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SELECTED CASES OF BI-AREA CONTACT FOR CYLINDRICAL ELEMENTS

M. CZERNIEC, C. KOMORZYCKI and R. ŁUCYSZYN (LUBLIN)

The paper presents a general solution to selected problems concerning the bi-area contact of cylindrical elements with small deviations from circular contour. Contact areas and pressure distribution were determined. In numerical calculations, three variants of joining elements in contact were taken into account. Influence of the ellipticity error on the contact area and the value of maximal contact pressures was presented in the figures.

1. Introduction

In the contact strength investigations of cylindrical joints composed of nominally circular elements, small shape errors, for example their ellipticity, are not taken into account. Their occurrence is usually caused by technological processes. These deviations are sometimes comparable with radial clearance and their considerable influence on the strength characteristics of contact is expected.

References [1, 2] describe a method of determining the contact pressures in cylindrical joints, the elements of which, with similar diameters, have small deviations from the circular shapes. In contact problems, solved so far by the mentioned method [3, 4], the case of bi-area contact has not been examined. This is the aim of the present paper.

2. Problem formulation and its basic equation

Problem of contact between cylindrical elements is considered as a plane problem of the elasticity theory. In a hole of an elastic isotropic plate 1, an elastic disk 2 is situated (Fig. 1). It is assumed that element contours differ somewhat from circles $L^{(1)}$ and $L^{(2)}$ (Fig. 1) having similar radii R_1 and R_2. Their deviations $\delta_k(\alpha) \ll R_k$ are described in the following manner:

$$\delta_k(\alpha) = (-1)^k[R_k(\alpha) - R_k]. \tag{2.1}$$

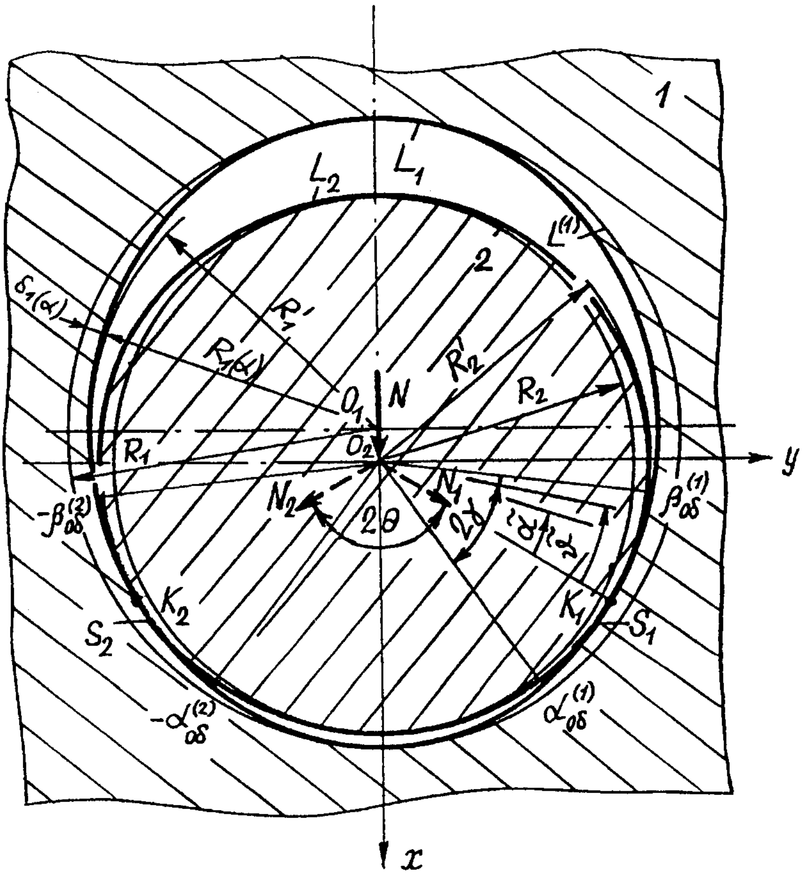

FIG. 1. Scheme of bi-area contact of cylindrical elements with elliptical deviations: 1 – the hole, 2 – the disk.

