closed form. Two examples for membrane and bending analysis of plate strips (Fig. 2a, 3a) and simply supported rectangular plate with sinusoidal load (Fig. 4a) are discussed.

The cantilever plate strip loaded by constant axial force F_1 (Fig. 2a) is analyzed in the first example. In this case, a membrane state of stress was con-

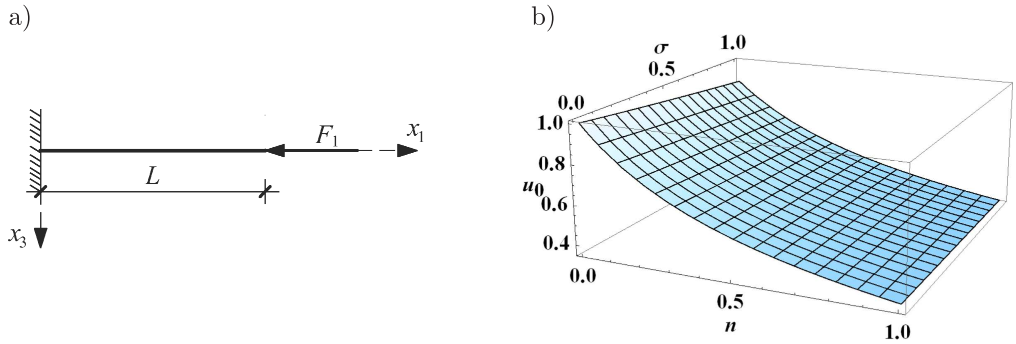

Fig. 2. a) Cantilever plate strips, b) graph of the horizontal displacement u_0.

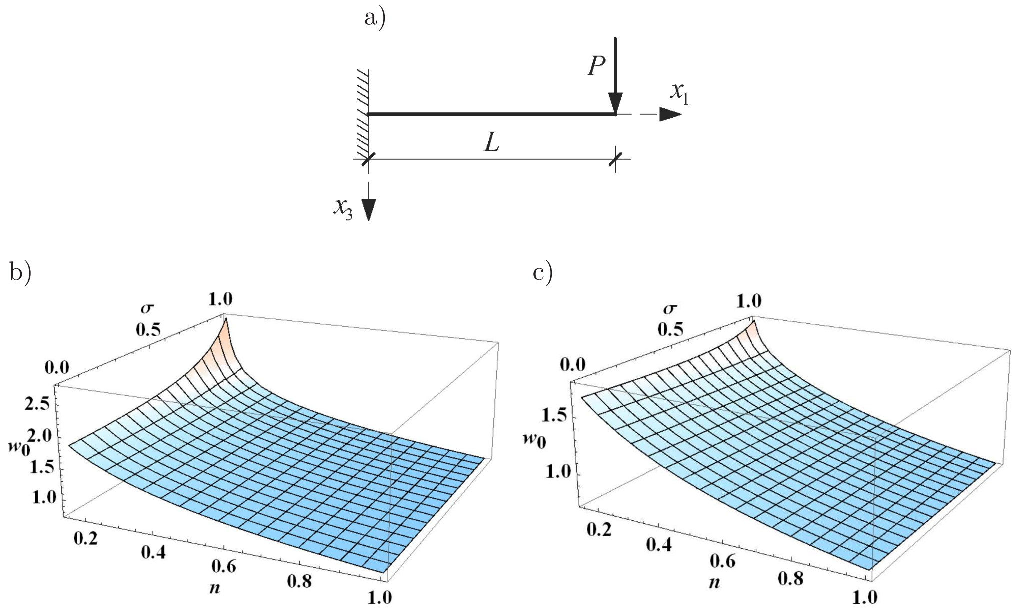

Fig. 3. Cantilever plate strips (a), and graphs of the vertical displacement w_0 for: $h/L = 0.2$ (b), $h/L = 0.1$ (c).

