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THE FULL SYSTEMS METHOD IN DYNAMICS PROBLEMS OF 3D BODIES

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A new method is proposed to solve the problems of stationary dynamics for inhomogeneous anisotropic 3D bodies of finite sizes with arbitrary conditions on bounding surfaces. It is the reduction of the initial three-dimensional boundary problem to the system of three correlated one-dimensional boundary-value problems. Thus the increase of the number of independent variables results in the linear (but not exponential!) increase of the required computer resources. This determines the method efficiency when solving multidimensional problems.
Several examples of solution for particular problems of mechanics of deformed bodies are presented.

1. The class of the problems

The paper deals with the problems of stationary dynamics of deformed three-dimensional bodies of finite sizes. The term ≪ problems of stationary dynamics≫ means linear problems of stressed-strained state and natural vibrations of elastic bodies. The mathematical model of the problems can be constructed both in variational and differential form. Here the presentation is limited to the boundary problems for inhomogeneous anisotropic 3D bodies written in the form of Lamé equations:

$$(L - \lambda B)\mathbf{U} = \mathbf{q}, \qquad (\alpha, \beta, \gamma) \in \Omega \tag{1.1}$$

$$R\mathbf{U} = \mathbf{f}, \qquad (\alpha, \beta, \gamma) \in \Omega^{+} \tag{1.2}$$

in the domain $\Omega + \Omega^{+}$ preset by curvilinear orthogonal parallelepiped

$$\Omega \cup \Omega^+ = \{\alpha,\beta,\gamma : \alpha \in [\alpha_0,\alpha_1]\,,\ \beta \in [\beta_0,\beta_1]\,,\ \gamma \in [\gamma_0,\gamma_1]\,;$$
$$\alpha_p = \text{const},\ \beta_p = \text{const},\ \gamma_p = \text{const},\ p=0,\ p=1\}\,.$$

Here $\mathbf{U} = \{\mathbf{u}_\alpha, \mathbf{u}_\beta, \mathbf{u}_\gamma\}$ is the vector of displacement of the body; L, R – differential matrices which are determined by fundamental relations of the theory of elasticity of anisotropic body [6]; B – matrix which defines the distribution of the density of material; $\mathbf{q}, \mathbf{f}$ – the given loads in the domain Ω and on the boundary Ω^+. In case of kinematic boundary conditions the elements of the matrix R are algebraic expressions.

On the boundary surface of the body any physically consistent conditions can be given, spatial distribution of loads is arbitrary, and the material has the lowest order of symmetry of the linear-elastic properties.

The equations (1.1) – (1.2) are a model of the following stationary problems:

• a problem of induced harmonic vibrations with the given frequency of external excitation ($B \neq 0,\ \mathbf{q} \neq 0,\ \mathbf{f} \neq 0,\ \lambda = \omega^2$);

• a problem of free vibrations with unknown natural frequencies ($B \neq 0,\ \mathbf{q} = 0,\ \mathbf{f} = 0,\ \lambda = \omega^2$);

• a problem of statical stressed-strained state ($B = 0,\ \mathbf{q} \neq 0,\ \mathbf{f} \neq 0,\ \lambda = 0$).

The projective methods are often used to solve both the general problem and its simpler 1D and 2D variants. Such a choice of methods is theoretically justified. But sometimes the use of the above methods is not a success. Such a situation can take place in the case of complicated 3D problems under unsuccessful choice of the system of basis functions of many variables and/or the approximation form of the unknown solution because there are no formal rules for their optimal choice. In this sense, the modification of the known methods seems to be expedient.

2. The fundamental points of the method

The method proposed called the Full Systems Method (FSM) is aimed at numerical solution of multidimensional boundary-value problems of mathematical physics. It can be considered as the extension of a class of projective methods. Therefore, one should point to its connection with the well-known projective methods and to its difference from these methods.

