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SENSITIVITY ANALYSIS OF THE BAR STRUCTURES RELIABILITY

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In the paper certain theoretical basis and practical approach for design sensitivity analysis of bar structure's reliability under a special type of excitation are presented. The analysis is carried out for the plane bar structures made of linearly elastic material. It is assumed that the structure's physical and geometrical parameters are deterministic variables and a non-Gaussian stochastic process describes the load. Examples for a beam loaded by a stream of moving random forces are presented.

Notations

A_k	the force amplitudes,
b	some design parameter,
c	the damping coefficient,
D_w	the standard deviation of the system response w,
e_b	the dimensionless measure of sensitivity,
$E[\cdot]$	means the expected value of the variable within the brackets,
E	the Young modulus,
f_c	the central factor of safety,
f_R	the probability density function of the capacity,
f_S	the probability density function of the load,
F_R	the cumulative distribution function of the capacity,
F_S	the cumulative distribution function of the load,
$H(x,t-\tau,b)$	the dynamic influence function,
I	the moment of inertia,
m	the mass density,
m_0	the second-order cumulant of structure's response,
m_2	the second-order cumulant of structure's response velocity,

m_4	the second-order cumulant of structure's response acceleration,
$N(0,t)$	a Poisson stochastic process,
$p(x)$	some deterministic function,
$p_e(\xi)$	the probability density function of the extreme peaks,
p_f	the probability of failure,
p_{fu}	the upper bound of the probability of failure,
$p_M(\eta)$	describes a probability density function of single maxima,
p_r	the probability of not failure,
$p_X(\eta)$	a probability density function of the stochastic process X,
R	structure's capacity (limit strength, limit displacements, ...),
S	the load (stresses, displacements, ...),
t_k	the random time points,
v	the velocity of moving force,
$w(x,t)$	the displacement of the structure,
$\dot{w}(x,t,b)$	structure's response velocity,
$\ddot{w}(x,t,b)$	response acceleration,
$W_n(x)$	the normal mode of the system,
α_b	the sensitivity measure,
β	the reliability index,
$\delta(\cdot)$	means the Dirac delta function,
η_R	the coefficient of variation of the capacity R,
η_S	the coefficient of variation of the load S,
$\kappa_w^k(t)$	the k-th order cumulants of structure's response,
$\kappa_{\dot{w}}^k(t)$	the k-th order cumulants of structure's response velocity,
$\kappa_{\ddot{w}}^k(t)$	the k-th order cumulants of structure's response acceleration,
λ	a parameter of Poisson stochastic process,
μ_0	the frequency of maxima,
v	the average frequency of the maxima in the time interval of length T,
$\sigma(x,t)$	a stress,
$\varphi_\gamma(x,t,b)$	the parameter of the objective function that defines the reliability measure,
$\psi(b,\varphi_\gamma(x,t,b))$	some objective function,
ω_n	the radian natural frequency of the undamped structure,

1. Introduction

Because of large uncertainties in reliability-based design of engineering structures, especially of structures loaded dynamically, the sensitivity analysis of the structure reliability can be useful for engineering decision-making proceses. It is very important to know how the reliability of bridges, offshore platforms, towers, chimneys and other structures changes due to variations of design parameters, and the passage of time. As the design variables, the material and element properties are assumed.

The objective of the present study is to give a theoretical basis and practical approach for design sensitivity analysis of bar structure's reliability under a special type of excitation. The analysis is carried out for the plane bar structures made of linearly elastic material. It is assumed that the structure's physical and geometrical parameters are deterministic variables and a non-Gaussian stochastic process describes the load. Some results for a beam loaded by a stream of moving random forces are presented.

