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A CERTAIN APPROXIMATE SOLUTIONS OF NONLINEAR FLUTTER EQUATION

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A nonlinear integro-differential flutter equation of a thin airfoil placed in an incompressible flow is solved by two different methods. The first method involves the center-manifold reduction and gives the asymptotic limit cycle amplitude and frequency in terms of power series expansions. The second method replaces the integro-differential equation by an approximate set of first-order ordinary differential equations which are solved by using bifurcation and continuation software package. A comparison of these two methods shows that the domain of a good agreement between them varies significantly depending on the parameters of the problem.

1. Introduction

Self-excited flutter oscillations of nonlinear aircraft structure immersed in subsonic flow are governed by an integro-differential equation. This is due to the influence of the history of motion on unsteady aerodynamic forces and causes some difficulty, since the theory of integro-differential equations provides much less tools to solve that problem than that of ordinary differential equations. It is well known from the theory of dynamical systems [1] that the loss of stability of the steady solution, such as a horizontal flight of an aircraft, is described by the mechanism called the Hopf bifurcation. Asymptotically, self-excited oscillations tend to a limit cycle which is always two-dimensional. In [2] a new algorithm was presented for calculation of the limit cycle amplitude and frequency based on center-manifold reduction [1] with application to integro-differential equations. Some numerical examples were given in [3] and [4], compared mainly to the harmonic balance method which is an approximate method itself. This paper tries to give more comparison in order to estimate applicability of the center-manifold reduction to aeroelastic systems. The most widely used approach for such a case is to perform a direct numerical integration, and then compare the results with those of the investigated method. Unfortunately, this cannot be done for most of the aeroelastic systems, because of lack of necessary information. The integro-differential flutter equation contains the convolution integral involving the impulsive response matrix function which is unknown, except for the simplest case of two-dimensional airfoil in incompressible flow. In general, impulsive response function can be calculated as an inverse Fourier transform of the transfer function for harmonic motion. However, the transfer function is calculated only numerically and only for a finite frequency range. Therefore, any method based on impulsive response function cannot be used. The only approach is to replace the integro-differential equation by an approximate set of ordinary differential equations by using a rational approximation of the transfer function, which gives the exponential approximation for the impulsive response function. This method is applied thereunder for an aeroelastic system composed of a thin airfoil two degrees of freedom, despite the fact that the impulsive response function for this system is known.

2. Two-dimensional airfoil

The geometry of a two-dimensional thin airfoil is shown in Fig. 1.

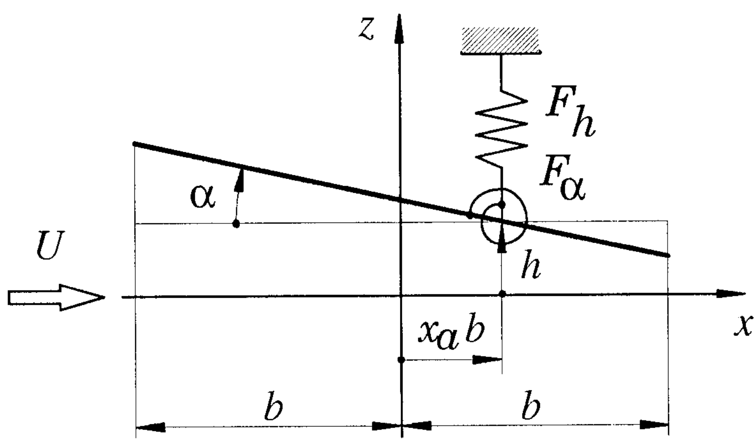

FIG. 1. Two-dimensional thin airfoil.

The motion of an airfoil is described by two-dimensional vector $\{\mathrm{u}(t)\}$ composed of physical coordinates being functions of time t:

$$\{\mathrm{u}(t)\} = \left\{ \begin{array}{c} h(t) \\ \alpha(t) \end{array} \right\}, \tag{2.1}$$

where $h(t)$ and $\alpha(t)$ denote the plunge displacement and the pitch angle, respectively. Both springs are assumed to be nonlinear and produce cubic restoring forces F_h and F_α in the plunge and pitch degree of freedom, respectively:

$$\begin{aligned} F_h &= K_h(h + c_h h^3), \\ F_\alpha &= K_\alpha(\alpha + c_\alpha \alpha^3), \end{aligned} \tag{2.2}$$

where K_h and K_α are linear stiffness coefficients while c_h and c_α are known constants. The flutter equation written in physical coordinates is:

$$[\mathbf{M}]\{\ddot{u}\} + [\mathbf{B}_u]\{\dot{u}\} + [\mathbf{K}]\{u\} + \{k_u(u)\} = \left\{f_A^{(u)}\right\},$$

where $[\mathbf{M}]$ and $[\mathbf{K}]$ are mass and stiffness matrices, respectively:

$$[\mathbf{M}] = \begin{bmatrix} m & -S_\alpha \\ -S_\alpha & I_\alpha \end{bmatrix}, \qquad [\mathbf{K}] = \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix},$$

with m, S_α, and I_α being the mass of an airfoil, its static and inertia moments about the elastic axis (per unit span), respectively. The matrix $[\mathbf{B}_u]$ is the viscous damping matrix. The nonlinear term $\{k_u(u)\}$ is composed of nonlinear parts of Eqs. $(2.2)_1$ and $(2.2)_2$.

