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CHARITONOV THEOREM AND STABILITY OF PARAMETRICALLY EXCITED SYSTEMS

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The present paper concerns the application of the Charitonov theorem to an analysis of stability of parametrically excited mechanical or physical systems with intervally changing parameters of systems. In such systems the problems of stable solutions of the equation of motion also arise. In some methods the stability analysis of the parametrically excited systems with intervally changing parameters transformes into the analysis of stability of some n-th degree interval polynomials. On the basis of ChT we can check that the solution is stable in the whole interval of changing parameters, without constructing of the boundary of instability regions. Examples of application of the ChT to the analysis of stability of some special systems in steady states of the periodic parametric resonance are considered.

1. Introduction

The present paper concerns the application of the Charitonov theorem (ChT) to an analysis of stability of parametrically excited mechanical or physical systems with intervally changing parameters of systems or surroudings (e.g. physical or geometrical parameters). Examples of application of the ChT to the analysis of stability of some special systems in steady states of the periodic parametric resonance are considered.

Parametric excitation of a physical or mechanical resonance system is a self-excitation caused by a periodic variation of some parameters of the system. These systems (parametric systems) are described by equations with periodically changing coefficients. In many problems the motion of these systems is described by Mathieu-Hill equation. In papers [1 – 5], the brief history of parametric phenomena is given. In recent years many new applications of Mathieu equation have appeared [6, 7], FORYŚ [8 – 10]. When proper relations between external excitation frequency and natural frequencies of the system hold, the instability regions (unstable solution regions) can occur.

Resonances in mechanical systems are usually undesirable phenomena – especially the parametric resonance is very dangerous. Hence our aim is to avoid the unstable solutions. In some methods [15, 16], the stability analysis of the systems in the above resonance states transforms to the analysis of stability of some n-th degree polynomials.

More often in physical and mechanical applications, the parameters of parametrically excited systems are changing in some intervals. In such systems the problems of stable solutions of the equation of motion also arise. The methods of obtaining the instability regions are approximative and time-consuming, especially in the cases of intervally changing parameters.

So the main problem of the paper can be formulated as follows: changing some parameters of the systems in periodic parametric resonance state, we require the system to be stable in the whole interval of changing parameters. In these cases one of the methods of verifying the stability of solutions in the whole interval of changing parameters is application of ChT. In some methods the stability analysis of the parametrically excited systems with intervally changing parameters transformses into the analysis of stability of some n-th degree interval polynomials. On the basis of ChT we can check that the solution is stable in the whole interval of changing parameters, without constructing of the boundary of instability regions.

2. Stability of interval polynomials. Charitonov theorem

First of all we consider the real polynomial of n-th degree

(2.1) $$f(z) = z^n + a_1 z^{n-1} + \dots\dots\dots + a_n, \qquad a_i \in R, \qquad i = 1, ..., n.$$

We say that such a polynomial is stable if and only if all its roots have negative real parts. In paper [11] some methods are proposed for checking stability of the polynomials. The most useful ones are necessary and sufficient conditions such as the Michajlov method which is a graphical method for polynomials with real coefficients or Routh criterion which is based on the Sturm method. As a generalization of the above theorems for polynomials whose coefficients are not necessarily real, we should recall the Schur criterion. Finally let us recall the

Routh-Hurwitz theorem for polynomials with real coefficients. In this method of checking stability we have to verify whether all the Hurwitz determinants are positive. In [12] we find a generalization of the Routh method for the case when there exists a coefficient of polynomial in the Sturm sequence which is equal to zero. Additionally, in [12] we find a generalization of the Routh-Hurwitz theorem for a case when some of the Hurwitz determinants are equal to zero.

Next we consider the family of n-th degree polynomials – called the interval polynomial

$$F_n = \{f(z) = z^n + a_1 z^{n-1} + \ldots + a_n, \qquad a_i \in [\alpha_i, \beta_i], \qquad \alpha_i \leq \beta_i, \qquad i = 1, \ldots, n\}. \tag{2.2}$$

We say that an interval polynomial is stable if all the polynomials of the family (2.2) are stable. The elegant and useful method of checking stability of interval polynomials was published by CHARITONOV in [13] in 1978. Charitonov's theorem gives a necessary and sufficient condition for stability of the interval polynomial, which requires checking only four polynomials of the family (2.2). If these four polynomials are stable, we are sure that the whole family of polynomials (2.2) is stable. The coefficients of four polynomials $f_1(z)$, $f_2(z)$, $f_3(z)$, $f_4(z)$ are given by the following relations.

For $f_1(z)$:

$$a_{n-2k} = \begin{cases} \alpha_{n-2k} & \text{for} \quad k - \text{odd}, \\ \beta_{n-2k} & \text{for} \quad k - \text{even}, \end{cases}$$

$$a_{n-2k-1} = \begin{cases} \alpha_{n-2k-1} & \text{for} \quad k - \text{odd}, \\ \beta_{n-2k-1} & \text{for} \quad k - \text{even}, \end{cases} \tag{2.3}$$

For $f_2(z)$:

$$a_{n-2k} = \begin{cases} \alpha_{n-2k} & \text{for} \quad k - \text{even}, \\ \beta_{n-2k} & \text{for} \quad k - \text{odd}, \end{cases}$$

$$a_{n-2k-1} = \begin{cases} \alpha_{n-2k-1} & \text{for} \quad k - \text{even}, \\ \beta_{n-2k-1} & \text{for} \quad k - \text{odd}. \end{cases}$$

For $f_3(z)$:

$$a_{n-2k} = \begin{cases} \alpha_{n-2k} & \text{for} \quad k - \text{even}, \\ \beta_{n-2k} & \text{for} \quad k - \text{odd}, \end{cases}$$

$$a_{n-2k-1} = \begin{cases} \alpha_{n-2k-1} & \text{for} \quad k - \text{odd}, \\ \beta_{n-2k-1} & \text{for} \quad k - \text{even}, \end{cases}$$

For $f_4(z)$:

$$a_{n-2k} = \begin{cases} \alpha_{n-2k} & \text{for} \quad k - \text{odd}, \\ \beta_{n-2k} & \text{for} \quad k - \text{even}, \end{cases}$$

$$a_{n-2k-1} = \begin{cases} \alpha_{n-2k-1} & \text{for} \quad k - \text{even}, \\ \beta_{n-2k-1} & \text{for} \quad k - \text{odd}. \end{cases}$$