2. Reliability and sensitivity analysis – general formulation

Reliability analysis is a complex process that comprises several stages, beginning with collecting data about probabilistic characteristics of the material and structure capacity, calculating the probabilistic characteristics of the structure response, choosing the reliability measure, identifying the reliability system and ending with calculation of the reliability. Usually, as the reliability measures, the probability that within a time interval $[0, t]$ the capacity of the structure is greater than the load (i.e. that the structure will be still within the safe domain), is assumed.

$$p_r = 1 - p_f = P\{R - S > 0\} = \int_{-\infty}^{\infty} f_R(x) F_S(x) dx \tag{2.1}$$

$$= 1 - \int_{-\infty}^{\infty} f_S(x) F_R(x) dx,$$

where p_f is the probability of failure, R is the structure's capacity (limit strength, limit displacements, ...), S is the load (stresses, displacements, ...), f_R, F_S are the probability density functions of the capacity and the cumulative distribution function of the load, respectively. In the FORM (First Order Reliability Method) and SORM (Second Order Reliability Method) methods, the reliability is measured by the reliability index β, which corresponds to the probability of failure p_f. From the reliability point of view, the systems are divided into serial, parallel and mixed ones.

In some cases it is possible to calculate only the two first probabilistic moments of system's response. Therefore, for the practical structural reliability analysis, M. Ichikawa proposed the following formula for the upper bound of the probability of failure based on mean values of variances of strength R and stress S:

$$p_{fu} = (4/9)(f_c^2\eta_R^2 + \eta_S^2)/(f_c - 1)^2, \tag{2.2}$$

where f_c is the central factor of safety, η_R and η_S are the coefficients of variation of R and S, respectively.

After choosing the reliability measure, its sensitivity can be also calculated. The sensitivity problem is formulated here as the investigation how some objective function $\psi(b, \varphi_\gamma(x,t,b))$ changes due to the change of a given design parameter b. The function $\varphi_\gamma(x,t,b)$ which is the parameter of the objective function, defines the reliability measure (for example the probability of non-failure, Eq. (2.1), the combination of the probabilistic characteristics of the system response and load applying Eq. (2.2), etc.).

The sensitivity measures are [1]

$$\alpha_b = \frac{d\psi(b,\varphi_\gamma(x,t,b))}{db} = \sum_{\gamma=1}^{n} \frac{\partial\psi(b,\varphi_\gamma(x,t,b))}{\partial\varphi_\gamma(x,t,b)} \frac{\partial\varphi_\gamma(x,t,b)}{\partial b} + \frac{\partial\psi(b,\varphi_\gamma(x,t,b))}{\partial b} \tag{2.3}$$

and (the dimensionless measure)

$$e_b = \frac{d\ln(\psi(b,\varphi_\gamma(x,t,b)))}{d\ln(b)} = \frac{b}{\psi(b,\varphi_\gamma(x,t,b))} \frac{\partial\psi(b,\varphi_\gamma(x,t,b))}{\partial b} + \sum_{\gamma=1}^{n} \frac{\partial\psi(b,\varphi_\gamma(x,t,b))}{\partial\varphi_\gamma(x,t,b)} \frac{\partial\varphi_\gamma(x,t,b)}{\partial b}. \tag{2.4}$$

As it follows from the expressions (2.3) and (2.4), for calculating the sensitivity of some further defined objective function $\psi(b, \varphi_\gamma(x,t,b))$ we must calculate the derivatives $\dfrac{\partial\psi(b,\varphi_\gamma(x,t,b))}{\partial b}$, $\dfrac{\partial\psi(b,\varphi_\gamma(x,t,b))}{\partial\varphi_\gamma(x,t,b)}$ as well as the derivatives $\dfrac{\partial\varphi_\gamma(x,t,b)}{\partial b}$. The two first types of derivatives can be calculated through direct differentiation, but the third type of derivatives, which describe the design sensitivities of the probabilistic characteristics of the structure response, is not easy to calculate.

3. Formulation of the particular problem

Let us consider vibrations of a linearly elastic beam due to a load modelled by a random stream of point forces moving with the same constant speed along the beam.