For an arbitrary motion, the vector of unsteady aerodynamic forces is given by a convolution integral:

$$\left\{f_A^{(u)}\right\} = \frac{\rho U^2}{2} \int_{-\infty}^{0} [\mathbf{G}_u(-\tau)] \left\{u\left(t + \frac{b}{U}\tau\right)\right\} d\tau,$$

where U is the flow velocity, b is the semi-chord, and elements of the matrix $[\mathbf{G}_u(-\tau)]$ are response functions corresponding to the impulsive changes of physical coordinates. For a thin airfoil in a two-dimensional incompressible flow, these functions can be expressed in terms of well-known Wagner function [5]:

$$\phi(\tau) = \int_0^\infty \frac{e^{-x\tau}\,dx}{x^2\left((K_0(x) - K_1(x))^2 + \pi^2(I_0(x) + I_1(x))^2\right)}, \tag{2.3}$$

where $K_0(x)$, $K_1(x)$, $I_0(x)$, and $I_1(x)$ are modified Bessel functions of order 0 and 1.

It is convenient to introduce modal coordinates since further calculations are considerably simplified. In the absence of aerodynamic and damping forces and for linear springs, the natural frequencies w_j and modes $\{\varphi_j\}$ $(j = 1, 2)$ can be calculated from the eigenvalue problem:

$$\omega_j^2[\mathbf{M}]\{\varphi_j\} = [\mathbf{K}]\{\varphi_j\}. \tag{2.4}$$

The vector $\{q(t)\}$ of modal coordinates is defined by the relation:

$$\{u(t)\} = [\mathbf{\Phi}]\{q(t)\}, \tag{2.5}$$

where the square matrix $[\mathbf{\Phi}]$ is composed of eigenvectors of the eigenproblem (2.4):

$$[\mathbf{\Phi}] = [\boldsymbol{\varphi}_1 \vdots \boldsymbol{\varphi}_2]. \tag{2.6}$$

The flutter equation written in modal coordinates is:

$$\{\ddot{q}(t)\} + [\mathbf{B}_\omega]\{\dot{q}(t)\} + [\mathbf{K}_\omega]\{q(t)\} + [\mathbf{\Phi}]^T\{k(q)\} = \{f_A\}, \tag{2.7}$$

where $\{f_A\}$ is the vector of generalized unsteady aerodynamic forces given by

$$\{f_A\} = \frac{\rho U^2}{2} \int\limits_{-\infty}^{0} [\mathbf{G}_q(-\tau)] \left\{ q\left(t + \frac{b}{U}\tau\right) \right\} d\tau, \tag{2.8}$$

and the remaining matrices are given by:

$$\begin{aligned} [\mathbf{G}_q(-\tau)] &= [\mathbf{\Phi}]^T[\mathbf{G}_u(-\tau)][\mathbf{\Phi}], \\ [\mathbf{B}_\omega] &= [\mathbf{\Phi}]^T[\mathbf{B}_u][\mathbf{\Phi}], \\ [\mathbf{K}_\omega] &= \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}. \end{aligned}$$

The nonlinear term is written as:

$$\{k(q)\} = [\varphi_1 \vdots \varphi_2]^T \left\{ \begin{array}{c} K_h c_h \left(\varphi_1^{(1)} q_1 + \varphi_1^{(2)} q_2\right)^3 \\ K_\alpha c_\alpha \left(\varphi_2^{(1)} q_1 + \varphi_2^{(2)} q_2\right)^3 \end{array} \right\},$$

where $\varphi_j^{(i)}$ denotes the i-th component of the eigenvector $\{\varphi_j\}$. At the flutter velocity U_0 the Hopf bifurcation occurs and the limit cycle oscillations are determined by the oscillatory branch of bifurcating solutions of Eq. (2.7) [1].

3. Center-maifold reduction

The main advantage of using center-manifold reduction is a small number of variables describing asymptotic motion. The limit cycle oscillations of nonlinear flutter, which is the Hopf bifurcation, are described by only two variables. Although the vector $\{q(t)\}$ is finite-dimensional, the problem is really of infinite dimensions, since the space of initial conditions is an infinite functional space. This is due to the influence of the history of motion expressed in Eq. (2.8) by the convolution integral with infinite delay. This also causes some mathematical difficulties pointed out in [6]. Consequently, the nonlinear flutter problem has to be reformulated and results in the generalized flutter equation [2]:

$$\frac{d\{x_t(\Theta)\}}{dt} = \mathcal{L}\{x_t(\Theta)\} + \mathcal{R}\{x_t(\Theta)\}, \tag{3.1}$$

where

$$\{x(t)\} = \left\{ \begin{array}{c} \mathbf{q}(t) \\ \cdots\cdots \\ \dot{\mathbf{q}}(t) \end{array} \right\}$$

is the vector of new variables, and

$$\{x_t(\Theta)\} = \{x(t+\Theta)\}.$$

The operator $\mathcal{L}$ acting on a continuous vector-function $\{\varphi(\Theta)\}$ from the interval $(-\infty, 0]$ is given by

$$\mathcal{L}\{\varphi(\Theta)\} = \begin{cases} \dfrac{d\{\boldsymbol{\varphi}(\Theta)\}}{d\Theta}, & \text{for } -\infty < \Theta < 0, \\[2ex] [\mathbf{D}]\{\boldsymbol{\varphi}(0)\} + \displaystyle\int_{-\infty}^{0} [\mathbf{G}(-\tau, U)]\{\varphi(\tau)\}d\tau, & \text{for } \Theta = 0, \end{cases}$$

and the nonlinear term is

$$\mathcal{R}\{x_t(\Theta)\} = \begin{cases} 0, & \text{for } -\infty < \Theta < 0, \\ \{f(x_t(0))\}, & \text{for } \Theta = 0. \end{cases}$$