For example, on the basis of Eqs. (2.2) and (2.3), for a 4-th degree interval polynomial

$$F_4 = \{f(z) = z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4, \qquad a_i \in [\alpha_i, \beta_i], \quad \alpha_i \le \beta_i, \quad i = 1, ..., 4\}, \tag{2.4}$$

we have to check stability of the following polynomials (cf. [13]):

$$\begin{aligned} f_1(z) &= z^4 + \alpha_1 z^3 + \alpha_2 z^2 + \beta_3 z + \beta_4, \\ f_2(z) &= z^4 + \beta_1 z^3 + \beta_2 z^2 + \alpha_3 z + \alpha_4, \\ f_3(z) &= z_4 + \alpha_1 z^3 + \beta_2 z^2 + \beta_3 z + \alpha_4, \\ f_4(z) &= z^4 + \beta_1 z^3 + \alpha_2 z^2 + \alpha_3 z + \beta_4. \end{aligned} \tag{2.5}$$

For the third order interval polynomial

$$F_3 = \{f(z) = z^3 + a_1 z^2 + a_2 z + a_3, \qquad a_i \in [\alpha_i, \beta_i], \quad \alpha_i \le \beta_i, \quad i = 1, ..3\} \tag{2.6}$$

we have to check the stability of the following four polynomials:

$$\begin{aligned} f_1(z) &= z^3 + \alpha_1 z^2 + \beta_2 z + \beta_3, \\ f_2(z) &= z^3 + \beta_1 z^2 + \alpha_2 z + \alpha_3, \\ f_3(z) &= z^3 + \beta_1 z^2 + \beta_2 z + \alpha_3, \\ f_4(z) &= z^3 + \alpha_1 z^2 + \alpha_2 z + \beta_3. \end{aligned} \tag{2.7}$$

3. Parametrically excited systems – equations of motion

The equation of motion of undamped parametrically excited mechanical systems (e.g. elastic elements) has the following form (cf. [9]):

$$\mathbf{M}(\mathbf{h})\left[\frac{\partial^2 w}{\partial t^2}\right] + \mathbf{S}(\mathbf{h})[w] + \beta(t)\hat{P}_\beta[w] = 0, \tag{3.1}$$

where $\mathbf{h}$ is the vector of parameters (physical parameters, e.g. stiffness, or geometrical cross-sectional parameters such as area of the cross-section of the rods or thickness of the plate), $\mathbf{M}$, $\mathbf{S}$, $\mathbf{P}_\beta$ – are the inertia, elasticity and stability linear operators. The form of these operators depends on the kinds of mechanical elements to be considered (beams, plates etc.), $w(x,t)$ is a transversal displacement of a vibrating system, $\beta(t)$ is a periodic function of t.

We look for an approximate solution of the above problem in the form

$$w = \sum_{k=1}^{N} f_k(t)\Phi_k(x,y,z), \tag{3.2}$$

where $f_k(t)$ are unknown functions of time and Φ are eigenfunctions of the eigenvalue problem

$$\left[\mathbf{S}(\mathbf{h}) - \omega^2\mathbf{M}(\mathbf{h})\right]\Phi^{(h)} = 0. \tag{3.3}$$

Inserting (3.2) to (3.1) and applying Galerkin's method, one obtains the system of ordinary differential equations of the second order in the matrix form (cf. [1])

$$\mathbf{M}(\mathbf{h})\frac{d^2 f}{dt^2} + [S(\mathbf{h}) + \beta(t)\mathbf{P}_\beta]\,\mathbf{f} = 0, \tag{3.4}$$

where $\mathbf{M}$, $\mathbf{S}$, $\mathbf{P}_\beta$ are the inertia, elasticity and parametric excitation matrices which depend on $\mathbf{h}$, $\mathbf{f} = \operatorname{col}[f_1(t), f_2(t)......f_n(t)]$ is the column matrix of the generalized coordinates. The elements of matrices are

$$\begin{aligned}
M_{ik} &= \int_D \Phi_i \mathbf{M}(\mathbf{h})[\Phi_k]d\tau = (\Phi_i, \mathbf{M}(\mathbf{h})[\Phi_k]) = J_1^{(ik)},\\
S_{ik} &= \int_D \Phi_i \mathbf{S}(\mathbf{h})[\Phi_k]d\tau = (\Phi_i, \mathbf{S}(\mathbf{h})[\Phi_k]) = J_1^{(ik)},\\
P_{\beta ik} &= \int_D \Phi_i \mathbf{P}_\beta[\Phi_k]d\tau = (\Phi_i, \mathbf{P}_\beta(\mathbf{h})[\Phi_k]) = J_3^{(ik)}.
\end{aligned} \tag{3.5}$$

Introducing the following matrices, cf. FORYŚ, BOLOTIN: $\mathbf{B(h)} = \mathbf{S}^{-1}\mathbf{P}_\beta$, $\mathbf{C(h)} = \mathbf{S}^{-1}\mathbf{M}$, the Eq. (3.4) assumes the form

$$\frac{d^2 f}{dt^2} + \mathbf{C}(h)^{-1}[E + \beta(t)\mathbf{B}(h)]f = 0. \tag{3.6}$$

For example for parametrically excited beams, the inertia, elasticity and stability operators have the form

$$\begin{aligned} \mathbf{M}(h) &= m(h) = \rho(x)h(x), \\ \mathbf{S}(h) &= \frac{\partial^2}{\partial x^2}\left[K_\alpha h^\alpha \frac{\partial^2}{\partial x^2}\right], \\ \mathbf{P}_\beta &= \frac{\partial^2}{\partial x^2}, \end{aligned} \tag{3.7}$$

the function of state Φ satisfies the equation of state (3.2) given below:

$$[K_\alpha h^\alpha(x)\Phi''(x)]'' - \rho h(x)\omega^2\Phi(x) = 0, \tag{3.8}$$

where $h(x)$ is a geometrical parameter, the area of the cross-section, $K_\alpha = EA_\alpha, E$ is Young's modulus, A_α is a constant connected with the geometry of cross-sections and depending on $\alpha(\alpha = 1, 2, 3)$, and ρ is the mass density.