The equation of motion under such excitation in the domain $[0, L]$ of x is as follows:

$$EJ(b)\frac{\partial^4 w(x,t,b)}{\partial x^4} + c(b)\frac{\partial w(x,t,b)}{\partial t} + m(b)\frac{\partial^2 w(x,t,b)}{\partial t^2} = p(x)\sum_{k=1}^{N(0,t)} A_k \delta[x - \nu(t - t_k],\tag{3.1}$$

where $w(x,t)$ is the displacement of the structure, EI is the bending stiffness of the beam, m is the mass density and c is the damping coefficient, $p(x)$ is some deterministic function, A_k are mutually independent random variables, and t_k are the random time points, which constitute a Poisson stochastic process $N(0,t)$ with parameter λ; the symbol $\delta(\cdot)$ denotes the Dirac delta function, and b is some design parameter.

The solution of the equation of motion (3.1) can be presented in the form of the STIELTJES integral [4]

$$w(x,t,b) = \int_0^t A(\tau)H(x, t-\tau, b)dN(\tau).\tag{3.2}$$

The dynamic influence function $H(x, t-\tau, b)$ is a solution of the equation

$$EJ(b)\frac{\partial^4 H(x,t-\tau,b)}{\partial x^4} + c(b)\frac{\partial H(x,t-\tau,b)}{\partial t} + m(b)\frac{\partial^2 H(x,t-\tau,b)}{\partial t^2} = \delta(x - \nu(t-\tau)).\tag{3.3}$$

Using the normal mode approach, we can present the dynamic influence function $H(x, t-\tau, b)$ in the form

$$H(x,t-\tau,b) = \sum_{n=1}^{\infty} r_n(t-\tau,b)W_n(x),\tag{3.4}$$

where

$$\Omega_n^2(b) = \omega_n^2(b) - \alpha(b), \quad \alpha(b) = \frac{c(b)}{m(b)}, \quad p_n(b) = \frac{1}{m_n(b)}\int_0^L W_n^2(x)dx,$$

$$r_n(t-\tau,b) = \frac{p_n(b)}{\Omega_n(b)}e^{-\alpha(b)(t-\tau)}\sin\Omega_n(b)(t-\tau), \quad m_n(b) = m(b)\int_0^L W_n^2(x)dx,$$

ω_n is the radian natural frequency of the undamped structure, $W_n(x)$ is the normal mode of the system.

The k-th order cumulants of the structure's response $w(x,t,b)$ can be calculated from the relationships

$$\kappa_w^k(t) = E\left[A^k\right] \int_0^t H^k(x, t-\tau, b)\lambda(\tau)d\tau. \tag{3.5}$$

By analogy, the k-th order cumulants of the structure's response velocity $\dot{w}(x,t,b)$ and acceleration $\ddot{w}(x,t,b)$ can be calculated from the relationships

$$\kappa_{\dot{w}}^k(t) = E[A^k] \int_0^t \dot{H}^k(x, t-\tau, b)\lambda(\tau)d\tau, \tag{3.6}$$

$$\kappa_{\ddot{w}}^k(t) = E[A^k] \int_0^t \ddot{H}^k(x, t-\tau, b)\lambda(\tau)d\tau. \tag{3.7}$$

Let us recall that: $\kappa_w^2(t) = D_w^2 = m_0$, $\kappa_{\dot{w}}^2(t) = D_{\dot{w}}^2 = m_2$, $\kappa_{\ddot{w}}^2(t) = D_{\ddot{w}}^2 = m_4$. In the approach presented we obtain the mean value, the variance function and other probabilistic characteristics of the output, but in practical engineering applications we need the maximum response or the probability that the maximum response does not exceed a certain precarious limit value.

4. Distribution of extreme peaks of the structure's response

We assume for the structure's response (stresses $\sigma(x,t)$ or displacements $w(x,t)$), which in general is described by some stochastic processes, the following extreme peak distribution [2]

$$p_E(\xi) = -e^{-\psi}\frac{d\Psi}{d\xi} \tag{4.1}$$

where

$$\Psi = vT\exp\left(-\frac{1}{2}\xi^2\right), \quad \frac{d\Psi}{d\xi} = -vT\xi\exp\left[-\frac{1}{2}\xi^2\right], \quad \xi = \frac{\sigma(x,t,b) - E[\sigma(x,t,b)]}{D_\sigma}$$

or

$$\xi = \frac{w(x,t,b) - E[w(x,t,b)]}{D_w}, \quad v = \mu_0\sqrt{1-\varepsilon^2}$$

is the average frequency of the maxima in the time interval of length T, $\mu_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{m_4}{m_2}}$ is the frequency of maxima, $\varepsilon^2 = \dfrac{m_0 m_4 - m_2^2}{m_0 m_4}$. Various extreme peak distributions for different values of vT are shown in Fig. 1.