The remaining matrices are following:

$$[\mathbf{D}] = \left[\begin{array}{c:c} \mathbf{0} & \mathbf{I} \\ \cdots\cdots & \cdots \\ -\mathbf{K}_\omega & 0 \end{array} \right],$$

$$[\mathbf{G}(-\tau; U)] = \left[\begin{array}{c:c} 0 & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots \\ \dfrac{\rho U^3}{2b}\left[\mathbf{G}_q\left(-\dfrac{U}{b}\tau\right)\right] & 0 \end{array} \right],$$

$$\{f(\mathbf{x}_t(0))\} = \left\{ \begin{array}{c} 0 \\ \cdots\cdots\cdots\cdots \\ -[\Phi]^T\{k(q)\} \end{array} \right\},$$

The classical flutter equation is obtained from Eq. (3.1) by setting $\Theta = 0$. At the bifurcation point ($U = U_0$), the operator $\mathcal{L}$ has a pure imaginary pair of eigenvalues $\pm i\omega_0$ and the corresponding eigenvectors span a two-dimensional linear subspace $\mathrm{E_c}$ which is tangent to the two-dimensional center-manifold which in turn contains an asymptotic motion. The idea of center-manifold reduction consists in splitting the vector $\{\mathbf{x}_t(\Theta)\}$ into two parts

$$\{x_t(\Theta)\} = \{\nu(z_1, z_2, \Theta)\} + \{w(t, \Theta)\}$$

in such way, that while the vector $\{\mathbf{x}_t(\Theta)\}$ remains all the time on the center-manifold, the vector $\{\nu(z_1, z_2, \Theta)\}$ belongs all the time to the subspace $\mathrm{E_c}$ and depends only on two variables $z_1(t)$ and $z_2(t)$ being functions of time. Both vectors must be "orthogonal" in a certain sense, and obtaining the relation between them is the essence of the method. The algorithm of center-manifold reduction, described in details in [2], provides the method of deriving the first-order ordinary differential equation describing the limit cycle oscillation in terms of a multi-variable power series with respect to new variables z_1 and z_2:

$$\left\{ \begin{array}{c} \dot{z}_1 \\ \dot{z}_2 \end{array} \right\} = [\mathbf{D}_1(\omega_0, U)] \left\{ \begin{array}{c} z_1 \\ z_2 \end{array} \right\} + \sum_{\mu \geqslant 2} \frac{1}{\mu!} [\mathbf{D}_\mu(U)] \{z^\mu\}, \tag{3.2}$$

where the matrices $[\mathbf{D}_\mu(U)] (\mu \geqslant 2)$ are composed of known polynomials with respect to the velocity U, and the first matrix is

$$[\mathbf{D}(\omega_0, U)] = \left[\begin{array}{cc} i\omega_0 + P(U) & 0 \\ 0 & -i\omega_0 + \overline{P}(U) \end{array} \right],$$

where $P(U)$ is also the known polynomial of U ($\overline{P}$ is a complex conjugate of P).

It is worth noting that the method of center-manifold reduction does not require the explicit knowledge of impulsive response matrix but only its Fourier transform – called aerodynamic transfer matrix – for different values of the frequency ω:

$$[\mathbf{A}(i\omega; U)] = \frac{\rho U^2}{2} \int_0^\infty [\mathbf{G}_q(\tau)] e^{-\frac{i\omega b}{U}\tau} d\tau.$$

Contrary to the response matrix, there are many numerical methods available for calculating the transfer matrix which corresponds to the harmonic motion.

Since Eq. (3.2) is an ordinary differential equation, it can be easily transformed to the so-called Poincaré normal form [7] by introducing new variables $\{\zeta(t)\}$ related to $\{z(t)\}$ by the near-identity transformation

$$\{z(t)\} = \{\zeta(t)\} + \sum_{\mu \geqslant 2} \frac{1}{\mu!}[\mathbf{H}_\mu]\{\zeta^\mu(t)\}.$$

Calculation of elements of matrices $[\mathbf{H}_\mu]$ is aimed at making as many coefficients $[\mathbf{D}_\mu]$ in Eq. (3.2) equal to zero as possible. The resulting normal form for the Hopf bifurcation has the phase-shift symmetry, and in polar coordinates

$$\zeta_1 = r(t)e^{i\theta(t)}, \qquad \zeta_2 = \overline{\zeta}_1$$

can be written as:

$$\begin{aligned} \frac{dr}{dt} &= r\sum_{j=0}^{\infty} a_j(U)r^{2j}, \\ \frac{d\theta}{dt} &= \omega_0 + \sum_{j=0}^{\infty} b_j(U)r^{2j}, \end{aligned} \tag{3.3}$$

where all functions $a_j(U)$ and $b_j(U)$are real and have the form of power series expansions with respect to U. In practice, calculations are implemented up to some finite order $j \leqslant n$. Therefore, the amplitude r_H of the limit cycle oscillations satisfies an algebraic equation obtained from Eq. $(3.3)_1$ by setting $r = 0$:

$$r_H \sum_{j=0}^{n} a_j(U)r_H^{2j} = 0. \tag{3.4}$$

For any given U, the left-hand side of Eq. (3.4) is of the form of polynomial with respect to r_H. Hence all possible limit cycle amplitudes are determined by the real positive roots of this polynomial. Since limit cycle oscillations $\zeta_1 = \zeta_H(t)$ on the center-manifold are pure harmonic [1]:

$$\zeta_H = r_H e^{i\omega_h t},$$

then for each amplitude r_H the corresponding frequency ω_H is calculated from

$$\omega_H = \omega_0 + \sum_{j=0}^{n} b_j(U)r_H^{2j}. \tag{3.5}$$

The sequence of transformations

$$r_H, \omega_H \to \{\zeta(t)\} \to \{z(t)\} \to \{x_t(\Theta)\} \to \{q(t)\} \to \{u(t)\}$$

gives finally the limit cycle oscillations of physical variables. Note that due to the nonlinearity of some of them, the resulting oscillations are no more harmonic in time. Moreover, the use of infinite power series results in asymptotic character of Eqs. (3.3) which may not converge. Nevertheless, it will be seen that the first few terms of these equations give often a satisfactory result.