When $\mathbf{C}^{-1} = \mathrm{diag}[\omega^2]$ is the diagonal matrix, we can describe the parametrically excited systems with damping by equations

$$\frac{d^2 f_k}{dt^2} + 2\varepsilon_k(\mathbf{h})\frac{df_k}{dt} + \omega_k^2(h)\left[f_k + \beta(t)\sum_{j=1}^{N} b_{kj} f_j\right] = 0, \quad k = 1, 2, ..., N, \tag{3.9}$$

where we introduce the damping matrix $\mathbf{E}$ and where $\varepsilon_{kk} = \varepsilon_k$ are the damping matrix elements. The Eqs. (3.9) form a set of coupled linear equations with variable coefficients.

Equations (3.6) and (3.9) are sets of coupled Mathieu equations for multiple-degree of freedom systems, which have been studied by BOLOTIN [1], CARTMEL [2], NAYFEH and MOOK [3], HSU [14] and TAKAHASHI [15, 16].

When the matrix $\mathbf{B(h)} = \mathbf{S}^{-1}\mathbf{P}_\beta$ is additionally diagonal, the analized systems are described by a non-coupled set of Mathieu-Hill equations (cf. [9])

$$\frac{d^2 f_k}{dt^2} + 2\varepsilon_k(\mathbf{h})\frac{df_k}{dt} + \omega_k^2(\mathbf{h})\left[1 + \beta(t)b_k(\mathbf{h})\right] f_k = 0, \quad k = 1, 2, ..., n, \tag{3.10}$$

where in Eqs. (3.9) and (3.10) the quantities:

$$\omega_k^2 = \frac{(\Phi_k, \mathbf{S}(\mathbf{h})[\Phi_k])}{(\Phi_k, \mathbf{M}(\mathbf{h})[\Phi_k])} = \frac{J_2^{(k)}}{J_1^{(k)}} \quad \text{and} \quad \beta_{cr} \cong -\frac{1}{b_k} = -\frac{(\Phi_k, \mathbf{S}(\mathbf{h})[\Phi_k])}{(\Phi_k, \mathbf{P}_\beta[\Phi_k])} = -\frac{J_2^{(k)}}{J_3^{(k)}} \tag{3.11}$$

are eigenvalues of proper eigenvalue problems; the functionals $J_1^{(k)}$, $J_2^{(k)}$, $J_3^{(k)}$ were introduced in (3.5). The quantities ε_k, ω_k, b_k depend on $\mathbf{h}$.

In many physical problems, the systems are described by one of Mathieu-Hill equations like (3.10) (cf. [7, 16]). The Equations (3.6), (3.9), (3.10) are the examples of equations with periodically changing coefficients.

3.1. Floquet theorem. Solutions of equations of motion and stability

Next we determine the behavior of systems governed by linear ordinary differential equations with periodic coefficients ([1 – 3, 15]). The Floquet theory may be applied to characterizing the functional behavior of such systems, [17]. On the basis of the analysis of Eq. (3.6), the solution takes the form, [1, 2, 16]:

$$f_k(t) = e^{\frac{t}{T}\ln\rho_k} g_k(t), \tag{3.12}$$

where $g_k(t)$ are periodic vector functions with period T and ρ_i are characteristic roots. Now we define the characteristic exponent

$$H = \frac{1}{T}\ln\rho. \tag{3.13}$$

Let us take $\beta(t) = \beta_t \cos\theta t$ in Eq. (3.6). We look for the solution of Eq. (3.6) in the form

$$\mathbf{f}(t) = e^{Ht}\left[\frac{1}{2}\mathbf{b}_0 + \sum_{k=1,2,3...}^{\infty} (\mathbf{a}_k \sin k\theta t + \mathbf{b}_k \cos k\theta t)\right], \tag{3.14}$$

where $\mathbf{b}_0$, $\mathbf{a}_k$, $\mathbf{b}_k$ are vectors which do not depend on the time variable.

Inserting (3.14) into Eq. (3.6) and applying the harmonic balance method we obtain the system of homogeneous algebraic equations

$$([\mathbf{N}_0] - H[\mathbf{N}_1] - H^2[\mathbf{N}_2])\mathbf{X} = \mathbf{0},$$

or in a short form

$$\mathbf{G}\mathbf{X} = \mathbf{0}, \tag{3.15}$$

where $[\mathbf{N}_0]$, $[\mathbf{N}_1]$, $[\mathbf{N}_2]$ are coefficient matrices of the zero (constant), first and second powers of $\mathbf{H}$, respectively, and $\mathbf{X}$ is the column vector consisting of $\mathbf{b}_0$, $\mathbf{a}_k$, $\mathbf{b}_k$.

To find any solution to this system, not including the trivial one, the determinant of the coefficient matrix must be equal to zero. Thus the equation for determining the characteristic exponent is obtained as follows (cf. [14, 15]):

$$\det\mathbf{G} = \det([\mathbf{N}_0] - H[\mathbf{N}_1] - H^2[\mathbf{N}_2]) = 0. \tag{3.16}$$

To obtain the value H in Eq. (3.16), one can make use of the method which determines the eigenvalue of the special double-size matrix (cf. [15]).

When the analysed system is described by Eq. (3.6) the algebraic Eq. (3.15) takes the form (one parametr h)

$$\begin{aligned}
&(H^2\mathbf{C}(h) + \mathbf{E})\mathbf{b}_0 + \beta_t\mathbf{B}(h)b_1 = 0,\\
&(H^2 - k^2\theta^2)\mathbf{C}(h)\mathbf{a}_k + 2Hk\theta\mathbf{C}(h)\mathbf{b}_k + \mathbf{E}\mathbf{a}_k + \beta_t\frac{1}{2}\mathbf{B}(h)(a_{k-1} + a_{k+1}) = 0,\\
&(H^2 - k^2\theta^2)\mathbf{C}(h)\mathbf{b}_k + 2Hk\theta\mathbf{C}(h)\mathbf{a}_k + \mathbf{E}\mathbf{b}_k + \beta_t\frac{1}{2}\mathbf{B}(h)(b_{k-1} + b_{k+1}) = 0,\\
&a_0 = 0, \quad k = 1, 2, 3, ...
\end{aligned} \tag{3.17}$$

