ENGINEERING TRANSACTIONS • *Engng. Trans.* • **49**, 2–3, 283–313, 2001
Polish Academy of Sciences • Institute of Fundamental Technological Research
10.24423/engtrans.561.2001

MODELLING ELECTRIC AND ELASTIC PROPERTIES OF CARTILAGE

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The aim of the paper is to propose a novel approach to modelling the macroscopic electromechanical behaviour of cartilage within the framework of linear response. The cartilage is treated as multiphase material with four constituents: anions, cations, viscous fluid and piezoelectric skeleton. The macroscopic equations were derived by using homogenization methods. Only stationary flow was studied. The elastic macroscopic moduli were determined by assuming, after BROOM [60], the honeycomb microstructure of the cartilage. Mathematical developments are preceded by a review of structure and properties of a cartilage.

1. INTRODUCTION

All cells and tissues of the organism are subjected to different stimuli, among others to mechanical forces. These forces arise from various reasons, such as blood circulation, inertial forces created during motion and gravity forces that act ceaselessly in normal conditions. The last two kinds of the forces are the most important for loading of the cartilage. It is easy to estimate that the pressure exerted on the cartilage during the walk is of the order of several atmosphers, and during the jump – of tens of atmospheres. The action of external force on the cartilage is twofold: on the one side, the moderate forces ensure proper functioning of the cartilage, on the other side, an overload degenerates the cartilage, as examples of sportsmen and race horses show, cf. [1 – 7].

In the last decades biomedical research has led to elucidation of the role of applied stresses on a physiological activity of the cartilage and its influence on its pathology, cf. [8, 9].

The aim of this contribution is to elaborate a general framework for modelling passive flows of electrolytes through the cartilage, treated as porous dielectric elastic media, and to provide a method of estimation of the properties of this material. The electrolytes involved consist of two-ion species. Similar approach can be applied to modelling the inervertebral disc.

The passive transport of electrolyte solution through a tissue is the process by which the solution moves under the influence of the gradients of electrochemical potentials. In such a process the electrolyte moves without expenditure of cellular metabolic energy. The passive transport processes play an important role in physiological functions of biological tissues, from cells to organs. The electrolytes-like water solutions of acids, bases or salts are electrically neutral in the bulk; they manifest, however, electric charge separation in the vicinity of contact surfaces where a double layer arises as a result of lowering the symmetry of the system. The appearance of charges leads to a series of elektrokinetic phenomena, especially important for metabolism in living tissues. Biological tissue as a whole is electroneutral under normal physiological conditions. Tissue is called to be charged when its matrix is charged. Then, in order to ensure the electrical neutrality, the interstitial fluid with dissolved electrolyte should be charged with charges of opposite sign.

A number of models of passive transport of electrolyte in charged porous biological tissues have been conceived. They are based on different variants of the mixture theory.

Here we propose a new method of description of the phenomenon. We start from more fundamental concepts, from elementary rules of motion of the components. By using the method of two-scale asymptotic expansions, the macroscopic phenomenological equations describing electrokinetics of such a (two + two) - phase structure are derived and the formulae for the effective mechanical and nonmechanical coefficients are given.

Porous materials, such as animal and human cartilage and bone, provide important classes of natural porous media, cf. [10 – 18]. From the mechanical point of view one considers deformable matrices.

The paper is organized as follows. Sections 1.1 – 1.2.6 are concerned with a presentation of cartilage structure. Sections 2 and 3 are devoted to the study of flow of electrolytes through a piezoelectric matrix. The piezoelectric effect is introduced because the dry collagen fibers exhibit piezoelectric properties and we may suppose that also the wet collagen network conserves this property in some, even to a small extent. Thus the matrix is a deformable anisotropic dielectric exhibiting piezoelectric properties. By using the method of two-scale asymptotic expansions, equations of electrokinetics involving the piezoelectric effect have been derived. In this part of considerations the present contribution is a contin-

uation of our previous papers on homogenization of piezoelectric composites and of flow of electrolytes through porous media, cf. [10, 12, 13, 15, 17, 19]. However, now we assume that we deal with three fluxes comprising the water flux, and fluxes of anions and cations which mutually interact. We also assume that the deformable matrix reveals piezoelectric properties.

In Secs. 4 and 5 the mechanical properties of the cartilage are examined. Particularly, a two-dimensional honeycomb model of the cartilage matrix is considered. The results obtained are compared with available experimental data.

Our considerations are confined to linear phenomena. The next step would consist in an elaboration of a nonlinear model. Particularly, a nonlinear response of the cartilage matrix combined with the homogenization is a challenging problem. Also, our results are restricted to the stationary flow. The study of non-stationary case would be similar, though technically more involved. Then the Darcy-Wiedemann law of filtration would be nonlocal in time, cf. [18 – 20].