4. Approximate set of ordinary differential equations

Two-dimensional thin airfoil is the only case when the response matrix can be expressed analytically in terms of the Wagner function (2.3). This gives the following expression for the unsteady aerodynamic forces (2.8):

$$(4.1)\qquad \{f_A\} = -\pi\rho U^2 b^2 \Bigg([\mathbf{P}_0]\{q\} + \frac{b}{U}[\mathbf{P}_1]\{\dot{q}\} + \left(\frac{b}{U}\right)^2 [\mathbf{P}_2]\{\ddot{\mathrm{q}}\} + \frac{U}{b}[\mathbf{P}_3]\int_0^t \frac{d\phi\left(\frac{U}{b}\tau\right)}{d\tau}\{\mathrm{q}(t-\tau)\}d\tau + \frac{U}{b}[\mathbf{P}_4]\int_0^t \frac{d^2\phi\left(\frac{U}{b}\tau\right)}{d\tau^2}\{q(t-\tau)\}d\tau \Bigg).$$

Since only an asymptotic motion is considered, it is assumed that all terms arising from initial conditions are damped out, and aerodynamic forces do not depend on time explicitly. The matrices $[\mathbf{P}_i](i = 0, 1, \ldots, 4)$ depend on the eigenmodes, location of the elastic axis x_a, and the initial values of Wagner function $\phi(0)$ and $\dfrac{d\phi(0)}{d\tau}$.

The nonlinear integro-differential flutter equation (2.7) can be replaced by an approximate set of first-order ordinary differential equations, if the Wagner function (2.3) is approximated by the function given by Jones [8]:

$$(4.2)\qquad \phi(\tau) \approx 1 - a_1 e^{-e_1\tau} - a_2 e^{-e_2\tau},$$

where $a_1 = 0.165$, $a_2 = 0.335$, $e_1 = 0.0455$, and $e_2 = 0.3$. After introducing the vector of new variables:

$$\{y(t)\} = \begin{Bmatrix} \mathrm{q}_1(t) \\ \mathrm{q}_2(t) \\ \mathrm{w}_1(t) \\ \mathrm{w}_2(t) \end{Bmatrix},$$

where:

$$\{q_1(t)\} = \{q(t)\}, \qquad \{q_2(t)\} = \{\dot{q}_1(t)\},$$

$$\{w_i(t)\} = [\mathbf{A}_1] \int\limits_0^t e^{-e_i \frac{U}{b}\tau} \{q(t-\tau)\} d\tau, (i = 1, 2),$$

$$[\mathbf{A}_i] = [\mathbf{P}_3] a_i e_i + [\mathbf{P}_4] a_i e_i^2, \quad (i = 1, 2),$$

the flutter equation (2.7) takes the form:

$$\{\dot{y}(t)\} = \left([\mathbf{C}_0] + [\mathbf{C}_1] \frac{U}{b} + [\mathbf{C}_2] \left(\frac{U}{b} \right)^2 + [\mathbf{C}_3] \left(\frac{U}{b} \right)^3 \right) \{y(t)\} + \{\nu(y)\}, \tag{4.3}$$

where dependence on the velocity is expressed explicitly as a polynomial of the third degree and the matrices composed of numbers are given by:

$$[\mathbf{C}_0] = \begin{bmatrix} 0 & [\mathbf{I}] & 0 & 0 \\ -[\mathbf{D}]^{-1}[\mathbf{K}_\omega] & -[\mathbf{D}]^{-1}[\mathbf{B}_\omega] & 0 & 0 \\ [\mathbf{A}_1] & 0 & 0 & 0 \\ [\mathbf{A}_2] & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathbf{C}_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\overline{\rho}[\mathbf{D}]^{-1}[\mathbf{P}_1] & 0 & 0 \\ 0 & 0 & -e_1[I] & 0 \\ 0 & 0 & 0 & -e_2[\mathbf{I}] \end{bmatrix},$$

$$[\mathbf{C}_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\overline{\rho}[\mathbf{D}]^{-1}[\mathbf{P}_0] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathbf{C}_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\overline{\rho}[\mathbf{D}]^{-1} & -\overline{\rho}[\mathbf{D}]^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nonlinear term is given by

$$\{\boldsymbol{\nu}(y)\} = \begin{Bmatrix} 0 \\ -[\mathbf{D}]^{-1}[\boldsymbol{\Phi}]^T \{k(q_1)\} \\ 0 \\ 0 \end{Bmatrix}.$$

The remaining matrices are:

$$[\mathbf{D}] = [\mathbf{I}] + \overline{\rho}[\mathbf{P}_2], \qquad \overline{\rho} = \pi\rho b^4,$$

$$[\mathbf{P}_0] = [\overline{\Phi}]^T\left([\mathbf{B}_3](1-a_1-a_2) + [\mathbf{B}_4](a_1e_1+a_2e_2)\right)[\overline{\Phi}],$$

$$[\mathbf{P}_1] = [\overline{\Phi}]^T\left([\mathbf{B}_2] + [\mathbf{B}_4](1-a_1-a_2)\right)[\overline{\Phi}],$$

$$[\mathbf{P}_2] = [\overline{\Phi}]^T[\mathbf{B}_1][\overline{\Phi}], \qquad [\mathbf{P}_3] = [\overline{\Phi}]^T[\mathbf{B}_3][\overline{\Phi}], \qquad [\mathbf{P}_4] = [\overline{\Phi}]^T[\mathbf{B}_4][\overline{\Phi}],$$

$$[\mathbf{B}_1] = \begin{bmatrix} 1 & -x_a \\ -x_a & \dfrac{1}{8}+x_a^2 \end{bmatrix}, \qquad [\mathbf{B}_2] = \begin{bmatrix} 0 & 1 \\ 0 & \dfrac{1}{2}-x_a \end{bmatrix},$$

$$[\mathbf{B}_3] = \begin{bmatrix} 0 & 2 \\ 0 & -1-2x_a \end{bmatrix}, \qquad [\mathbf{B}_4] = \begin{bmatrix} 2 & 1-2x_a \\ -1-2x_a & 2x_a^2-\dfrac{1}{2} \end{bmatrix}.$$