Here α is the polar co-ordinate, $k = (1;2)$ denote the element numbers, and $R_k(\alpha)$ are the radius-vectors of element contours.

First, the contact between the disk and the hole is reached at two points K_1 and K_2 situated at an angle 2Θ (Fig. 1). When the disk is loaded by a constant force N in the contact zones S_1 and S_2, situated symmetrically in relation to the initial contact points, radial contact stresses σ_r and tangent ones τ_{r_α} occur. Forces N_1 and N_2, applied to points K_1 and K_2, are resultants of those stresses. External load N and forces N_1, N_2 are related by the formula:

$$N_1 = N_2 = 1/2\, N \cos^{-1} \Theta.$$

The problem consists in determining the contact pressures $p(\alpha, \delta) = -\sigma_r(\alpha, \delta)$, their distribution, the contact angle 2γ and the initial contact angle 2Θ for the following boundary conditions:

$$
(2.2)\qquad
\begin{aligned}
\sigma_r^{(1)}(\alpha,\delta) &= \tau_{r\alpha}^{(1)}(\alpha,\delta)=0, & \beta_{0\delta}^{(1)} &\le \alpha \le 2\pi+\alpha_{0\delta}^{(1)},\\
\sigma_r^{(2)}(\alpha,\delta) &= \tau_{r\alpha}^{(2)}(\alpha,\delta)=0, & -\beta_{0\delta}^{(2)} &\le \alpha \le -(2\pi+\alpha_{0\delta}^{(2)}),\\
\tau_{r\alpha}^{(j)}(\alpha,\delta) &= f\sigma_r^{(j)}(\alpha,\delta), & \sigma_r^{(j)}(\alpha,\delta) &= -p^{(j)}(\alpha,\delta),\\
\alpha_{0\delta}^{(1)} &\le \alpha \le \beta_{0\delta}^{(1)}, & -\alpha_{0\delta}^{(2)} &\le \alpha \le -\beta_{0\delta}^{(2)},
\end{aligned}
$$

where $\alpha_{0\delta}^{(1)}$, $\beta_{0\delta}^{(1)}$ and $-\alpha_{0\delta}^{(2)}$, $-\beta_{0\delta}^{(2)}$ are angles bounding the contact zones S_1 and S_2, respectively (Fig. 1); $j = 1$ for zone S_1 and $j = 2$ for zone S_2; $\sigma_r^{(j)}$, $p^{(j)}$ are the contact stresses and pressures in zones S_1 and S_2, respectively; f is the coefficient of the sliding friction.

Problem is symmetric with respect to N. By using the method given in [1, 2], we obtain the governing equation for calculation of the contact pressures in the following form:

$$
(2.3)\qquad m_1\int_{\alpha_{0\delta}}^{\beta_{0\delta}} \operatorname{ctg}\frac{\alpha-\vartheta}{2}\left[p'(\vartheta,\delta)+fp(\alpha,\vartheta)\right]d\vartheta = m_2\left[p(\alpha,\delta)-fp'(\alpha,\delta)\right]
$$

$$
+\,m_3\int_{\alpha_{0\delta}}^{\beta_{0\delta}} p(\alpha,\delta)\,d\alpha + m_4N'\cos\alpha+\frac{\varepsilon}{R_1R_2}
$$

$$
-\,\sin\alpha\left\{2\left[f_{1x}'/R_1^2-f_{2x}'/R_2^2\right]-\left[f_{1y}''/R_1^2-f_{2y}''/R_2^2\right]\right\}
$$

$$
+\,\cos\alpha\left\{2\left[f_{1y}'/R_1^2-f_{2y}'/R_2^2\right]+\left[f_{1x}''/R_1^2-f_{2x}''/R_2^2\right]\right\},
$$

where:

$$
\begin{aligned}
m_1 &= \frac{1}{8\pi}\left(\frac{1+\kappa_1}{G_1R_1}+\frac{1+\kappa_2}{G_2R_2}\right), & m_2 &= \frac{1}{4}\left(\frac{1-\kappa_1}{G_1R_1}-\frac{1-\kappa_2}{G_2R_2}\right),\\
m_3 &= \frac{1+\kappa_1}{8\pi G_1R_1}, & m_4 &= \frac{1}{2\pi}\left(\frac{\kappa_1}{G_1R_1}+\frac{1}{G_2R_2}\right),\\
f_k' &= \frac{df_k(\alpha)}{d\alpha}, & f_k'' &= \frac{d^2f_k(\alpha)}{d\alpha^2}, \qquad p'(\vartheta,\delta)=\frac{dp'(\vartheta,\delta)}{d\vartheta},
\end{aligned}
$$

$$
(2.4)\qquad N' = -\int_{\alpha_{0\delta}}^{\beta_{0\delta}}\left[p(\alpha,\delta)\cos\alpha - fp(\alpha,\delta)\sin\alpha\right]d\alpha,
$$

$$
(2.5)\qquad f_{kx}\equiv f_{kx}(\alpha)=x_k^{(e)}(\alpha)-x_k(\alpha),\qquad f_{ky}\equiv f_{ky}(\alpha)=y_k^{(e)}(\alpha)-y_k(\alpha),
$$

and G, ν are the shear modulus and Poisson's ratio, respectively, $\kappa = 3-4\nu$ is the coefficient of the plane strain state, $\kappa = (3-\nu)/(1-\nu)$ is the coefficient of the plane stress state, $x_k^{(e)} = x_k^{(e)}(\alpha)$, $y_k^{(e)} = y_k^{(e)}(\alpha)$ – denote the parametric equations of non-circular contours, $x_k(\alpha) = A_k \cos\alpha$, $y_k(\alpha) = A_k \sin\alpha$ – parametric equations of the initial circles: ($A_1 = R_1$ for $k = 1$, $A_2 = R_2$ for $k = 2$), the circle (with radius R_2) inscribed in the disk, and the circle (with radius R_1) circumscribed about the hole, $\varepsilon = R_1 - R_2 > 0$ – radial clearance.

3. General solution of certain cases of contact

Solving equation (2.3) we assume that friction does not occur in the contact zone ($f = 0$), materials of the elements are identical ($G_1 = G_2$, $\nu_1 = \nu_2$) and $R_1 \cong R_2 = R$, and $\varepsilon > 0$.

Let us consider the following cases of contact.

Case 1.

Elements of the system are elliptical (Fig. 1). Parametric equations take the form:

$$\begin{aligned} x_1^{(e)}(\alpha) &= a_1 \cos\alpha, \qquad & y_1^{(e)}(\alpha) &= b_1 \sin\alpha, \\ x_2^{(e)}(\alpha) &= b_2 \cos\alpha, \qquad & y_2^{(e)}(\alpha) &= a_2 \sin\alpha, \end{aligned} \tag{3.1}$$

where semi-axes of the elliptical elements are:

$$a_1 = R_1, \qquad b_1 = R_1', \qquad a_2 = R_2', \qquad b_2 = R_2, \qquad \text{and} \qquad a_1 > b_1, \qquad a_2 > b_2.$$

According to (2.5), the out-of-roudness characteristics $f_{kx}(\alpha)$, $f_{ky}(\alpha)$ have been determined.

$$\begin{aligned} f_{1x}(\alpha) &= a_1 \cos\alpha - a_1 \cos\alpha = 0, \\ f_{1y}(\alpha) &= b_1 \sin\alpha - a_1 \sin\alpha = -(a_1 - b_1)\sin\alpha, \\ f_{2x}(\alpha) &= b_2 \cos\alpha - b_2 \cos\alpha = 0, \\ f_{2y}(\alpha) &= a_2 \sin\alpha - b_2 \sin\alpha = (a_2 - b_2)\sin\alpha. \end{aligned} \tag{3.2}$$