Let us remind that in the classical Ritz and Bubnov-Galyerkin methods for the solution of N – dimensional problems, the approximation was taken in the following form:

$$u(x_1, x_2, ..., x_N \cong F_M \equiv \sum_{i=1}^{M} \alpha_i \varphi_i(x_1, x_2, ..., x_N), \tag{2.1}$$

where α_i are unknown numerical coefficients, $\varphi_i(x_1, x_2, ..., x_N)$ $(i = \overline{1,N})$ are basis functions of N variables. The specific choice of these functions depends on individual experience of the researcher, while coefficients α_i are determined from the solution of the corresponding system of the linear algebraic equations. The required solution accuracy can be achieved with the corresponding increase of the number of terms M in the approximation (2.1).

The generalization of the approximation (2.1) has resulted in the methods of Kantorovich-Vlasov type. Here the function sought for is represented by the following expression:

$$u(x_1, x_2, ..., x_N) \cong F_M \equiv \sum_{i=1}^{M} X_{ni}(x_n)\varphi_i(x_1, x_2, ..., x_N), \tag{2.2}$$

where the functions of one arbitrary variable (x_n here) are unknown and the basis functions ($\varphi_i = \varphi_i(x_1, ..., x_{n-1}, x_{n+1}, ..., x_N)$) can be independent of this variable. The unknown functional coefficients determined from the solution of the corresponding 1D boundary-value problem appear in these method instead of the constant coefficients of formula (2.1). The methods of Kantorovich-Vlasov type allow one to obtain the more exact solution of the boundary-value problem than the Ritz type methods under the same number of terms M in (2.1) and (2.2).

The logical development of the projective methods in this direction is connected with the further modification of the form (2.2). The functions $X_{1i}(x_1, X_{2i}(x_2, ..., X_{n-1i}(x_{n-1}), X_{n+1i}(x_{n+1}), ..., X_{Ni}(x_N)$, which depend on the rest of the domain variables, are introduced additionally into (2.2) as the unknown ones. Such a procedure leads to the nonlinear approximation of the sought – for solution in the following form:

$$u(x_1, x_2, ..., x_N) \cong F_M \equiv \sum_{i=1}^{M} X_{1i}(x_1)X_{2i}(x_2)...X_{Ni}(x_N)\varphi_i(x_1, x_2, ..., x_N). \tag{2.3}$$

It is equivalent to the renunciation of using the basis in its usual sense, i.a. we may assume $\varphi_i \equiv 1$ $(i = \overline{1,M})$.

The relation (2.3) is the basic one for the FSM. Its fundamental points in conformity to the variational problems of mechanics of deformed bodies have been formulated in the paper [2]. Here the structure of the method under its adaptation for the problems of stationary dynamics of 3D bodies is presented in differential form.

Let us consider the 3D problem (1.1), (1.2) with respect to the scalar function:

$$(L - \lambda B)u = q, \qquad (\alpha, \beta, \gamma) \in \Omega \tag{2.4}$$

$$Ru = f, \qquad (\alpha, \beta, \gamma) \in \Omega^+. \tag{2.5}$$

The solution of any multidimensional problem by FMS consists of three main stages.

The first stage is connected with the choice of the concrete form of solution approximation. For the problem (2.4), (2.5), following (2.3) it has the form:

$$u(\alpha,\beta,\gamma) \cong F_M \equiv \sum_{i=1}^{M} X_i(\alpha)Y_i(\beta)Z_i(\gamma). \tag{2.6}$$

Here all the functions $X_i(\alpha), Y_i(\beta), Z_i(\gamma)$ which can be considered as the components of three vector-functions

$$\mathbf{X} = \left\{X_i(\alpha),\ i=\overline{1,M}\right\}, \quad \mathbf{Y} = \left\{Y_i(\beta),\ i=\overline{1,M}\right\}, \quad \mathbf{Z} = \left\{Z_i(\gamma),\ i=\overline{1,M}\right\}, \tag{2.7}$$

are the unknowns. The value M is determined with the required approximation accuracy.