The matrix $[\overline{\Phi}]$ is a modified eigenmode matrix (2.6):

$$[\overline{\Phi}] = [\overline{\varphi}_1 \vdots \overline{\varphi}_2],$$

containing nondimensional plunge displacements:

$$\{\overline{\varphi}_j\} = \begin{Bmatrix} h_j/b \\ \alpha_j \end{Bmatrix}, \qquad (j=1,2),$$

where b is the semichord of an airfoil. Upon solving the Eq. (4.3), the sequence of transformations

$$\{y(t)\} \to \{q(t)\} \to \{u(t)\}$$

gives the limit cycle oscillations of physical variables.

5. Results

The values (per unit span) of the following basic parameters of the system were constant during calculations: $b = 0.1$ m, $m = 3.313$ kg/m. The values of the remaining parameters were changed. All results correspond to zero damping.

The method of center-manifold reduction gives a sequence of solutions depending on the number of terms n retained in Eqs. (3.4) and (3.5). The only way to estimate convergence of the series is to calculate and compare several consecutive solutions for $n = 1, 2, \ldots$, up to the desired value. Since the motion of an airfoil is not necessarily harmonic in time, the limit cycle amplitudes were calculated as maximum absolute values of the plunge and pitch displacements during one period. The method of center-manifold reduction was applied to both the integro-differential equation (3.1) (identified in figures as "IDE") and the ordinary differential equation (4.3) (identified in figures as "ODE"). In general, all results of calculations fall into two groups. The first group contains solutions corresponding to rather fast convergence of the series of Eqs. (3.3) in a wide range of the velocity (at least as only six terms are considered). A sample of these results is shown in Figs. 2, 3 and 4 as functions of nondimensional velocity U/U_0. The second group contains solutions revealing rather poor convergence, covering at most 5% of the velocity changes. A sample of such results is shown in Figs. 7, 8 and 9.

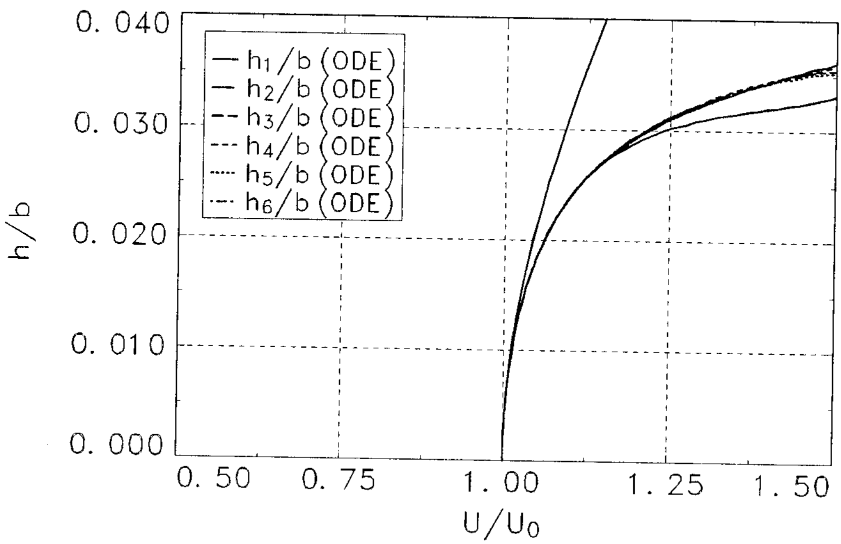

FIG. 2. Limit cycle amplitude in plunge. $x_a = 0$, $S_\alpha = -0.025$ kg, $I_\alpha = 0.043$ kgm, $K_h = 2160$ N/m^2, $K_\alpha = 39$ N, $c_h = 10000$, $c_\alpha = 100$.

The set of ordinary differential equations (4.3) was solved by using the continuation and bifurcation software AUTO97 (available at ftp://ftp.cs.concordia.ca/pub/doedel/auto). This very useful free software gives all desired limit cycle periodic solutions for different values of the velocity (identified in figures as "AUTO").

A comparison of the results was performed in two steps. First, the results given by the AUTO97 software applied to the approximate set of ordinary differential equations were compared to those of center-manifold reduction of the same set of equations (Figs. 6 and 10). The three-term center-manifold approximation was taken in this case. It can be seen that the agreement is very good in the velocity range where there is a good convergence of the series (3.3). Unfortunate-