Denoting by $\alpha = \tilde{\alpha} - \Theta$, we obtain equation for $p(\tilde{\alpha}, \delta)$:

$$\begin{aligned} \frac{1}{\pi}\int_{\gamma_1}^{\gamma_2} \operatorname{ctg}\frac{\tilde{\alpha} - \tilde{\vartheta}}{2} p'(\tilde{\vartheta}, \delta)\, d\tilde{\vartheta} &= \frac{1}{2\pi}\int_{\gamma_1}^{\gamma_2} p(\tilde{\alpha}, \delta)\, d\tilde{\alpha} \\ &+ \frac{2}{\pi}\cos\tilde{\alpha}\int_{\gamma_1}^{\gamma_2} p(\tilde{\alpha}, \delta)\cos\tilde{\alpha}\, d\tilde{\alpha} + \frac{4G}{R(1+\kappa)}\left[\varepsilon - \varepsilon_1\cos^2\alpha - \varepsilon_2\sin^2\alpha\right], \end{aligned} \tag{3.3}$$

where:

$$\varepsilon_1 = 2(\delta_1 + \delta_2), \qquad \varepsilon_2 = -(\delta_1 + \delta_2),$$

$$\delta_1 = a_1 - b_1, \qquad \delta_2 = a_2 - b_2, \qquad \tilde{\vartheta} = \vartheta + \Theta, \qquad 0 \le \alpha \le \frac{\pi}{2},$$

$$\delta_1 + \delta_2 < \varepsilon, \qquad a_2 < b_1,$$

$$\gamma_1 \le \tilde{\alpha} \le \gamma_2, \qquad \gamma_1 = \Theta - 0.5\left(\beta_{0\delta}^{(1)} - \alpha_{0\delta}^{(1)}\right), \qquad \gamma_2 = \Theta + 0.5\left(\beta_{0\delta}^{(1)} - \alpha_{0\delta}^{(1)}\right).$$

The contact half-angle is determined from the equation:

$$N' = R \int_{\gamma_1}^{\gamma_2} p(\tilde{\alpha}, \delta) \cos\tilde{\alpha}\, d\tilde{\alpha}. \tag{3.4}$$

CASE 2

Element 1 is elliptic and element 2 is circular. Then $\varepsilon_1 = 2\delta_1$, $\varepsilon_2 = -\delta_1$, $\delta_1 < \varepsilon$, $R_2 < b_1$.

CASE 3

Element 1 is circular and element 2 is elliptic. Then $\varepsilon_1 = 2\delta_2$, $\varepsilon_2 = -\delta_2$, $\delta_2 < \varepsilon$, $a_2 < R_1$.

Approximate solution of equation (3.3) can be obtained by using the collocation method and assuming the distribution of the contact pressures in the following form:

$$p(\tilde{\alpha}, \delta) \approx \left(C_0 + C_2 \operatorname{tg}^2\frac{\xi}{2}\right)\sqrt{\operatorname{tg}^2\frac{\gamma}{2} - \operatorname{tg}^2\frac{\xi}{2}}, \tag{3.5}$$

where:

$$\xi = \tilde{\alpha} - \Theta, \qquad \gamma = 0.5\left(\beta_{0\delta}^{(1)} - \alpha_{0\delta}^{(1)}\right),$$

C_0, C_2 – unknown collocation indices.

Substituting (3.5) into (3.3) and introducing the notations

$$x = \operatorname{tg}\left[(\tilde{\vartheta} - \Theta)/2\right], \qquad y = \operatorname{tg}\left(\frac{\xi}{2}\right)$$

we obtain

$$\begin{aligned} \frac{1}{\pi}\int_{-a_0}^{a_0} \frac{p'(2\operatorname{arctg} x)}{x-y}\,dx = {} & -\frac{2}{\pi}\frac{1-y^2}{1+y^2}\int_{-a_0}^{a_0} \frac{(1-y^2)p(2\operatorname{arctg} y)}{(1+y^2)^2}\,dy \\ & - \frac{1}{2\pi}\int_{-a_0}^{a_0} \frac{p(2\operatorname{arctg} y)}{(1+y^2}\,dy + \frac{1}{\pi}\int_{-a_0}^{a_0} \frac{y p'(2\operatorname{arctg} y)}{1+y^2}\,dy \\ & - \frac{2G}{R(1+\kappa)}\left[\varepsilon - \varepsilon_1 D_1(\xi) - \varepsilon_2 D_2(\xi)\right], \end{aligned} \tag{3.6}$$