On the second stage of the method the 3D problem (2.4), (2.5) is reduced to the system of 1D boundary-value problems with the help of the special procedure $P = \{P_\alpha, P_\beta, P_\gamma\}$. Each of the components $P_\alpha, P_\beta, P_\gamma$ is the operation of projecting which is usually used in the projective methods for reduction to ordinary differential equations. The component subscript means a variable of the 1D problem to which the problem (2.4), (2.5) is reduced. This operations may be as follows:

$$\begin{aligned}
P_\alpha(.) &= \left\{P_{\alpha k}(.) = \int_{\beta_0}^{\beta_1}\int_{\gamma_0}^{\gamma_1} (.)Y_k Z_k ds_\beta ds_\gamma, \quad k=\overline{1,M}\right\},\\
P_\beta(.) &= \left\{P_{\beta k}(.) = \int_{\alpha_0}^{\alpha_1}\int_{\gamma_0}^{\gamma_1} (.)X_k Z_k ds_\alpha ds_\gamma, \quad k=\overline{1,M}\right\},\\
P_\gamma(.) &= \left\{P_{\gamma k}(.) = \int_{\alpha_0}^{\alpha_1}\int_{\beta_0}^{\beta_1} (.)X_k Y_k ds_\alpha ds_\beta, \quad k=\overline{1,M}\right\},
\end{aligned} \tag{2.8}$$

where the symbol (.) means the object of transformation.

The operation of (2.8) being applied successively to the problems (2.4) – (2.5) with relations (2.6) and (2.7), we arrive to the system of 1D problems of the following structure:

$$(2.9)\quad \begin{aligned} P_\alpha\left[(L-\lambda B)F_M-q\right] &\equiv (L_\alpha-\lambda B_\alpha)\mathbf{X}-\mathbf{q}_\alpha = 0, \quad \alpha\in(\alpha_0,\alpha_1),\\ P_\alpha\left[RF_M-f\right] &\equiv R_{\alpha p}\mathbf{X}-\mathbf{f}_{\alpha p} = 0, \quad \alpha=\alpha_p(p=0,1), \end{aligned}$$

$$(2.10)\quad \begin{aligned} P_\beta\left[(L-\lambda B)F_M-q\right] &\equiv (L_\beta-\lambda B_\beta)\mathbf{Y}-\mathbf{q}_\beta = 0, \quad \beta\in(\beta_0,\beta_1),\\ P_\beta\left[RF_M-f\right] &\equiv R_{\beta p}\mathbf{Y}-\mathbf{f}_{\beta p} = 0, \quad \beta=\beta_p(p=0,1), \end{aligned}$$

$$(2.11)\quad \begin{aligned} P_\gamma\left[(L-\lambda B)F_M-q\right] &\equiv (L_\gamma-\lambda B_\gamma)\mathbf{Z}-\mathbf{q}_\gamma = 0, \quad \gamma\in(\gamma_0,\gamma_1),\\ P_\gamma\left[RF_M-f\right] &\equiv R_{\gamma p}\mathbf{Z}-\mathbf{f}_{\gamma p} = 0, \quad \gamma=\gamma_p(p=0,1). \end{aligned}$$

The system (2.9) – (2.11) consists of three 1D problems of the order 2M. Consider, for example, the problem (2.9). It has been formulated only with respect to the unknown vector-function **X**. The vector-functions **Y** and **Z**, which are transformed using the procedure P_α, are taken into account in the coefficients of operators L_α, B_α, $R_{\alpha p}$ and in the components of vectors $\mathbf{q}_\alpha$, $\mathbf{f}_\alpha$. The problems (2.10) and (2.11) under cyclical permutation of the vectors **X**, **Y**, **Z** and variables α, β, γ in (2.9) possess the analogous structure. Thus, each unknown vector-function (2.7) is sought for in one problem of the system (**X** – in (2.9), **Y** – in (2.10), **Z** – in (2.11)), in two other problems it is contained in the form of constants. Such is the correlation of all 1D problems in the system (2.9) – (2.11).