The non-zero solution of the set of algebraic Eqs. (3.17) exists if the following determinant is equal to zero (cf. [1]):

$$\begin{vmatrix}
(H^2 - \theta^2)\mathbf{C}(h) + \mathbf{E} & \frac{1}{2}\beta_t\mathbf{B}(h) & 2H\theta\mathbf{C}(h)\\
\beta_t\mathbf{B}(h) & H^2\mathbf{C}(h) + \mathbf{E} & 0\\
-2H\theta\mathbf{C}(h) & 0 & (H - \theta^2)\mathbf{C}(h) + \mathbf{E}
\end{vmatrix}. \tag{3.18}$$

If eigenvalues H_i are distinct, the solution is stable if all roots have non-positive real parts.When parameter h which characterises the parametric systems takes values in the interval

$$h \in < h_1, h_2 >$$

then Eq. (3.18) has the form of the interval polynomial (2.2). The analysis of stability of Eq. (3.6) is reduced to the analysis of stability of interval polynomials (3.18), and ChT plays an important role (cf. Sec. 2).

Our main problem is the analysis of the paramertrically excited system with intervally changing parameters. We require stable solutions of the equation of motion of parametrically excited system in the whole interval of variable parameters.

4. Examples

Now we consider the one-dimensional parametrically excited systems, (Fig. 1) – the elastic or viscoelastic rods with variable cross-section and with different boundary conditions (cf. [9]).

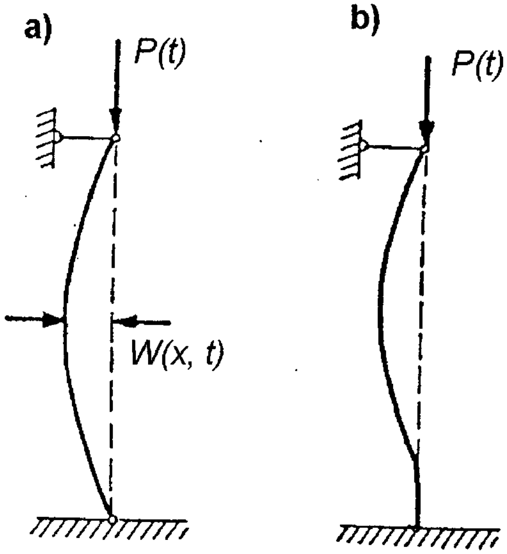

Fig. 1. Parametrically excited systems (beams) with different boundary conditions: a) simple supported beam, b) fixed-simple supported beam.

Equation of motion of a non-prismatic rod excited by force $\beta(t) = P(t)$ which is longitudinal and periodically varying in time, has the form

$$\frac{\partial^2}{\partial x^2}\left[K_\alpha h^\alpha \frac{\partial^2 w}{\partial x^2} + \tau K_\alpha h^\alpha \frac{\partial^3 w}{\partial x^2 \partial t}\right] + \rho h(x)\frac{\partial^2 w}{\partial t^2} + \beta(t)\frac{\partial^2 w}{\partial x^2} = 0, \tag{4.1}$$

where $w(x,t)$ is a transverse displacement of the cross-section x at time $t, h(x)$ is cross-sectional area, (cf. (15)), ρ mass density, $\tau = \eta/E$, η is a coefficient of internal damping.

For example we consider two different cases (Fig. 1): simply supported beam (Fig. 1a) and partly fixed, partly simply supported beam (Fig. 1b). The boundary conditions are as follows:

$$\begin{aligned} w(0,t) &= 0, \quad \left[K_\alpha h^\alpha \frac{\partial^2 w}{\partial x^2} + \tau K_\alpha h^\alpha \frac{\partial^3 w}{\partial x^2 \partial t}\right](0,t) = 0,\\ w(l,t) &= 0, \quad \left[K_\alpha h^\alpha \frac{\partial^2 w}{\partial x^2} + \tau K_\alpha h^\alpha \frac{\partial^3 w}{\partial x^2 \partial t}\right](l,t) = 0, \end{aligned} \tag{4.2}$$

and

$$(4.3)\qquad \begin{aligned} w(0,t) &= \frac{\partial w}{\partial x}(0,t) = 0, \\ w(l,t) &= 0, \quad \left[K_\alpha h^\alpha \frac{\partial^2 w}{\partial x^2} + \tau K_\alpha h^\alpha \frac{\partial^3 w}{\partial x^2 \partial t}\right](l,t) = 0. \end{aligned}$$

We look for approximate solutions of the above problems in the form of a series of eigenfunctions of natural vibrations of prismatic beams (cf. (3.2) and (3.8)). For the beam in Fig. 1a, the eigenfunction Φ_i has the form:

$$(4.4)\qquad \Phi_i = \sin i\pi x/l.$$

For the beam in Fig. 1b, the eigenfunction Φ_i has the form:

$$(4.5)\qquad \Phi_i = \cos\gamma_i(\mathrm{sh}\gamma_i x/l - \sin\gamma_i x/l) - \sin\gamma_i(\mathrm{ch}\gamma_i x/l - \cos\gamma_i x/l),$$

$$\gamma_1 = 3.9266.$$

After discretization (Galerkin's method) and some rearrangement we get the systems (3.4) of ordinary differential equations with the following matrix elements:

$$(4.6)\qquad M_{ik} = \rho\int_0^l h(x)\Phi_i(x)\Phi_k(x)dx, \qquad S_{ik} = \int_0^l I(x)\Phi_i'(x)\Phi_k''(x)dx,$$

$$P_{\beta ik} = \int_0^l \Phi_i''(x)\Phi_k(x)dx.$$

Now we introduce some parameters $\overline{\alpha}$, $\overline{\kappa}$ describing the rod's shape, (Fig. 2). For the case such as that in Fig. 1a we assume that the side $a(\overline{\alpha},\overline{\kappa},x)$ of quadratic section of the rod changes as a quadratic function of x (Fig. 2a) and has the form

$$(4.7)\qquad a(\overline{\alpha},\overline{\kappa},x) = \overline{\alpha}\left\{4\overline{\kappa}\left(\frac{x^2}{l^2} - \frac{x}{l}\right) + 1\right\} = \overline{\alpha}\varphi(\overline{\kappa},x),$$

where

$$\kappa = \frac{\overline{\alpha} - \overline{\beta}}{\overline{\alpha}}, \qquad a(0) = a(1) = \overline{\alpha}, \qquad a(1/2) = \overline{\beta}, \qquad \overline{\kappa} \in (-\infty, 1].$$