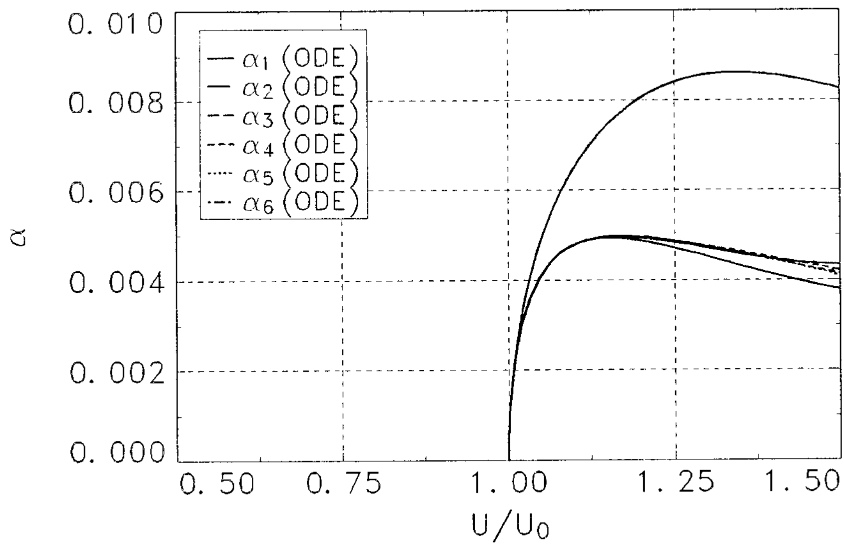

FIG. 3. Limit cycle amplitude in pitch $x_a = 0$, $S_\alpha = -0.025$ kg, $I_\alpha = 0.043$ kgm, $K_h = 2160$ N/m^2, $K_\alpha = 39$ N, $c_h = 10000$, $c_\alpha = 100$.

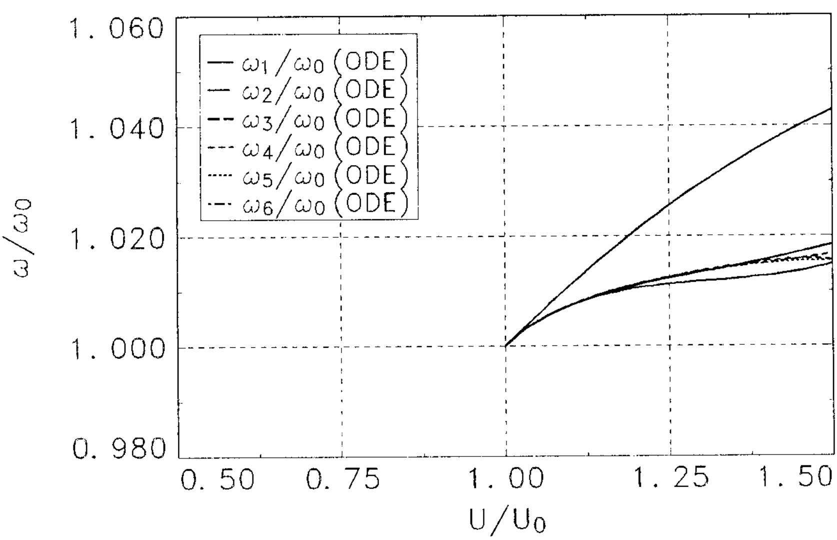

FIG. 4. Limit cycle frequency. $x_a = 0$, $S_\alpha = -0.025$ kg, $I_\alpha = 0.043$ kgm, $K_h = 2160$ N/m^2, $K_\alpha = 39$ N, $c_h = 10000$, $c_\alpha = 100$.

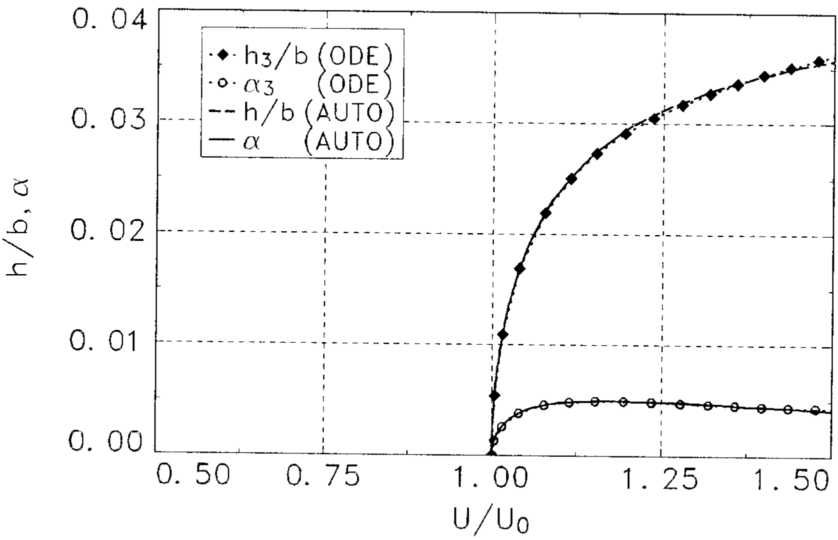

FIG. 5. Limit cycle amplitude. $x_a = 0$, $S_\alpha = -0.025$ kg, $I_\alpha = 0.043$ kgm, $K_h = 2160\ \mathrm{N/m^2}$, $K_\alpha = 39$ N, $c_h = 10000$, $c_\alpha = 100$.