Solution of the system (2.9) – (2.11) is a final stage of FMS. The following iterative process is used for this purpose: each of the problems (2.9) – (2.11) is solved separately, independently of the other ones with respect to its unknown vector-function under the assumption that the functions of other variables are known after fulfilling the previous step of iteration:

$$(2.12)\quad \begin{aligned} (L_\alpha^{s-1}-\lambda B_\alpha^{s-1})\mathbf{X}-\mathbf{q}_\alpha^{s-1} &= 0, \quad \alpha\in(\alpha_0,\alpha_1),\\ R_{\alpha p}^{s-1}\mathbf{X}^s-\mathbf{f}_{\alpha p}^{s-1} &= 0, \quad \alpha=\alpha_p(p=0,1), \end{aligned}$$

$$(2.13)\quad \begin{aligned} (L_\beta^{s-1}-\lambda B_\beta^{s-1})\mathbf{Y}^s-\mathbf{q}_\beta^{s-1} &= 0, \quad \beta\in(\beta_0,\beta_1),\\ R_{\beta p}^{s-1}\mathbf{Y}^s-\mathbf{f}_{\beta p}^{s-1} &= 0, \quad \beta=\beta_p(p=0,1), \end{aligned}$$

$$(2.14)\quad \begin{aligned} (L_\gamma^{s}-\lambda B_\gamma^{s})\mathbf{Z}^s-\mathbf{q}_\gamma^{s} &= 0, \quad \gamma\in(\gamma_0,\gamma_1),\\ R_{\gamma p}^{s}\mathbf{Z}^s-\mathbf{f}_{\gamma p}^{s} &= 0, \quad \gamma=\gamma_p(p=0,1). \end{aligned}$$

Here $s = 1, 2..$ is the number of the step of iteration, and notations $p = 0$ or $p = 1$ are correlated to the initial or to the terminal points of the integration interval in each coordinate direction.

If the iterative process begins, for example, from solution of the problem (2.12), any system of linearly independent functions of one variable may be used for the initial approximation $\mathbf{Y}_i^{(0)}$, $\mathbf{Z}_i^{(0)}$ $(i = \overline{1,M})$. Solution of a separate 1D boundary-value problem is considered a priori to be an elementary operation which can be always implemented with the sufficient degree of accuracy. Therefore the choice of the method of the 1D problem solution is not discussed here.

The proof of convergence of the given iterative process (2.12) – (2.14) is based on the physical condition that the initial boundary-value problem is conservative.

The FSM testing on many problems of mechanics of a deformed body and analysis of its computational aspects in a number of rather complicated problems (e.g. [3]) allow the following conclusions to be made.

1. The convergence of iterative process (2.12) – (2.14) is rather fast (5-7 iterations) under a fixed number of terms M in the representation (2.6).

2. Practical solution accuracy is achieved under a small number of terms M in (2.6) ($M \leq 4$).

3. The required computer resources increase according to the linear law with the increase of independent variables of the domain.

The above FSM pattern may be applied to solution of the problem (1.1), (1.2), when the stressed-strained state of 3D bodies under statical and harmonic loads is analyzed. To solve the problem of free vibrations one can use the method of inverse iteration with construction of the Rayleigh ratio [5] and FSM to solve a sequence of intermediate auxiliary multidimensional problems.

3. Examples

Let us consider the following three examples. The examples 3.1 and 3.2 illustrate the peculiarity of FSM, and example 3.3 contains some results of the specific mechanical problem.

3.1. Bending of a square isotropic plate

This example helps to visualize the formation of two-dimensional solution in the course of the iterative process (2.12) – (2.14).

