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Analysis of Non-stationary Heat Transfer in a Hollow Cylinder with Functionally Graded Material Properties Performed by Different Research Methods

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A unidirectional non-stationary heat conduction in two-phase hollow cylinder is considered. The conductor is made of two-phase composites and has a smooth gradation of effective properties in the radial direction. Thus, we deal with a special case of functionally graded materials, FGM [1, 3, 4, 11, 12]. The mathematical model of the conductor is based on a tolerance averaging approach (TAA) [12]. Comparison of the results of tolerance model published in the article [7] with finite element method (FEM) calculated by Autodesk Simulation Multiphysics program is shown. The effect of geometry and material properties of conductor on the temperature field as well as different boundary conditions is examined.

Key words: heat transfer, tolerance averaging method, composite, FEM.

1. Introduction

The analysis of heat conduction in a hollow cylinder with functionally graded material properties is a very important problem taking into account the possibility of its use in a wide range of industries starting from civil engineering and finishing on heat shields, that are used in the aviation and aerospace industry. The analysis mentioned above has been researched in many different ways so far. An analytical approach is presented in papers [1, 7, 13]. The numerical approach can be divided into meshfree methods [9] and mesh methods [2, 3]. The innovative approach of this paper is the comparison between analytical analysis made by tolerance averaging technique published in work [7] and numerical analysis in which finite element method was used.

The main aim of this paper is an analysis of the influence of geometry and material properties of a conductor and initial-boundary conditions on the temperature field for non-stationary heat transfer. The thermal stress behaviour of functionally graded hollow circular cylinders is shown in paper [6], where there are given results for a homogenous cylinder. The composite conductor that is under consideration has a deterministic microstructure and is λ-periodic along its angular axis and has smooth and slow gradation of effective properties in the radial direction (Fig. 1). Accordingly, we cope with a special case of functionally graded materials, (FGM) [1, 3, 4, 10, 11].

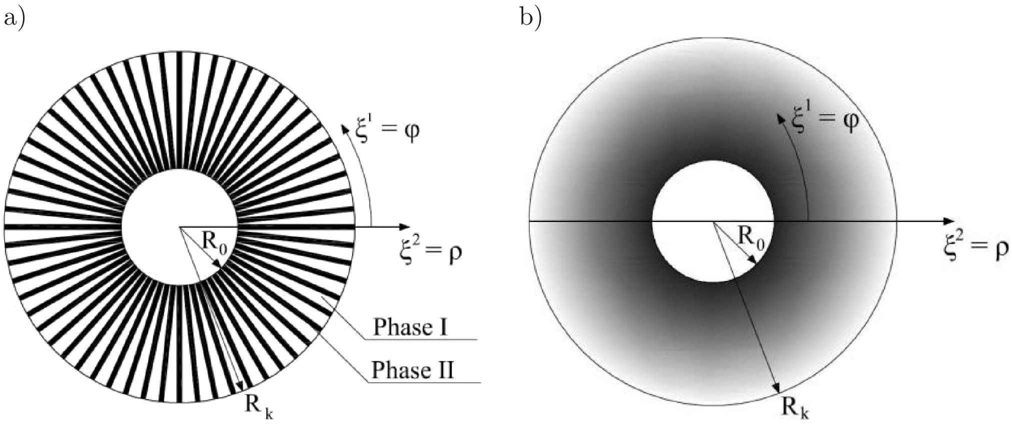

FIG. 1. Structure of two-phase functionally graded composites on a) micro- and b) macro-scale.

The heat conduction phenomenon is described by well known Fourier equation:

$$c\dot{\Theta} - \nabla(K \cdot \nabla\Theta) = Q_v, \tag{1.1}$$

where K – heat conduction tensor, c – specific heat and Q_v – internal heat sources. The selection of different materials gives different values of specific heat and heat conduction. The coefficients can have radically different values, for instance the material that represents insulating functions as polystyrene performs $c = 1460\ \frac{\mathrm{J}}{\mathrm{kg \cdot K}}$, $k = 0.045\ \frac{\mathrm{W}}{\mathrm{m \cdot K}}$ while the material that is typical conductor performs $c = 440\ \frac{\mathrm{J}}{\mathrm{kg \cdot K}}$, $k = 58\ \frac{\mathrm{W}}{\mathrm{m \cdot K}}$. The temperature field $\Theta(\varphi, \rho, t)$ depends on the geometry of the composite and on the time where $\rho(R_0, R_k)$ and $t \geq 0$ [s]. The above-mentioned equation, which contains highly oscillating and discontinuous coefficients caused by micro-heterogeneous composite structure, is of little use to solve engineering problems. Hence, there is a need to create mathematical models described by averaged equations which contain continuous and smooth coefficients. In this paper certain averaged heat transfer equations obtained in the article [7] are analysied by tolerance averaging technique.