where:

$$\xi_1 = \theta, \qquad \xi_2 = \theta + 0.65\,\mathrm{tg}\,\frac{\gamma}{2}, \qquad a_0 = \mathrm{tg}(\gamma/2),$$

$$D_1(\xi_i) = \left(1-\xi_i^2\right)^2\left(1+\xi_i^2\right)^{-2}, \qquad D_2(\xi_i) = 4\xi_i^2\left(1+\xi_i^2\right)^{-2}, \qquad i = 1;2.$$

Integration of equation (3.6) yields the system of algebraic equations with unknown C_0, C_2 indices.

$$(3.7) \qquad \begin{aligned} C_0A_{01}(y_1) + C_2A_{21}(y_2) &= -\frac{GK_1(\xi_1)}{(1+\kappa)R}, \\ C_0A_{02}(y_2) + C_2A_{22}(y_2) &= -\frac{GK_2(\xi_2)}{(1+\kappa)R}, \end{aligned}$$

where $y_1 = 0$, $y_2 = 0.65a_0$ – collocation nodes.

$$\begin{aligned} A_{0i}(y_i) &= -0.25(1+y_i^2) + \frac{(b-1)(1-y_i^2)}{b(1+y_i^2)} + \frac{b-1}{4}, \\ A_{2i}(y_i) &= 0.125\left[(a_0^2 - 6y_i^2)(1+y_i^2)\right] + 0.125(b-1)^2 \\ &\quad + 0.5\frac{\left[4a_0^2 - (a_0^2+6)b + 6\right](1-y_i^2)}{b(1+y_i^2)}, \end{aligned}$$

where: $i = (1;2)$, $b = \sqrt{a_0^2+1}$,

$$\begin{aligned} K_1(\xi_1) &= \varepsilon - \varepsilon_1 D_1(\xi_1) - \varepsilon_2 D_2(\xi_1), \\ K_2(\xi_2) &= \varepsilon - \varepsilon_1 D_1(\xi_2) - \varepsilon_2 D_2(\xi_2). \end{aligned}$$

Then, we obtain respectively:

$$(3.8) \qquad \begin{aligned} C_0 &= \frac{A_{21}(y_1)K_2(\xi_2) - A_{22}(y_2)K_1(\xi_1)}{A_{01}(y_1)A_{22}(y_2) - A_{02}(y_2)A_{21}(y_1)}\,\frac{\varepsilon G}{R(1+\kappa)}, \\ C_2 &= \frac{A_{02}(y_2)K_1(\xi_1) - A_{01}(y_1)K_2(\xi_2)}{A_{01}(y_1)A_{22}(y_2) - A_{02}(y_2)A_{21}(y_1)}\,\frac{\varepsilon G}{R(1+\kappa)}. \end{aligned}$$

Equation for the contact half-angle γ takes the form:

$$(3.9) \qquad N' = 2\pi R(C_0B_0 + C_2B_2),$$

where

$$B_0 = 1 - \frac{1}{b}, \qquad B_2 = \frac{a_0^2(4-b) + 6(1-b)}{2b}.$$

4. Determination of the initial contact angle Θ

Evaluation of the initial contact angle is obtained by solving the equations 1 and 2 (element contours) and using condition of the common tangents in the contact points K_1 and K_2.

In co-ordinates (xO_2y), the equations take the form:

$$\frac{(x_0+\Delta)^2}{a_1^2}+\frac{y_0^2}{b_1^2}=1, \qquad \frac{x_0^2}{b_2^2}+\frac{y_0^2}{a_2^2}=1, \qquad \frac{b_1^2}{a_1^2}(x_0+\Delta)=\frac{a_2^2}{b_2^2}x_0\,, \tag{4.1}$$

where: Δ – distance between points O_1 and O_2, x_0, y_0 – co-ordinates of the contact point of the contours,