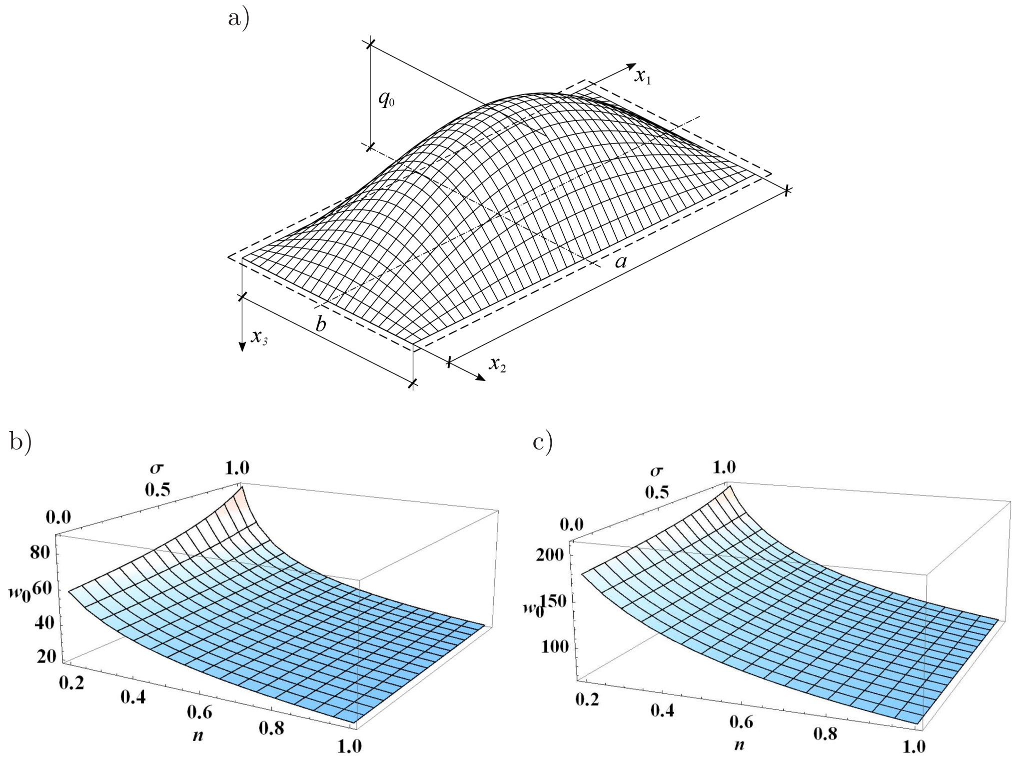

FIG. 4. Simply supported rectangular plate (a), and graph of the vertical displacement w_0 for: $h/a = 0.2$ (b), $h/a = 0.1$ (c).

sidered. It was assumed that the force mass f_1 is neglected. Boundary conditions are the following: $u(0) = 0$, $N_{11}(L) = -F_1$. The maximum displacement can be described with the form:

$$u_{1,\max} = u_1(L) = Cu_0, \qquad C = -\frac{F_1 L h}{2EA}, \qquad u_0 = \frac{1}{\delta_{11}}, \tag{4.1}$$

which, for constant geometrical parameters, depends only on coefficient (3.3). The parametric study for $m = n$ is presented graphically in Fig. 2b.

The second example is the cantilever plate strip loaded by constant transverse force P (Fig. 3a). In this case, a bending state of state was considered. It was assumed that the uniformly distributed external loads f_3 and m_1 are neglected.

Boundary conditions for this case are as follows: $w(0) = 0$, $\phi_1(0) = 0$, $N_{13}(L) = P$, $M_{11}(L) = 0$. The maximum displacement is described with the formula:

$$w_{\max} = w(L) = Cw_0, \qquad C = \frac{PL^3}{EAh}, \qquad w_0 = \left[\frac{2}{\delta_{11}} + \frac{6}{5}\left(\frac{h}{L}\right)^2 \frac{1}{\delta_{13}}\right]. \tag{4.2}$$

In this case, the displacement (4.2) depends on coefficients (3.3) and (3.6) and on the ratio of thickness h to width L of plate strip. It is very important that the displacement is valid only for the assumption that $\delta_{13} \neq 0 \rightarrow n \neq 0.1210128\sigma$. Some representative results are presented graphically in Figs. 3b and 3c.

Simply supported rectangular plate (Fig. 4a) is the third example. Sinusoidal load was applied:

$$f_3(x_1, x_2) = q_0 \sin\left(\frac{\pi}{a}x_1\right) \sin\left(\frac{\pi}{b}x_2\right). \tag{4.3}$$

The closed form solution for the maximum displacement, on the assumption of square plate ($b = a$), can be described with the following formula:

$$\begin{gathered} w_{\max} = w(0.5a, 0.5a) = Cw_0, \qquad C = \frac{q_0 a^2}{k^2 EAh\pi^4}, \\ w_0 = \frac{h^4\pi^4\left[\delta_{12}(2\delta_{22} - 3\delta_{12}) + 2\delta_{11}(2\delta_{22} + \delta_{12})\right] + 100a^4\delta_{13}\delta_{23} + a^*}{20a^2(\delta_{11} + \delta_{22} + 3\delta_{12})\delta_{13}\delta_{23} + b^*}, \end{gathered} \tag{4.4}$$

where

$$a^* = 10a^2h^2\pi^2\left[2\delta_{22}\delta_{13} + 2\delta_{11}\delta_{23} + \delta_{12}(\delta_{13} + \delta_{23})\right],$$

$$b^* = h^2\pi^2\left[\delta_{12}(2\delta_{22} - 3\delta_{12}) + 2\delta_{11}(2\delta_{22} + \delta_{12})\right]\left[\delta_{13} + \delta_{23}\right].$$

The results for selected geometrical parameters are presented graphically in Figs. 4b and 4c.

The instructive examples presented above show, in the closed form the influence of the geometric and physical parameters as well as the influence of self-stress level for the displacements of plate-like tensegrity structures. Similar analysis can be done for any tensegrity modules comprising the orthotropic structure.

5. Conclusion

The continuum flat-shell six-parameter model of tensegrity plate-like structures proposed in this paper allows to estimate in a simple way the influence of geometric and physical properties of cables and struts as well as the influence of self-stress state for average displacements, strains and internal forces of the structure. The model is valid for any structure composed of tensegrity modules with orthotropic properties.

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