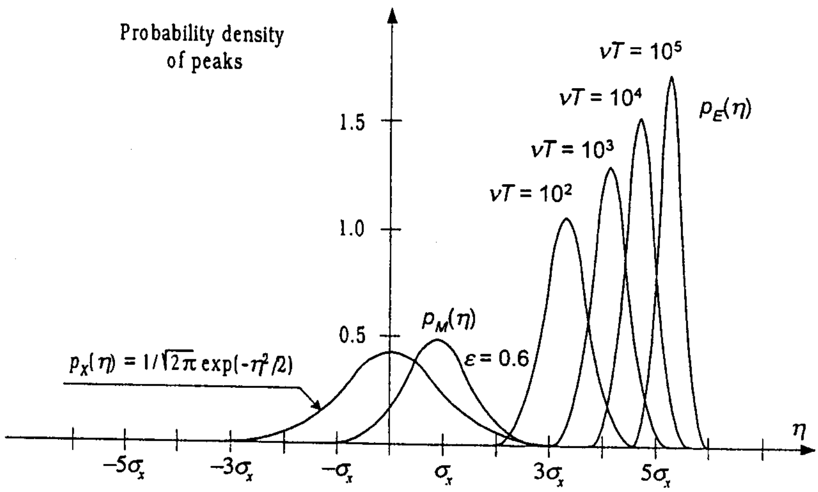

FIG. 1. Extreme peak distributions [2].

In the above figure $p_x(\eta)$ is a probability density function of the stochastic process X, $p_M(\eta)$ describes a probability density function of single maxima; it is assumed that all of the maxima observed within the time interval T have the some probability density functions.

5. EXAMPLES

To illustrate the approach presented, a bridge beam loaded by a traffic flow is considered. It is assumed that the beam is simply supported and the traffic flow is modelled by a stream of point forces moving at the some constant velocity, but with random amplitudes. The force arrival time points constitute a Poisson stochastic process with parameter λ.

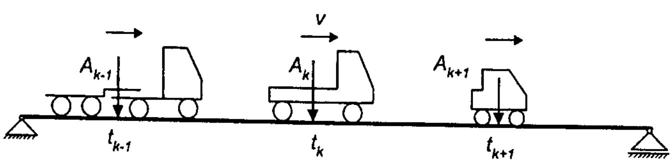

FIG. 2. A model of a bridge beam and load by traffic flow.

The sensitivity Eq. (2.3) of the beam's reliability described by Eq. (2.1) due to the change of the observation time T and the expected values of the stresses and displacements have been calculated. For calculating the reliability measure (Eq. (2.1)), the probability density function f_S of the extreme peaks of stresses as shown in Eq. (4.1) and the normal distribution F_R for the bearing capacity for the ultimate limit state have been assumed. For calculating the reliability measure (Eq. (2.1)) the probability density function f_S of the extreme peaks of displacements as shown in Eq. (4.1) and the uniform distribution F_R for the limit displacement for the serviceability limit state have been assumed. The cumulants of the displacements and stresses have been calculated using the formulae (3.6) and (3.7). More details about calculating the probabilistic characteristics of the bridge's beam response due to the traffic flow are given in [4]. All calculations have been made using the Mathematica software system.