In this paper heat transfer is analysed through an analytical method described in the article [7] and numerical method calculated by Autodesk Simulation Multiphysics program [14]. A similar comparison is presented in paper [2], where the finite element in conjunction with finite difference method is shown for heat conduction problems in non-homogeneous materials.

2. Calculation method

2.1. *Tolerance averaging method*

For the tolerance averaging method, let us build macroscopic models of heat transfer in micro-heterogeneous environment. These models are described by partial differential equations for averaged temperature and ordinary differential equations that contains the time derivative of some additional unknown called thermal internal variables. Those unknowns describe a disturbance of temperature caused by micro-heterogeneous structure of composite. Thereby, in average models the boundary conditions can be defined only for the average temperature field.

In the paper [7] the model presented in Fig. 1 is considered. Orthogonal curvilinear coordinate system O in the physical space Ω is established. Where $\lambda = 2\pi/N$ diameter of the unit cell, N – number of cells in considered composite, g – fibers width which is constant along radial axis. Let us define volume fractions of homogeneous layers: $v'(\rho) = \delta(\rho)/\lambda$ and $v''(\rho) = g/\lambda\rho$. The distribution of heterogeneity is denoted as dimensionless function.

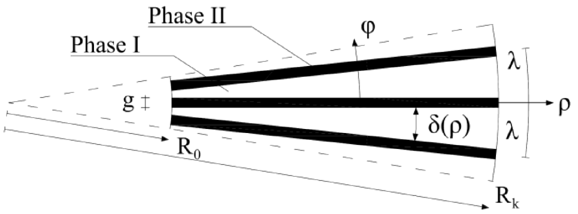

Fig. 2. Deterministic microstructure of composite.

The fundamental assumption of averaging tolerance technique is the temperature field decomposition

$$\Theta(\varphi, \rho, t) = \theta(\varphi, \rho, t) + h(\varphi, \rho) \cdot \psi(\varphi, \rho, t), \tag{2.1}$$

where $\varphi \in [0, 2\pi)$, $\rho \in [R_0, R_k]$ and $t \geq 0$ [s]. Equation (2.1) shows temperature field as the sum of averaged temperature and an oscillating part caused by micro-heterogeneous structure of composite. The functions θ and ψ are unknown while h function is given *a priori*. It is assumed that average temperature function θ and oscillation amplitude of the temperature ψ are slowly varying function, i.e. $\theta(\cdot, \rho, t)$, $\psi(\cdot, \rho, t) \in SV_d^1(\Omega, \Delta)$, whereas h function caused by discontinuity of the coefficients in (1.1) is provided by 'saw-type' locally periodic function, i.e., $h \in HO_d^1(\Omega, \Delta)$ which would be called the fluctuation shape function (Fig. 3).

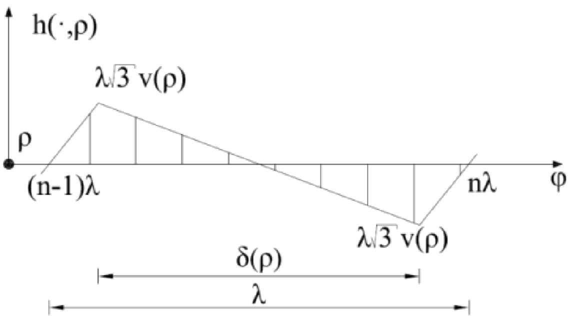

FIG. 3. Fluctuation shape function.

The averaging procedure of Eq. (1.1) is based on tolerance averaging technique which foundations and examples of applications can be found in papers [5, 7–9].

2.2. Numerical method

To compare the results obtained by tolerance averaging method [7] Autodesk Simulation Multiphysics program [14] is used. The program contains modules needed for i.a. thermal calculations and calculations of fluid and solid mechanics. Furthermore, this allows us to make analysis of the flow, heat transfer, analysis of steady and transition state, and analysis of transfer with heat exchange. In the present paper the problem of planar non-stationary heat transfer is a hollow cylinder. The analysis is limited to polarsymmetric issues. The boundary conditions are established as for constant or periodic function of heat flux density and temperature for any time t, and initial temperature distribution in the conductor in the time $t = 0$. The program mentioned above does the calculations by finite element method. The assumed mesh of considered model varies depending on examined cases from 25×10 to 12×10 elements for every part of the matrix and rib.