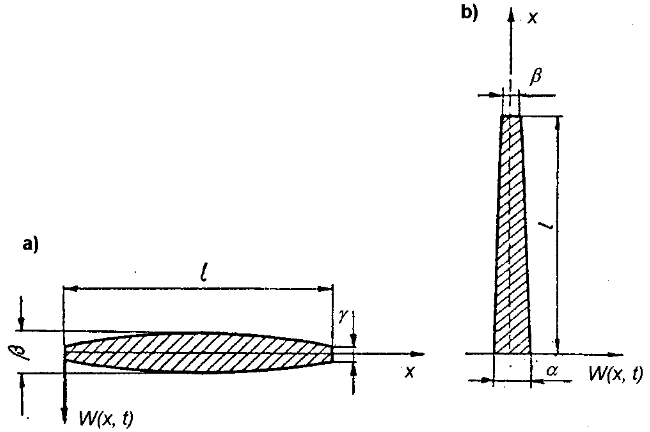

FIG. 2. The shapes and geometrical parameters of the beams.

For the case shown in Fig. 1b we assume that the side of quadratic section of the rod changes as a linear function of x (Fig. 2b):

$$(4.8)\qquad a(\overline{\alpha}, \overline{\kappa}, x) = \overline{\alpha}\left(1 - \overline{\kappa}\frac{x}{l}\right) = \overline{\alpha}\varphi(\overline{\kappa}, x),$$

where

$$\kappa = \frac{\overline{\alpha} - \overline{\beta}}{\overline{\alpha}}, \qquad a(0) = \overline{\alpha}, \quad a(l) = \overline{\beta}, \quad \overline{\kappa} \in (-\infty, 1].$$

So we obtain the area of cross-section and cross-sectional moment of inertia

$$(4.9)\qquad h(x) = [\overline{\alpha}\varphi(\overline{\kappa}, x)]^2, \qquad I = 1/12[\overline{\alpha}\varphi(\overline{\kappa}, x)]^4.$$

Inserting these formulas to (4.6) we get the system of ordinary equations of motion in the form

$$(4.10)\qquad \sum_{i=1}^{l} \left[\ddot{f} M_{ik} + \tau \dot{f} S_{ik} + (S_{ik} + \beta(t)P_{ik})f\right] = 0,$$

where

$$(4.11)\qquad \begin{aligned} M_{ik} &= \rho[\overline{\alpha}]^2 \int_0^l [\varphi(\overline{\kappa}, x)]^2 \Phi_i(x)\Phi_k(x)dx, \\ S_{ik} &= E\frac{1}{12}[\overline{\alpha}]^4 \int_0^l [\varphi(\overline{\kappa}, x)]^4 \Phi_i''(x)\Phi_k''(x)dx, \\ P_{\beta ik} &= \int_0^l \Phi_i''(x)\Phi_k(x)dx. \end{aligned}$$

The elements of matrix (4.11) for one degree of freedom and for Case I (Fig. 1a, Fig. 2a) have the form

$$M = \rho\overline{\alpha}^2 l f_M(\kappa), \qquad S = \frac{E\overline{\alpha}^4}{l^3} f_S(\kappa), \qquad P = -\frac{\pi^2}{2l}, \tag{4.12}$$

where

$$\begin{aligned} f_M &= 0.3910\overline{\kappa}^2 - 0.8693\overline{\kappa} + 0.500, \\ f_S &= 2.701\overline{\kappa}^4 - 11.64\overline{\kappa}^3 + 19.01\overline{\kappa}^2 - 14.12\overline{\kappa} + 4.058. \end{aligned} \tag{4.13}$$

The elements of matrix (4.11) for one degree of freedom and for Case II (Fig. 1b, Fig. 2b) have the form

$$M = \rho\overline{\alpha}^2 l f_M(\kappa), \qquad S = \frac{E\overline{\alpha}^4}{l^3} f_S(\kappa), \qquad P = -\frac{5.7518}{l}, \tag{4.14}$$

where

$$\begin{aligned} f_M &= 0.1747\overline{\kappa}^2 - 0.5680\overline{\kappa} + 0.500, \\ f_S &= 1.253\overline{\kappa}^4 - 7.153\overline{\kappa}^3 + 15.94\overline{\kappa}^2 - 17.06\overline{\kappa} + 9.89. \end{aligned} \tag{4.15}$$

4.1. Dynamic stability

For one degree of freedom, the system of Eq. (4.10) is reduced to one equation in the form

$$\ddot{f} + 2\varepsilon\dot{f} + (\omega^2 + \beta(t)c)f = 0, \tag{4.16}$$

where (cf. (3.11) and (4.6))

$$\begin{aligned} \omega^2 &= \frac{S}{M} = \frac{\frac{1}{12}E[\overline{\alpha}]^2 \int_0^l [\varphi]^4[\Phi'']^2 dx}{\rho \int_0^l [\varphi]^2\Phi^2 dx} = \frac{J_2}{J_1}, \qquad 2\varepsilon = \tau\omega^2, \\ C &= \frac{P}{M} = \frac{\int_0^l \Phi\Phi'' dx}{\rho[\overline{\alpha}]^2 \int_0^l [\varphi]^2\Phi^2 dx} = \frac{J_3}{J_1}. \end{aligned} \tag{4.17}$$

Substituting in (4.16) $\beta(t) = P(t) = P_0 + P_t \cos\theta t$, after some rearrangement (cf. [8]) we have

$$\ddot{f} + 2\varepsilon \dot{f} + \Omega^2(1 - 2\mu\cos\theta t) f = 0, \tag{4.18}$$

where we define the quantity

$$\Omega(h) = \omega(h)\sqrt{1 + \beta_0 b(h)} = \omega(h)\sqrt{1 - \frac{\beta_0}{\beta_{\rm cr}(h)}}, \tag{4.19}$$

and the excitation parameter μ

$$\mu(h) = -\frac{\beta_t b(h)}{2(1 + \beta_0 b(h))} = \frac{\beta_t}{2(\beta_{\rm cr}(h) - \beta_0)} \tag{4.20}$$

connected with constant part β_0 and amplitude β_t of the oscillating part of external parametric excitation $\beta(t)$, where

$$b(h) = \frac{c}{\omega^2} = \frac{P_\beta}{S}, \qquad b(h) = -1/\beta_{\rm cr}, \quad \beta_{\rm cr} \cong -\frac{1}{b_k} = -\frac{J_2}{J_3} = -\frac{S}{P_\beta}. \tag{4.21}$$