1.1. The structure of cartilage

From the medical point of view the tissue of cartilage (cf. Lat. cartilago, Gr. $\chi o \nu \delta \rho o \varsigma$ belongs to one of the simplest connective tissues. The level of metabolism in it is low, it is known as a privileged tissue from immunological point of view. It is easily preserved in tissue banks, the articular cartilage is devoid of nerves and is generally avascular, although a few blood vessels may be found in its deepest zones. The tissue consists of a relatively small number of cells and an abundant extracellular matrix. The matrix contains a large amount of water, a meshwork of collagen fibres and a non-fibrous "filler" substance. Together these form a stiff gel (gristle). Nevertheless, the cartilage is very complicated as an object of physical or chemical investigation.

The most widespread and most characteristic form of cartilage is described as *hyaline*, because of its glassy translucent appearance (cf. Gr. $\upsilon \acute{\alpha} \lambda o s$ – transparent stone, amber, glas). It is built from the intercellular substance and of the cellular elements – chondrocytes which are found in spaces called lacunae.

The characteristics of cartilage, responsible for its function as a supporting tissue, are given by its interstitial substance. This has a dense network of collagen fibres, embedded in the form of a very firm gel, three-fourth of the weight of which is water. This structure gives to cartilage the consistency and a considerable tensile strength, and enables it to bear weight while retaining a certain degree of elasticity. In adult individuals, except in certain locations, as in the sternal portions of the ribs, cartilage disappears from the bony structure, leaving only a thin layer at the joint surfaces. Here it provides a smooth surface, lubricated by

the joint fluid (synovial fluid), which makes it possible for the bones to carry the weight of the body, while moving easily against one another at the joints.

In addition to hyaline cartilage, two other varieties of cartilage are found in mammals. *Elastic cartilage* is found in the external ear, in the larynx, while *fibrocartilage* is found in the intervertebral disks, in the attachments of tendons and in some other similar areas. In the healing of fractures, the provisional callus, before its replacement by bone, is also formed of fibrocartilage.

Some of the information concerning the multilevel structure and physical properties of articular cartilage are given in the following two tables.

Table 1. Multiscaled structure of articular cartilage, after [21]

scale	m	
nano-	$10^{-10} - 10^{-9}$	Na^+, Ca^{2+}, SO_3^-, COO^-
ultra-	$10^{-8} - 10^{-6}$	collagen and proteoglycan, pores
micro-	$10^{-7} - 10^{-4}$	cells and extracellular matrix
tissue-	$10^{-4} - 10^{-2}$	articular cartilage and bone
macro-	$0.05 - 0.15$	diarthroidal joints

Table 2. Cartilage composition, after [21].

tissue	water	collagen (wet wt)	proteoglycan (wet wt)
articular cartilage	60 – 85%	15 – 22%	5 – 7%

1.2. Physical and chemical properties

The gel-like portion of the interstitial substance of articular cartilage, the chondromucoid, is characterized by a complex sugarlike substance containing sulphur and known as chondroitin sulfate (which is responsible for its staining properties), cf. [2, 22].

The extracellular component of cartillage consists primarily of collagen fibres embedded in a gel of proteoglycans and water. Since cells occupy a small fraction of the total volume of cartilage, the physicochemical properties of cartilage are defined mainly by the properties of this gel. Whilst the collagen network is responsible for the integrity of the tissue and its tensile strength, both fluid and solute transport depend principally on the properties of the proteoglycans – water gel. Because the proteoglycan molecule contains fixed negatively charged groups (carboxylate and sulphate due to chondroitin and keratan sulphates), the gel possesses properties of a polyelectrolyte solution.

1.2.1. Collagen

Collagen accounts for about half of the dry weight (which is about one third of the wet weight) of articular cartilage. This amount increases to almost whole of the dry weight in the superficial layer.

Collagen fibers are embedded in the amorphous substance of the extracellular matrix, consisting predominantly of proteoglycans. All components of the matrix are produced by cartilaginous cells, i.e. chondrocytes.

At least 20 different types of collagen have now been identified. The major collagen of cartilage is type II. The type II collagen is different from other interstitial collagens. The most significant variation is the high level of glycosylation of the hydroxylysine residues. Almost half of the hydroxylysine residues are bound to galactose or galactosyloglucose. The biological role of this high carbohydrate content remains obscure. It is speculated that glycosylation facilitates the interaction with proteoglycans and affects the steric alignment of the molecules, cf. [23].

Collagen fibres form a relatively coarse structure. The fibrils themselves are $30 - 80$ nm ($300 - 800$Å) in diameter and the gaps between them are of the order 100 nm or more. At physiological pH the cartillage collagen behaves as if it contained no unneutralized charged groups.

Collagen has a high degree of structural organization. The basic tropocollagen structural unit is composed of three collagen polypeptide chains (α-chains) coiled into left-handed helices which are further coiled about each other into right-handed helices. These rod-like tropocollagen collagen molecules (aproximately 1.4 nm in diameter and 300 nm long) are polymerized into large collagen fibrils. Without covalent intermolecular cross-links, collagen fibrils have drastically reduced mechanical strength. Both the coarseness of the fibres and their electrical neutrality render collagen neutral from the physicochemical point of view, and therefore of less interest than the proteoglycan constituent.