3. NUMERICAL EXAMPLES

3.1. Assumptions

The main aim of this paper is to compare the results from the analysis made by tolerance averaging method and numerical method of the model showed in Fig. 1. Here, two cases are considered, each of which has different geometric parameters as presented in Table 2. In both cases transient heat transfer dependent on the time t is considered.

The Eq. (3.1) describes thermal conductivity tensor of anisotropic material

$$K = k \begin{bmatrix} 1 & b \\ b & a \end{bmatrix}, \tag{3.1}$$

where $a \in (0,1]$, $b \in [0, \sqrt{a})$. For numerical calculations there is assumed $a = 1$ and $b = 0$. Material properties are described in Table 1.

Table 1. Material properties.

Type of the material	Conductivity $k\left[\frac{\text{W}}{\text{m}\cdot\text{K}}\right]$	Mass density $\rho\left[\frac{\text{kg}}{\text{m}^3}\right]$	Specific heat $c\left[\frac{\text{J}}{\text{kg}\cdot\text{K}}\right]$
Phase No. 1 (matrix – steel)	58	7800	440
Phase No. 2 (ribs – polystyrene)	0.045	10	1460

There are presented adopted geometric parameters in Table 2.

Table 2. Geometric parameters for considered cases.

A type of the analysis Case	Inner radius R_0 [m]	External radius R_k [m]	Width of the ribs g [m]	The number of the cells N [–]
Case I	1	3	$0.5\ \lambda R_0$	60
Case II.I	5	6	$\frac{\pi}{g} R_0$	100
Case II.II	10	11		200
Case A,B	1	3	$0.5\ \lambda R_0$	60

In the comparative analysis of tolerance and FEM model (case No. I, II) it is assumed:
– initial condition:

$$\theta(\rho, 0) = \psi(\rho, 0) = 0\ [^\circ\text{C}], \tag{3.2}$$

– boundary condition for every $t \geq 0$ [s]:

$$\begin{gathered} \theta(R_0, t) = 100\ [^\circ\text{C}], \\ \theta(R_k, t) = \psi(R_0, t) = \psi(R_k, t) = 0\ [^\circ\text{C}]. \end{gathered} \tag{3.3}$$

Additionally, there is analysed heat transfer of following initial-boundary conditions for FEM model (case No. A, B):
– initial condition as in (3.2),
– boundary condition for every $t \geq 0$ [s]:

THE CASE NO. A

$$(3.4)\qquad \begin{aligned} &\theta(R_0,t) = 100\ [^\circ\mathrm{C}] && \text{for}\quad \varphi \in \left[0, \frac{\pi}{2}\right),\\ &\theta(R_0,t) = 0\ [^\circ\mathrm{C}] && \text{for}\quad \varphi \in \left[\frac{\pi}{2}, 2\pi\right),\\ &\theta(R_k,t) = 0\ [^\circ\mathrm{C}]. \end{aligned}$$

THE CASE NO. B

$$(3.5)\qquad \begin{aligned} &\theta(R_0,t) = 100\ [^\circ\mathrm{C}] && \text{for}\quad \varphi \in \left[0, \frac{\pi}{2}\right),\\ &\theta(R_0,t) = 0\ [^\circ\mathrm{C}] && \text{for}\quad \varphi \in \left[\frac{\pi}{2}, 2\pi\right),\\ &\theta(R_k,t) = 100\ [^\circ\mathrm{C}]. \end{aligned}$$

3.2. The comparative analysis

3.2.1. The case No. I. The results respectively for analytical and numerical methods for averaged temperature (Fig. 4) and for oscillation amplitude (Fig. 5) according to the radial coordinate of hollow cylinder are presented below. It is assumed that the initial and boundary conditions are compatible with the Eqs. (3.2), (3.3). In the Autodesk Simulation Multiphysics program, the adopted mesh contains 25×10 to 12×10 elements for every part of matrix and rib.

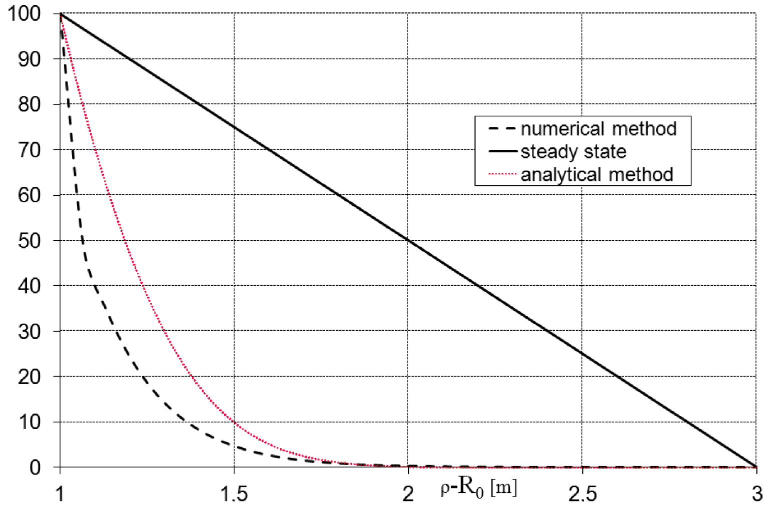

FIG. 4. The results of analytical and numerical method – Case No. I after $t = 1$ [h]. A diagram of averaged temperature.