$$\Delta=\frac{1}{b_1a_2}\sqrt{(b_1^2-a_2^2)(a_1^2a_2^2-b_1^2b_2^2)}\,,$$

$$x_0=\frac{b_1b_2^2}{a_2}\sqrt{\frac{b_1^2-a_2^2}{a_1^2a_2^2-b_1^2b_2^2}}\,,$$

$$y_0=\pm\sqrt{\frac{a_1^2a_2^4-b_1^4b_2^2}{a_1^2a_2^2-b_1^2b_2^2}}\,.$$

Finally we obtain an expression for Θ:

$$\Theta=\operatorname{arctg}\frac{a_2}{b_1b_2^2}\sqrt{\frac{a_1^2a_2^4-b_2^2b_1^4}{b_1^2-a_2^2}}\,. \tag{4.2}$$

Hence for case 2, assuming that $a_2=b_2=R_2$, we get

$$\Theta=\operatorname{arctg}\frac{1}{b_1}\sqrt{\frac{a_2^2R_2^2-b_1^4}{b_1^2-R_2^2}}\,. \tag{4.3}$$

and for case 3, when $a_1=b_1=R_1$,

$$\Theta=\operatorname{arctg}\frac{a_2}{b_2^2}\sqrt{\frac{a_2^4-b_2^2R_1^2}{R_1^2-a_2^2}}\,. \tag{4.4}$$

5. Numerical analysis of the problem

Solution of the problem was carried out for the following data: $N=10\,\text{kN}$; $R=R_2=25\,\text{mm}$; $G=8.1\cdot10^4\,\text{MPa}$; $\nu=0.3$; $\varepsilon=(0.11;0.21;0.41)$ mm; δ_1, $\delta_2=(0;0.05;0.1;0.2;0.3;0.4)$ mm; $\kappa=3-4\nu$.

Results for $p(\alpha, \delta)$ are presented in Figs. 2–4. In these figures curves 1 correspond to $\delta_2 = 0$, curves 2 – to $\delta_2 = 0.05$ mm, curves 3 – to $\delta_2 = 0.1$ mm, curves 4 – to $\delta_2 = 0.2$ mm, curves 5 – to $\delta_2 = 0.3$ mm, curves 6 – to $\delta_2 = 0.4$ mm.

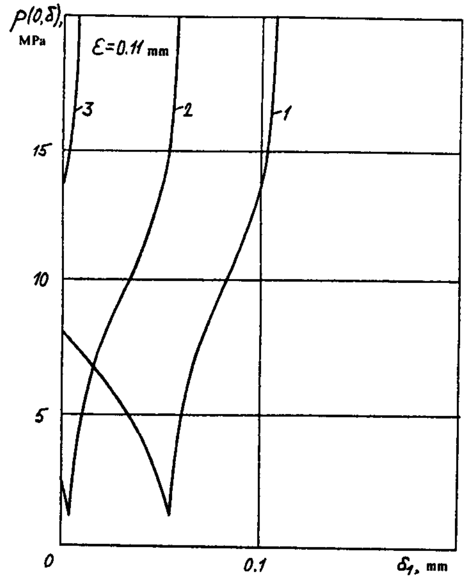

FIG. 2. Dependence of the maximal pressure $p(0, \delta)$ on the parameter δ_1 for $\varepsilon = 0.11$ mm in the contact area.

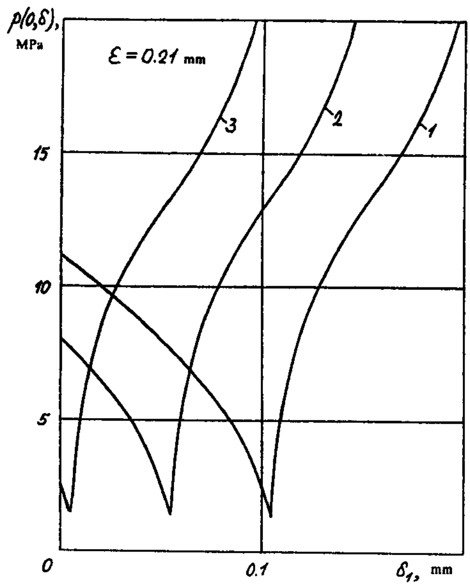

FIG. 3. Dependence of the maximal pressure $p(0, \delta)$ on parameter δ_1 for $\varepsilon = 0.21$ mm in the contact area.