We can analyse the stability of Eq. (4.18) in the $(\theta/2/\Omega,\ \mu)$ space. The most popular and very effective method of determining the instability region is Bolotin's method (cf. [1]). This method is valid only for $\mu \ll 1$ – it is a disadvantage of the method. The instability regions in the $(\theta/2/\Omega, \mu)$ space are illustrated in Fig. 3. The first most important instability region is illustrated in Fig. 4.

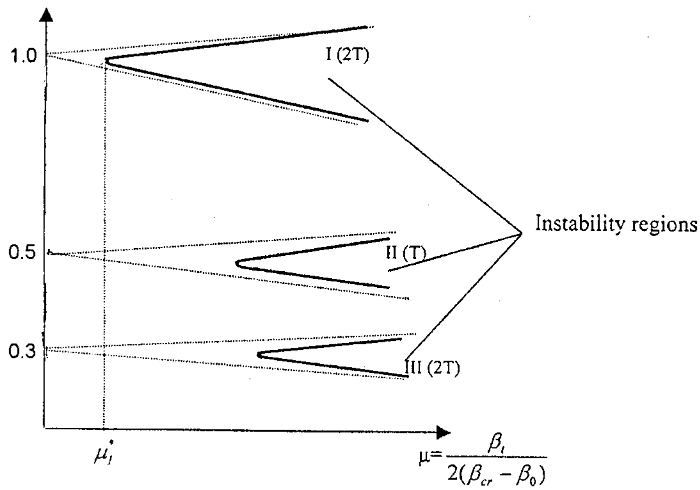

FIG. 3. The instability regions in the $(\theta/2/\Omega, \mu)$ space.

Now for $\beta(t) = P(t) = P_t \cos \theta t$ we transform the equation of motion (4.18) into classical form of the Mathieu equation

$$\frac{d^2u}{dz^2} + (p - 2q\cos 2z)u = 0, \tag{4.22}$$

where

$$p = \frac{4}{\theta^2}(\Omega^2 - \varepsilon^2), \qquad q = 4\frac{\mu\Omega^2}{\theta^2}. \tag{4.23}$$

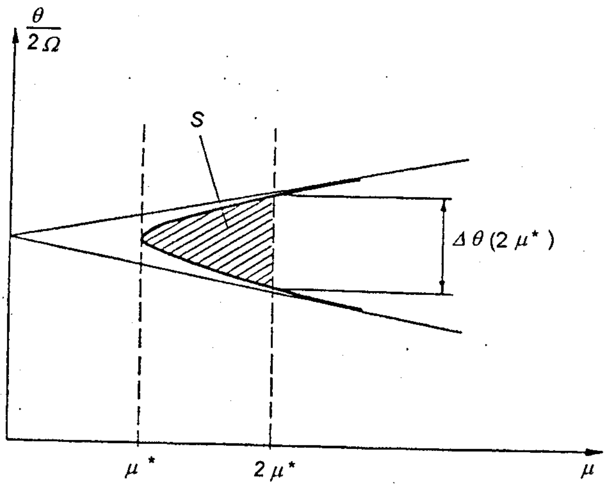

FIG. 4. The first most important instability region.

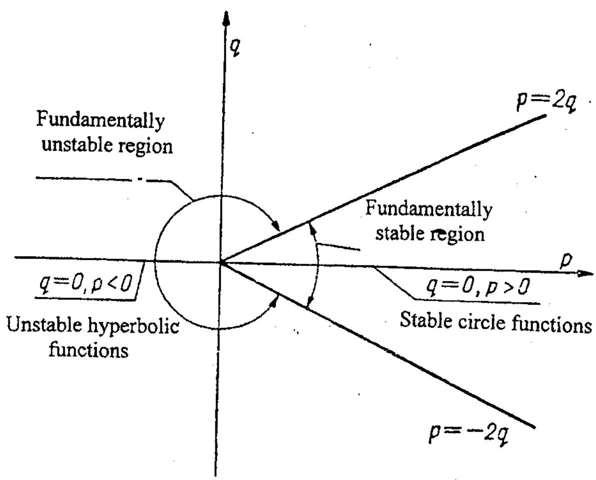

FIG. 5. Approximate graph of stability of Mathieu equation – qualitative consideration, (cf. [17]).

Now we analyse the stability of Eq. (4.22) in the (p, q) space. There are many methods of analysis of stability of Eq. (4.22) (e.g. the method of small parameter, method of multiple scales etc.). The determination of the boundary between the stable and unstable solution it is a difficult problem – we can do it on the basis of approximate methods. In Fig. 5 we illustrate the approximate graph of stability of the Mathieu equation obtained on the basis of qualitative consideration. In Fig. 6 we illustrate accurate graph of the boundary of instability regions obtained on the basis of the small parameter method valid for $q \ll 1$.

When the parameter of the systems changes is some interval, the values of p and q also change in intervals and this influences the instability region.

On the basis of the Floguet theory we look for the approximate solutions of Eq. (4.18) or (4.22) in the form

$$f(t) = e^{Ht}\left[\frac{1}{2}b_0 + \sum_{k=1,3,5\ldots}^{k} (a_k \sin k\theta t + b_k \cos k\theta t)\right],$$

and we determine the stability point by point for different values of parameters.