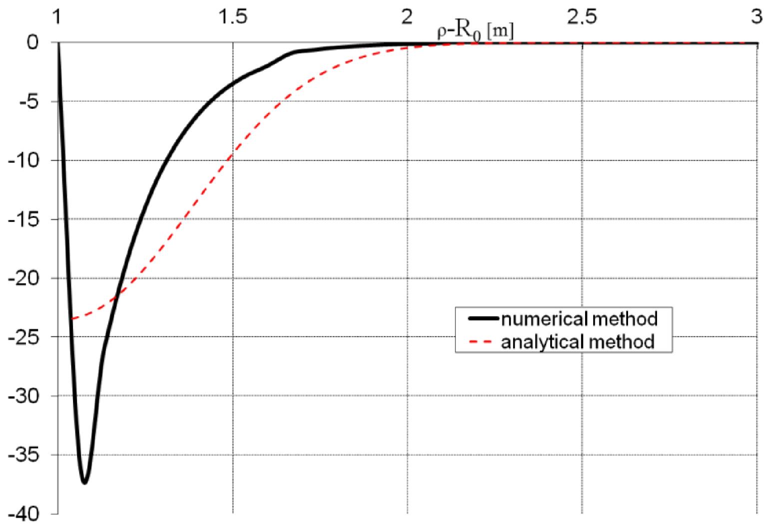

FIG. 5. The results of numerical method – Case No. I after t = 1 [h]. A diagram of oscillation amplitude.

3.2.2. The case No. II. Figure 6 shows in which way averaged temperature for non-stationary state approaches to averaged temperature for stationary state for calculations made by tolerance averaging method and by numerical analysis It is assumed that the mesh consists of 12×10 elements for every part of matrix and rib.

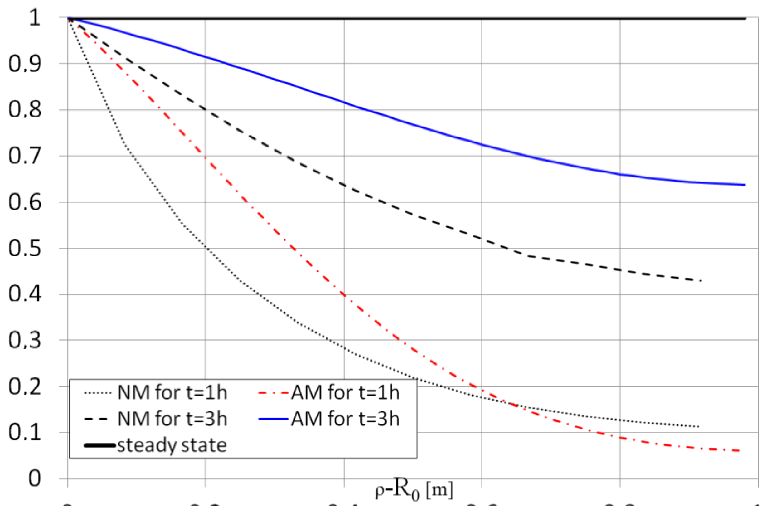

FIG. 6. Results of analytical and numerical method – Case No. II. The averaged temperature.

It is presented below how oscillation temperature approaches to obtain steady state in the case of different geometrical parameters of the composite.

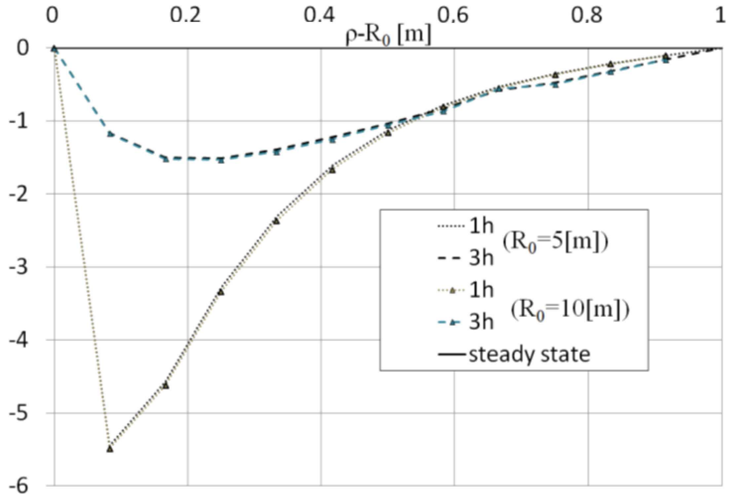

Fig. 7. Results of numerical method – Case No. II. The oscillation amplitude.