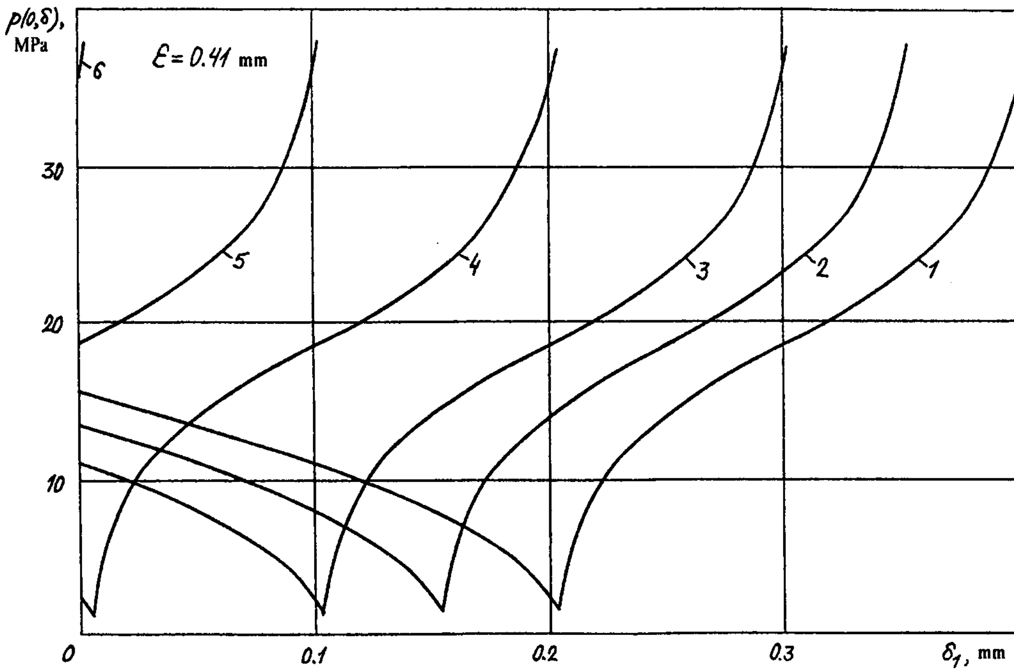

FIG. 4. Dependence of maximal pressure $p(0, \delta)$ on parameter δ_1 for $\varepsilon = 0.41$ mm in the contact area.

The graphs show that for $\delta_1 > 0$, $\delta_2 > 0$, the contact of elements with ellipticity in one or two contact zones is possible. In the presented figures, left (decreasing) parts correspond to the uni-area contact, whereas right (increasing) parts correspond to bi-area contact. In case of certain optimum quantities δ_1 and δ_2, minimal values of pressures $p(0, \delta)$ occur. For elements with ellipticity, always for $\delta_k \neq 0$, uni-area contact exists first, then following the change of δ_1 or δ_2, bi-area contact occurs. The results show that whether $\delta_1 = \text{const}$, and $\delta_2 \neq \text{const}$ or $\delta_2 = \text{const}$, and $\delta_1 \neq \text{const}$, quantities $p(0, \delta)$ reach the same values for the same values of $\delta_1 + \delta_2$.

6. Conclusions

1. In the presented contact problem, contact of elements in one and two contact zones takes place.

2. Depending on the value of radial clearance ε and deviations δ_1, δ_2, considerable change of the maximal contact pressures $p(0, \delta)$ is obtained.

3. Minimal value of the pressures $p(0, \delta)$ on the boundary between the uni-area and bi-area contact exists.

4. In case when difference between $\delta_1 + \delta_2$ and ε decreases, values of the pressures $p(0, \delta)$ are considerable as compared with their possible minimal (optimal) value.

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LUBLIN UNIVERSITY OF TECHNOLOGY
CHAIR OF APPLIED MECHANICS

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