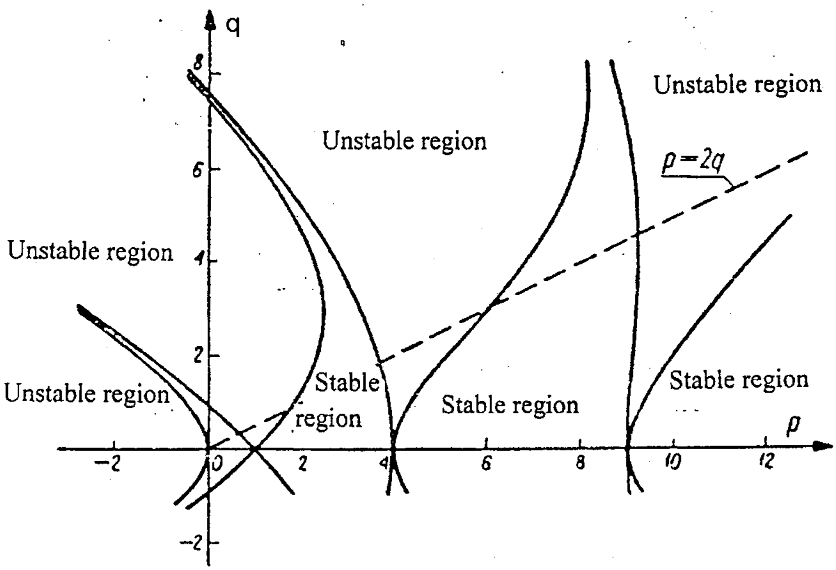

FIG. 6. Precise graph of the boundary of instability regions of Mathieu equation – small parameter method, (cf. [17]).

If we look for stable solutions of Eq. (4.18) or (4.22), we may also look for the conditions under which the algebraic equation like (3.17) (the proper polynomial for H) has no roots with positive real parts.

When the parameter h which characterises the parametric systems changes in the interval, e.g.

$$h \in < h_1, h_2 >,$$

the polynomial of H has the form of the interval polynomial (2.2). The analysis of stability of (4.18) or (4.22) reduced to the analysis of stability of proper interval polynomials, and ChT plays an important role.

Because our infinite determinants (3.18) belong to the class of convergent determinants, the so-called normal determinants, we limit ourselves to a finite sum in (3.14). Applying the procedure of Sec. 3.1 to the analysis of stability of the system described by Eq. (4.18), we have to find (for $k = 1$) the solution of the following polynomial of 6-th degree:

$$H^6 + a_1(h)H^4 + a_2(h)H^2 + a_3(h) = 0. \tag{4.24}$$

Changing the parameter h (e.g. geometrical parameters $\overline{\alpha}, \overline{\beta}$ of the shape of the beam or parameters of external longitudinal force β_0, β_t) in some interval, we have an interval polynomial of 3-rd degree to analyse the stability

$$F_3 = \{f(z) = z^3 + a_1 z^2 + a_2 z + a_3, \quad \text{where } H^2 = z, \quad a_i \in [\alpha_i, \beta_i], \quad \alpha_i \le \beta_i, \quad i = 1,..3\}. \tag{4.25}$$

When four polynomials (2.7) are stable, the parametrically excited system (4.18) or (4.22) is stable in the whole interval of variation of parameter h. Because $H^2 = z$, one concludes that if all roots of (4.25) are real, negative and different, the sixth order polynomial (4.24) has no roots with positive real parts. So we must check an interval polynomial of 3-rd degree to analyse the stability. So on the basis of ChT we must check the stability of four polynomials of the form (2.7).

$$\begin{aligned} f_1(z) &= z^3 + \alpha_1 z^2 + \beta_2 z + \beta_3, \\ f_2(z) &= z^3 + \beta_1 z^2 + \alpha_2 z + \alpha_3, \\ f_3(z) &= z^3 + \beta_1 z^2 + \beta_2 z + \alpha_3, \\ f_4(z) &= z^3 + \alpha_1 z^2 + \alpha_2 z + \beta_3. \end{aligned} \tag{4.26}$$

4.2. Numerical results and conclusions

In our first numerical example the analysed parametrically excited beams are prismatic and elastic. The interval parameter equals the amplitude of oscillating part of the external excitation $h = \beta_t$, and its values change in the interval

$h \in < \beta_t^1, \beta_t^2 >$, $\beta_t^1 < \beta_t^2$. The constat part of external excitation equals zero, $\beta_0 = 0$. Changing the parameter $h = \beta_t$ of the beam in periodic parametric resonance state in interval $< \beta_t^1, \beta_t^2 >$, we require the stable solutions, in the whole interval of the changing parameters. Under proper selection of the interval the solutions are stable in the whole interval of the changing parameters.

The methods of checking that the solutions are stable in the whole interval of changing parameters are based on application of ChT. Repeating the procedure of Sec. 3.1 to analyse the stability of beams, we analyse the stability of the polynomial of 6-th degree with respect to the characteristic exponent H (cf. (3.13)). In our example the determinant (3.18) takes the following form:

$$\det G = \begin{vmatrix} H^2 + \omega^2, & 0, & \beta_i c \\ 0 & H^2 + \omega^2 - \theta^2 & -2H\theta \\ \beta_i c/2 & 2H\theta & H^2 + \omega^2 - \theta^2 \end{vmatrix} = 0, \tag{4.27}$$

or in the form of the 3-rd degree interval polynomial (4.25), where

$$a_1 = 3\omega^2 + 2\theta^2, \qquad a_2 = 2k\omega^2 + k^2 + 4\theta^2\omega^2 - \beta_t^2 d, \quad a_3 = k^2\omega^2 - \beta_t^2 dk,$$
$$d = c^2/2, \qquad k = \omega^2 - \theta^2. \tag{4.28}$$

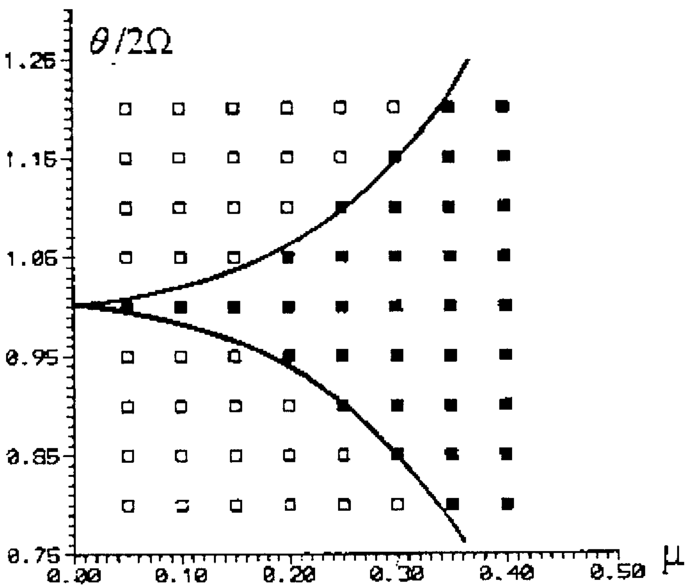

FIG. 7. Numerical results: stable (□) and unstable (■) points – characteristic exponent method. Instability regions.