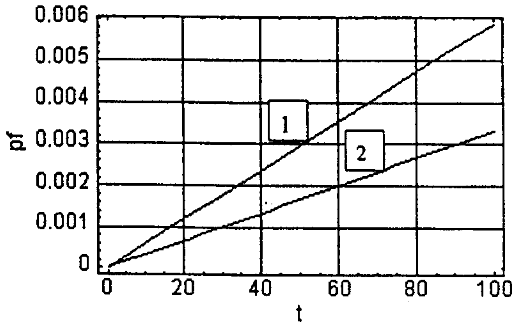

FIG. 3. The probability of failure vs. the time of observation (years), the curve "1" is calculated for the ultimate limit state, the curve "2" for the serviceability limit state.

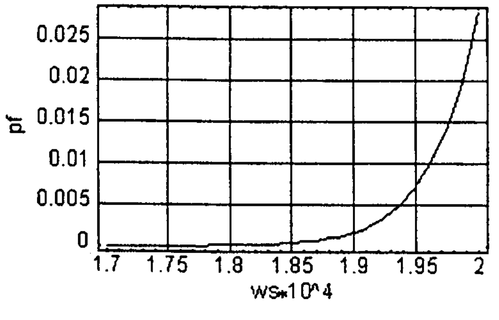

FIG. 4. The probability of failure calculated for the serviceability limit state vs. the expected value of the displacement.

The results are shown in Figs. 3 – 7. The Figures 3 and 4 show the values of the probability of failure as a function of time and as a function of the mean value of the displacement, respectively. Figures 5 and 6 show the values of the sensitivity of the reliability as a function of time and as a function of the mean value of displacement. Figure 7 shows the sensitivity of the probability of failure calculated for the stresses due to the time of observation. The sensitivity of the reliability versus time observation calculated for stresses has the same shape as that calculated for the displacements, only the values are smaller.

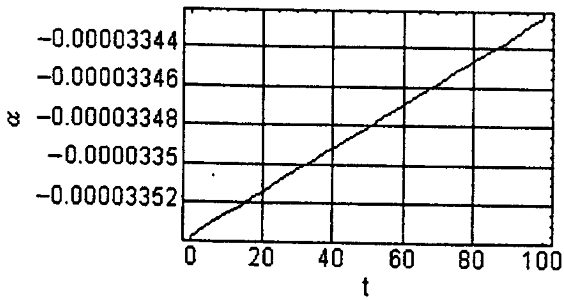

FIG. 5. The sensitivity of the reliability calculated for the serviceability limit state due to the time of observation.

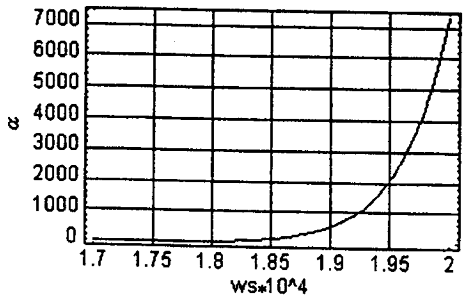

FIG. 6. The sensitivity of the reliability calculated for the serviceability limit state due to the expected value of the displacement.

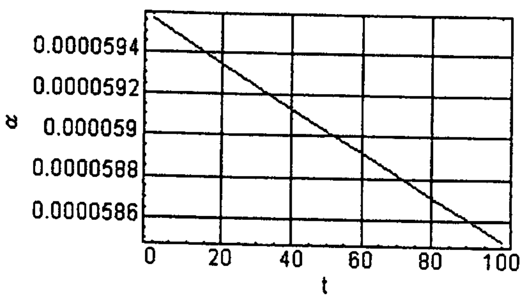

FIG. 7. The sensitivity of the probability of failure calculated for the ultimate limit state due to the time of observation.

6. Concluding remarks

• An approach for sensitivity analysis of the reliability of bar structures loaded by a special type of excitation, which is described by a non-Gaussian stochastic process, has been presented.

The extreme peak distribution for the structure's response as introduced by SOLNES [2] has been assumed. This assumption included in the reliability analysis seems to be interesting from the engineering point of view.

• As expected, from calculations made it follows that the reliability is very sensitive to the expected value of beam's response and very little sensitive to the time of observation.

• It is to be emphasised that the results are strongly dependent on the quality of data and the accuracy of load and structure models that are used in the sensitivity analysis.

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