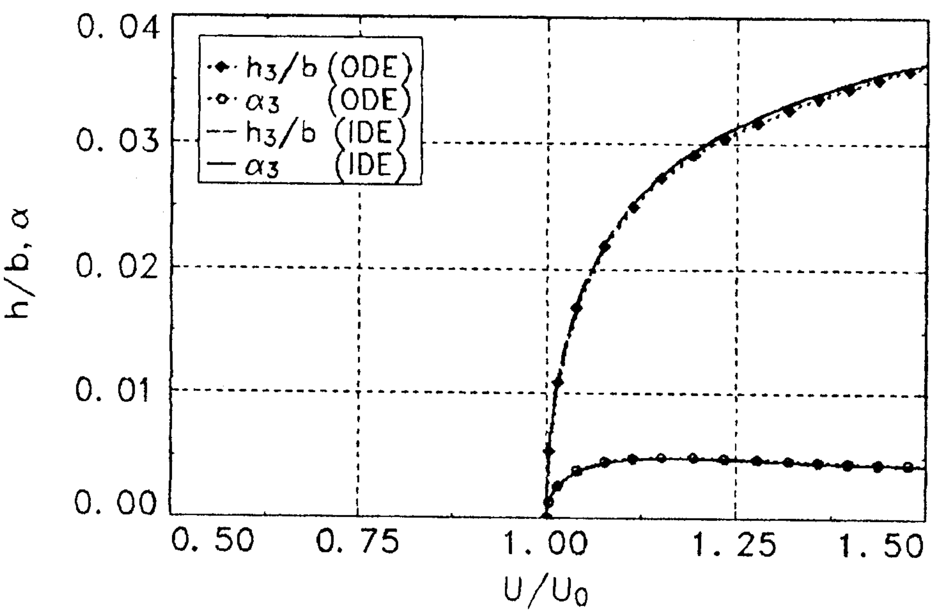

FIG. 6. Limit cycle amplitude. $x_a = 0$, $S_\alpha = -0.025$ kg, $I_\alpha = 0.043$ kgm, $K_h = 2160\ \mathrm{N/m^2}$, $K_\alpha = 39$ N, $c_h = 10000$, $c_\alpha = 100$.

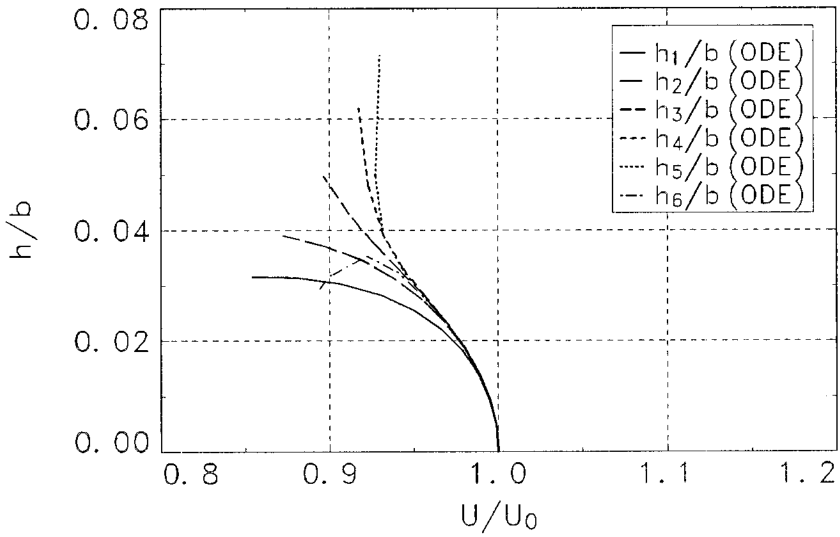

FIG. 7. Limit cycle amplitude in plunge. $x_a = -0.02$, $S_\alpha = -0$ kg, $I_\alpha = 0.042$ kgm, $K_h = 4000$ N/m^2, $K_\alpha = 72$ N, $c_h = 10000$, $c_\alpha = 100$.

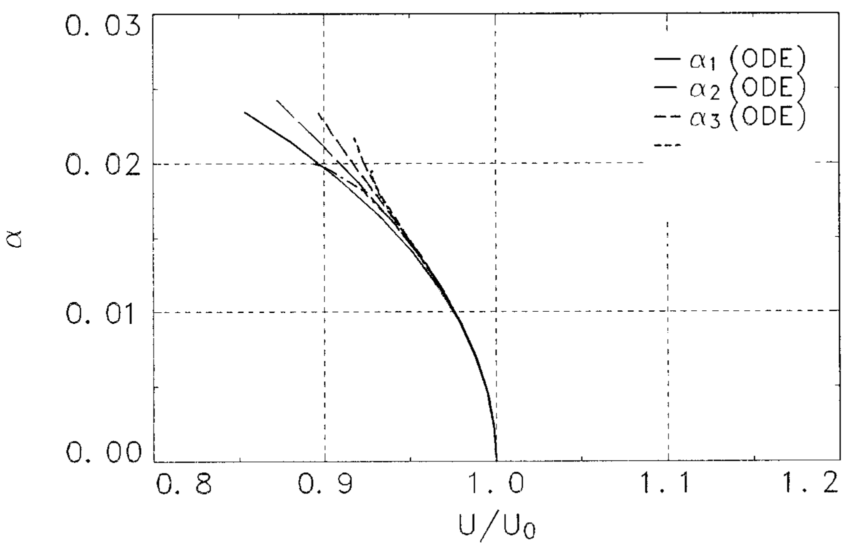

FIG. 8. Limit cycle in pitch $x_a = -0.02$, $S_\alpha = 0$ kg, $I_\alpha = 0.042$ kgm, $K_h = 4000$ N/m^2, $K_\alpha = 72$ N, $c_h = 10000$, $c_\alpha = 100$.

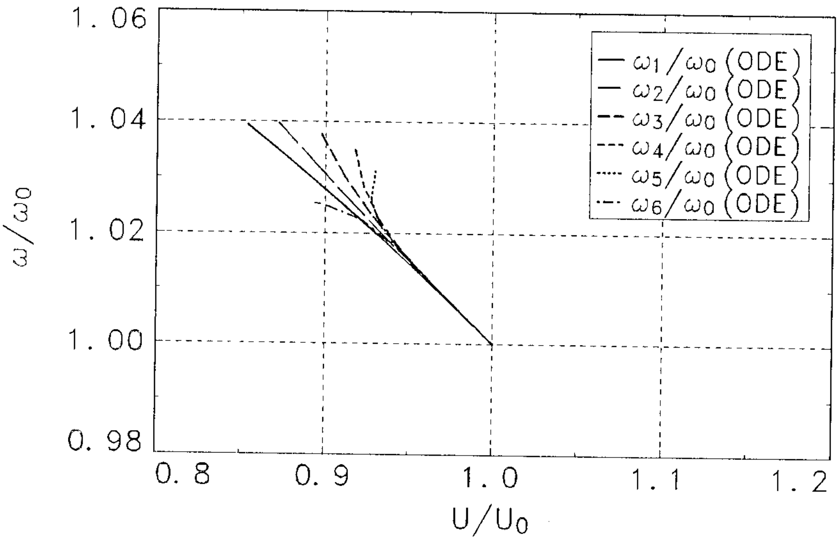

FIG. 9. Limit cycle frequency. $x_a = -0.02$, $S_\alpha = 0$ kg, $I_\alpha = 0.042$ kgm, $K_h = 4000$ N/m^2, $K_\alpha = 72$ N, $c_h = 10000$, $c_\alpha = 100$.