Because $H^2 = z$, one concludes that if all roots of interval polynomial (4.25) are real, negative and different, the sixth order polynomial (4.25) has no roots with positive real parts. So we must check the interval polynomial of 3-rd degree (4.28) to analyse the stability. On the basis of ChT we must check the stability of four polynomials of the form (2.7).

The numerical results are illustrated in the Fig. 7. In the $(\theta/2/\Omega, \mu)$ space we illustrate the stable (marked by □) points and unstable points (marked by ■) checked on the basis of the stability analysis of the polynomial (4.25) with coefficients (4.28). We can draw the boundary of the instability region. The main problem of the paper is changing the parameter $\beta_t(\mu)$ of the system, so that the system is stable in the whole interval of the changing parameters.

We can choose one of the stable points e.g. point 1, and verify how the ChT works. When we choose the interval of changing $\beta_t(\mu)$ so that the stable point 2 is the second end of the interval, the four polynomials (2.7) are stable (Fig. 8, Table 1, Table 2). If the end of the interval of changing of $\beta_t(\mu)$ is e.g. point 3 or 4, at least one of polynomials (2.7) is unstable (Fig. 8, Table 1, Table 2).

The main result of our considerations is that on the basis of ChT we can check that the solution of parametrically excited system (beam) is stable in the whole interval of changing parameters without construction of the boundary of instability regions. We can change the geometrical parameters of the shape, e.g. $\overline{\alpha}, \overline{\kappa}$, in any interval so that the solution is stable. On the basis of ChT we can check that the solution of the parametrically excited system (beam) is stable in the whole interval of changing of parameters of shape, without construction of the boundary of instability regions.

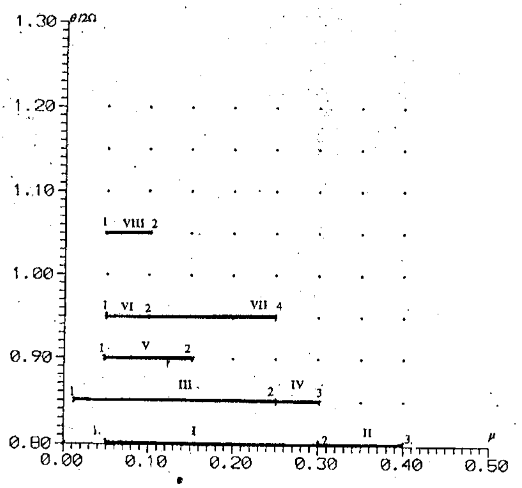

FIG. 8. Stable and unstable intervals of changing of parameters β_t, verified by ChT. Stability of four polynomials.

Table 1.

Interval	a_1	a_2	a_3	$\Theta/2\omega, \mu$	Polynomials (4.26)
I	$\alpha_1 = \beta_1$ $= 1.79 \cdot 10^5$	$\alpha_2 = 4.57 \cdot 10^9$ $\beta_2 = 4.65 \cdot 10^9$	$\alpha_3 = 2.63 \cdot 10^{13}$ $\beta_3 = 2.92 \cdot 10^{13}$	0.8, 0.05 – 0.3	$f_1(z)$ – stable $f_2(z)$ – stable $f_3(z)$ – stable $f_4(z)$ – stable
II	$\alpha_1 = \beta_1$ $= 1.79 \cdot 10^5$	$\alpha_2 = 4.57 \cdot 10^9$ $\beta_2 = 4.49 \cdot 10^9$	$\alpha_3 = 2.63 \cdot 10^{13}$ $\beta_3 = 3.16 \cdot 10^{13}$	0.8 0.05 – 0.4	$f_1(z)$ – unstable $f_2(z)$ – stable $f_3(z)$ – stable $f_4(z)$ – stable
III	$\alpha_1 = \beta_1$ $1.94 \cdot 10^5$	$\alpha_2 = 5.47 \cdot 10^9$ $\beta_2 = 5.53 \cdot 10^9$	$\alpha_3 = 3.84 \cdot 10^{13}$ $\beta_3 = 4.01 \cdot 10^{13}$	0.85, 0.01 – 0.25	$f_1(z)$ – stable $f_2(z)$ – stable $f_3(z)$ – stable $f_4(z)$ – stable
IV	$\alpha_1 = \beta_1$ $1.94 \cdot 10^5$	$\alpha_2 = 5.44 \cdot 10^9$ $\beta_2 = 5.53 \cdot 10^9$	$\alpha_3 = 3.84 \cdot 10^{13}$ $\beta_3 = 4.20 \cdot 10^{13}$	0.85, 0.01 – 0.30	$f_1(z)$ – stable $f_2(z)$ – stable $f_3(z)$ – stable $f_4(z)$ – unstable
V	$\alpha_1 =$ $2.09 \cdot 10^5$ $\beta_1 =$ $2.10 \cdot 10^5$	$\alpha_2 = 6.54 \cdot 10^9$ $\beta_2 = 6.58 \cdot 10^9$	$\alpha_3 = 5.41 \cdot 10^{13}$ $\beta_3 = 5.50 \cdot 10^{13}$	0.90 0.05 – 0.15	$f_1(z)$ – stable $f_2(z)$ – stable $f_3(z)$ – stable $f_4(z)$ – stable
VII	$\alpha_1 =$ $2.25 \cdot 10^5$ $\beta_1 =$ $2.10 \cdot 10^5$	$\alpha_2 = 7.75 \cdot 10^9$ $\beta_2 = 7.80 \cdot 10^9$	$\alpha_3 = 7.38 \cdot 10^{13}$ $\beta_3 = 7.67 \cdot 10^{13}$	0.95 0.05 – 0.25	$f_1(z)$ – ustable $f_2(z)$ – ustable $f_3(z)$ – ustable $f_4(z)$ – stable

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