Here the mathematical model is presented by the differential equation

$$D\Delta\Delta w = -q(x,y),$$

with following boundary conditions:

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, \quad x = a,$$

$$w = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = 0, \quad y = a.$$

A special function $q(x, y)$ in the right-hand side of the equation is chosen in such a way that the exact analytical solution of the problem has the form:

$$w(x, y) = \sin^2 (x/a) \sin^{20} (y/a) .$$

Here D and w have an ordinary sense for the problem of plate bending, variables x and y are defined in a square $a \times a$.

The problem solution being sought at each step s of the iterative process is written in the form

$$w_s(x, y) = X^{s-1}(x) Y^{(s)}(y), \qquad s = 1, 2.$$

The initial approximation being taken as $X^{(0)}(x) \equiv 1$, we obtain from solution of the system (2.12) – (2.13) a sequence of functions $w_s(x, y)$ $(s = 1, 2...)$. All steps of the iterative process are illustrated in Fig. 1: 1a is a function of initial approximation $X^{(0)}(x) \equiv 1$, 1b – the form of the normalized function $w_1 = w_1(x, y)/w_{1\,\max}(x, y)$ after the first iteration, 1c – presents the same function $w_2 = w_2(x, y)/w_{2\,\max}(x, y)$ after the second iteration. It is evident that it practically coincides with the exact solution 1d.

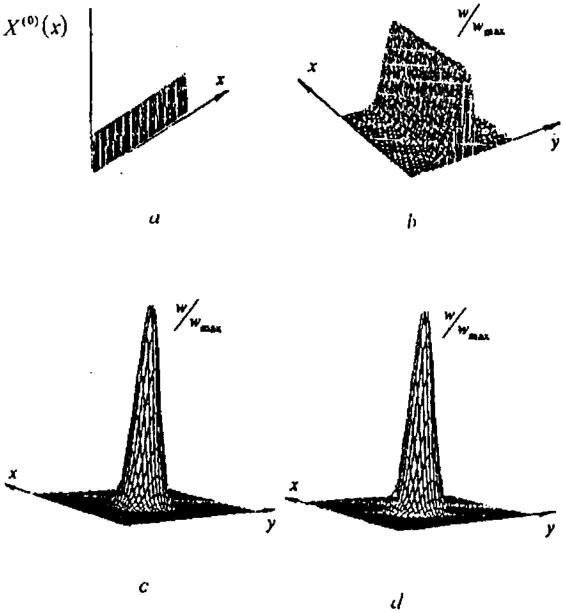

FIG. 1. The iterative process of determining the 2D solution: 1a – the form of initial approximation $X^{(0)}(x)$; 1b – the form of normalized surface $w_1 = w_1(x, y)/w_{1\,\max}(x, y)$, obtained after the first iteration; 1c – the function $w_2 = w_2(x, y)/w_{2\,\max}(x, y)$; 1d – exact solution.

3.2. Free lateral vibrations of a slope isotropic panel with local supports

The example is given to visualize the process of solution of a rather complicated problem depending on the number of approximation terms M in (2.6).

We consider a slope of cylindrical shape, quadratic in plan, panel with the side a, relative thickness $h/a = 0.01$ and relative radius of curvature $R/a = 3.14$. The panel is made fast along one curvilinear edge. Elastic supports localized in the square domain with the area 0.01 and relative rigidity $C/D = 10^{-3}$ m^{-4} are arranged in the vicinity the angular points between unfastened edges. The form of bending vibrations of the panel for minimum frequency depending on the number M in approximation (2.6) is shown in Fig. 2: $M = 1$ (Fig. 1a); $M = 2$ (Fig. 1b); $M = 3$ (Fig. 1c); $M = 4$ (Fig. 1d). The problem solutions coincide at $M = 3$ and $M = 4$.