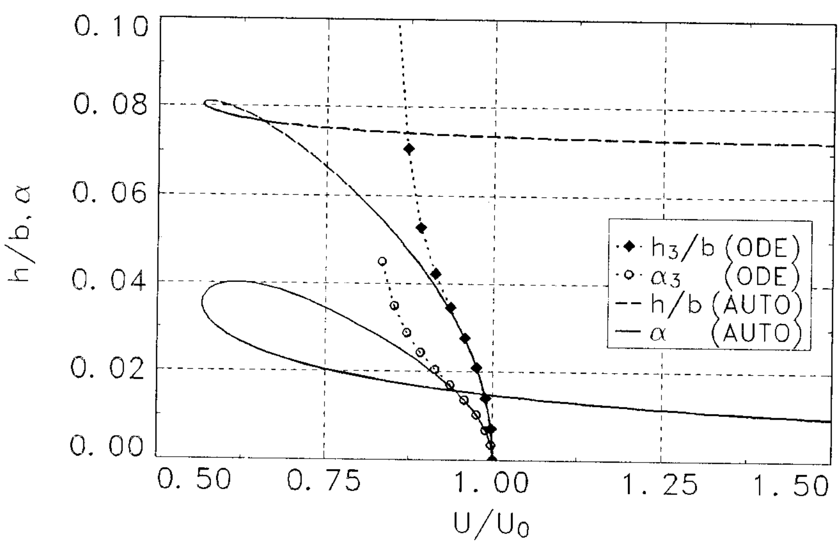

FIG. 10. Limit cycle amplitude. $x_a = -0.02$, $S_\alpha = 0$ kg, $I_\alpha = 0.042$ kgm, $K_h = 4000$ N/m^2, $K_\alpha = 72$ N, $c_h = 10000$, $c_\alpha = 100$.

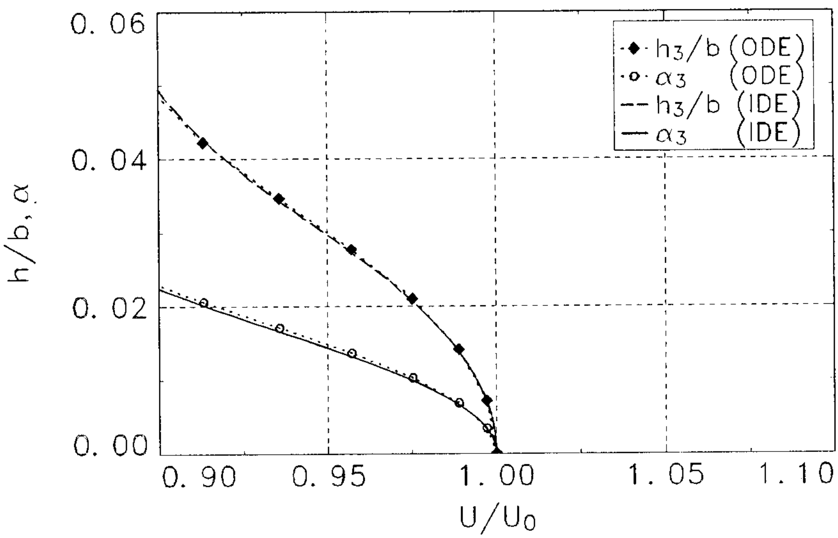

FIG. 11. Limit cycle amplitude. $x_a = -0.02$, $S_\alpha = 0$ kg, $I_\alpha = 0.042$ kgm, $K_h = 4000$ N/m^2, $K_\alpha = 72$ N, $c_h = 10000$, $c_\alpha = 100$.

ly, such range varies significantly (from 50% to 5%, according to plots presented herein). At present, no correlation between the magnitude of the area of a good agreement and physical parameters of the problem was found.

The second step consists of comparing results of the method of center-manifold reduction applied to the integro-differential equation (Eq. (3.1)) and the approximate set of ordinary differential equations (Eq. (4.3)) (Figs. 6 and 11). Essentially, this comparison gives an error caused by the Jones approximation (Eq. (4.2)) of Wagner function (Eq. (2.3)) which in the present case seems to be negligible.

6. CONCLUSIONS

Although the method of center-manifold reduction gives final results in a form of simple power series of Eqs. (3.3), the way of calculating its coefficients is rather complex and based on purely numerical procedure. Therefore, any evaluation of the method has also relay on numerical results. This is not a comfortable situation since no general conclusions can be drawn. Nevertheless, if there is a numerical convergence of the series observed in some interval of the velocity, it can be expected that such results are satisfactory within this interval. From that point of view Figs. 2 – 4 and 7 – 9 help to estimate easily the applicability of the method of center-manifold reduction which is additionally verified by the comparison with other method, as shown in Figs. 5 and 10. In both cases agreement is very good in the estimated range of applicability. The problem is that while in the case of Fig. 5 the method can be considered to be a practical tool for nonlinear flutter calculations since the velocity can vary in a wide range of values (about the results are almost of no practical importance because of the very small (less than 5%) usable range of velocity. This needs further investigations.

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