For calculations for all the above showed figures, initial-boundary conditions form the Eqs. (3.2), (3.3) are adopted.

3.3. Numerical analysis

3.3.1. The case No. A. Figure 8 presents heat transfer after time $t = 4$ [h] in the case of initial-boundary conditions stated in Eqs. (3.2), (3.4). The assumed

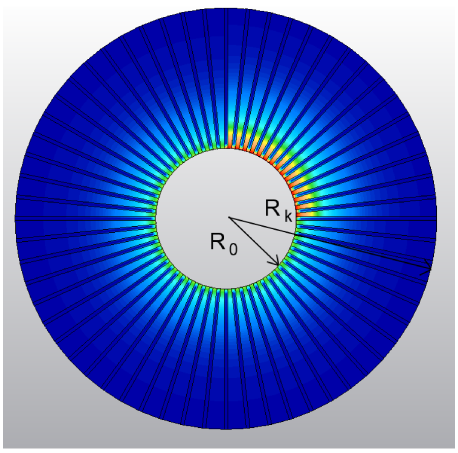

Fig. 8. The results of numerical method – a case No. A after $t = 4$ [h].

mesh is made of 25×10 elements for every part of matrix and rib. Geometrical and physical parameters are adopted as in the Table 1 and Table 2.

3.3.2. The case No. B. Figure 9 describes heat transfer after time $t = 1$ [h] in the case of initial-boundary conditions stated in Eqs. (3.2), (3.5). Mesh of finite elements and geometrical and physical parameters are assumed as above.

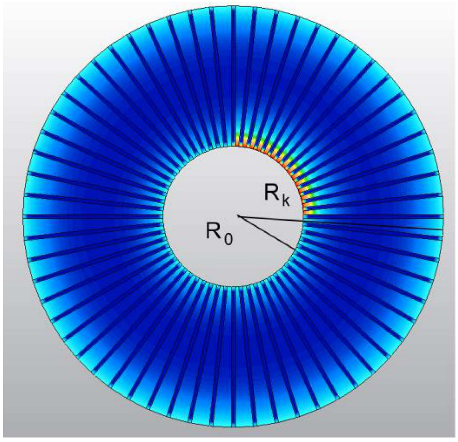

FIG. 9. The results of numerical method – a case No. B after $t = 1$ [h].

4. Summary

This paper consists of an analysis of non-stationary heat conduction in two-phase hollow cylinder. Concerning all of the experiments presented above there has been reached following conclusions:

- In the cases No. I, II it was carried out the comparative analysis of the unidirectional heat transfer.
  - the results, for the averaged temperature, obtained from analytical and numerical method (Figs. 4–6) present a similar character and comparable values,
  - a negligible difference can be the result of the density of the assumed mesh,
  - the denser mesh assumed more detail in the results,
  - an analogous character of the graphs is also obtained in the case of oscillation amplitude (Figs. 4, 5) but this time the values differ more significantly from each other,
  - in both methods the maximum values of the oscillation temperature are next to the boundary of the conductor. This phenomenon indi-

cates the presence of a boundary effect. The boundary conditions are determined only for the averaged temperature and as an effect of micro-heterogeneous structure of the conductor on the ribs and matrix the rapid variation occurs, which is seen as the occurrence of the maximum of oscillation amplitude of the temperature,
  - the averaged equations of the heat conduction of the tolerance model for conductor with functionally graded material properties contain continuous and smooth coefficients. Hence, they can be solved easily by numerical methods due to their slowly varying coefficients.

- In the cases no. A, B the problem of planar heat transfer was solved using finite elements method (Figs. 8 and 9). Unfortunately, this requires a lot of work regarding the creation of the mesh, separately for every rib and every part of matrix. This additional effort is not needed in the case of use of tolerance averaging methods.

Further investigation is planned, in which different kinds of mesh elements will be proposed simultaneously with a denser mesh. Furthermore, various boundary conditions will be introduced to research whether in the other cases the results would be as corresponding to each other as here.

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