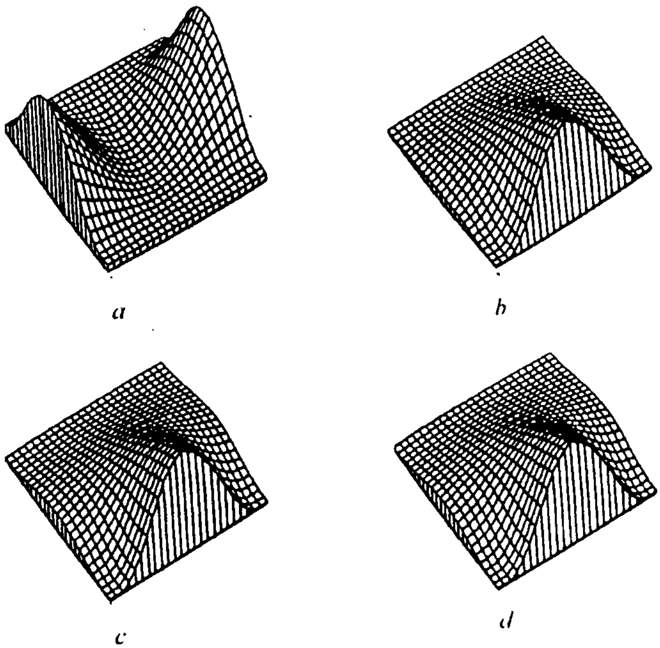

FIG. 2. The form of flexural vibrations of the panel depending on the number of approximation terms: 2a – one term ($M = 1$); 2b – two terms ($M = 2$); 2c – three terms ($M = 3$); 2d – four terms ($M = 4$).

3.3. Natural vibrations of an anisotropic plate

Some results of research of dynamic behaviour of the 3D body are presented. The rectangular parallelepiped with the base $L * L$ and thickness $h = L/10$ is the object of investigation. The mechanical properties of material, namely carbon fibre-reinforced plastic (CFRP), are given in the paper [1]. The fibres may be oriented at the angle θ to axes x, y in the plane. The results of calculation are given under the following boundary conditions: the sides $x = -L/2$, $y = -L/2$ are fixed and all the others are free of stresses. The θ – dependence for the first natural frequency is presented in Fig. 3a. The shape of deformed surface ($z = 0$) at $\theta = 45°$ is shown in Fig. 3b.

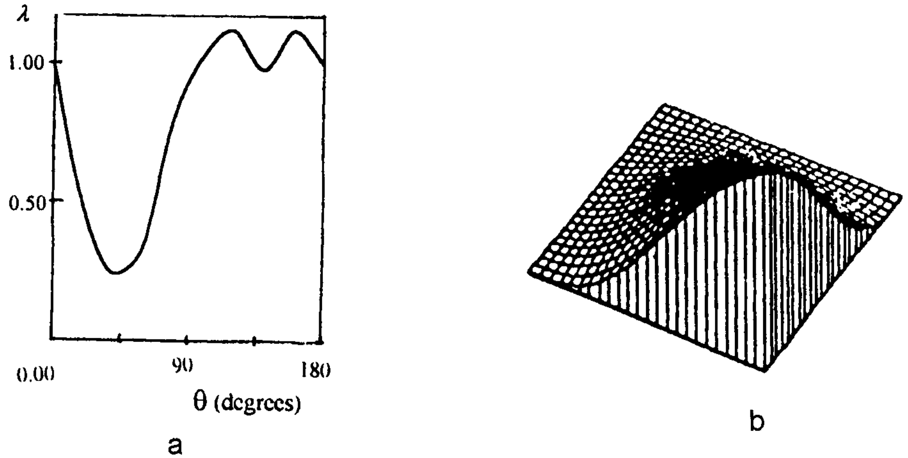

FIG. 3. Dependence of the frequency characteristic on angle θ: 3a – first natural frequency ($\lambda = \omega^2(\theta)/\omega^2(0)$) as a function of θ; 3b – the shape of vibrations surface $z = 0$ at $\theta = 45°$.

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