The collagen fibres in the intermediate zone are mostly randomly oriented in relation to the vertical direction, cf. Figs. 4, 5 and 6. Under physiological compressive load the fibres have become oriented at right angles to the direction of loading, cf. [24].

The attempts to determine the mechanical properties of collagen molecule were performed by several groups of researches, using the X-ray diffraction technique and Brillouin light scattering, beginning from COWAN *et al.* [25], until the last experiments by SASAKI and ODAJIMA [26].

1.2.2. Water

Although all the water in cartilage is accessible to small solutes and participates in the isotopic exchange, some of this water is associated with collagen

fibrils. As far as the cartilage is concerned, it is not known however, what fraction of water is present in the intrafibrillar form or whether this water is available to the proteoglycan molecules.

From hydraulic permeability experiments, the pore size in articular cartilage has been estimated to range from 2 to 6 nm. Such small pores provide an efective barrier against transport of large molecules through the tissue. Thus frictional interaction between interstitial water and the walls of the nano-sized pores of the solid matrix occurs within the nano-scale range.

It is estimated that a small percentage of the water is contained in the intracellular compartment, about 30% exists in the intrafibrillar compartment of the collagen fibers (and a small fraction of this water – 20% is localized within the helix itself), and the remainder exists in the solution domain of proteoglycan molecules. That rest is more or less equally distributed between pores (every one of which holds about 5 water molecules) and holes (every one of which holds about 50 water molecules). Thus most of the collagen water shows limited accessibility to larger molecules but is freely available to small solutes.

1.2.3. Proteoglycans

The ground substance forms the "filler" matrix between the fibrous component of the matrix; it contains a wide variety of chemical constituents, but is principally composed of protein-polysaccharide complexes. The filler substance endows the cartilage with its elasticity in compression and is made up of large molecules of carbohydrates and protein, cf. Fig. 1.

The cartilage contains polysaccharide molecules which have a considerable negative charge and a molecular weight up to 5×10^4. These are termed acid mucopolysaccharides or glycosaminoglycans; chondroitin sulphate (CS), keratan sulphate (KS) and a small amount of hyaluronic acid are found in cartilage. In cartilage, the SO_4 and COOH groups on the CS and KS chains become ionized. The distance between these charged groups is of the order of 1nm and is within the Debye length. The total number of these charge groups is measured in terms of "fixed charge density" (FCD). The value of the FCD in articular cartilage ranges from 0.05 to 0.3 mEq/ml of tissue, where 1Eq≡1 chemical equivalent. The FCD in cartilage determines the total counter- ion (mainly Na^+) concentration within the interstitium *via* the Donnan equilibrium, see Sec. 1.2.5.

Chondroitin sulphate and keratan sulphate chains are covalently bound as side-chains to a central core of protein. These protein-polysaccharides (known as proteoglycans) have a molecular weight of about $1 - 4 \times 10^6$. Some proteoglycan molecules aggregate to form very large complexes.

Altough present in rather small quantity (10% of wet weight), the proteoglycans are just this constituent which is responsible for a number of important

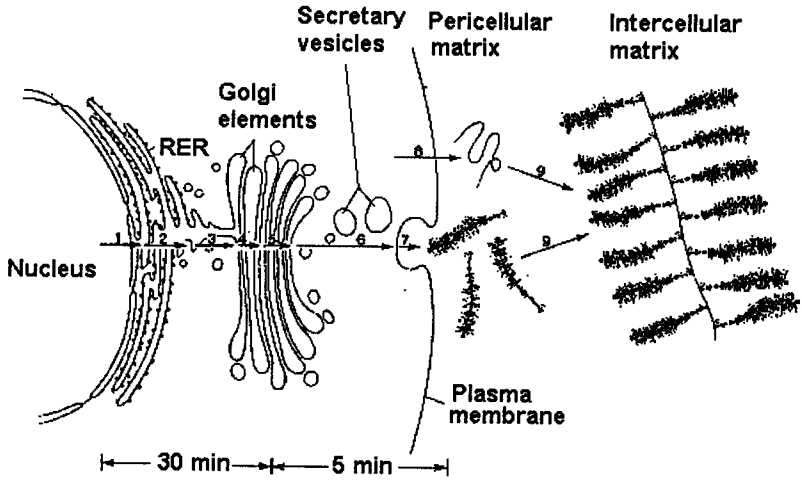

FIG. 1. A schematic representation of a series of metabolic events controlling the proteoglycans in cartilage. The chondrocytes synthesize and secrete proteoglycan, link protein and hyaluronan, which become incorporated into the matrix as functional aggregates. Enzymes released by the cells break down these aggregates into fragments, which are released from the matrix into the synovial fluid. The fragments are then taken up by the lymphatics and moved to the circulating blood, after [22].

properties of cartilage. The flexible, hydrophilic nature of the glycosaminoglycan chains and their high concentration of negatively charged fixed groups, lead to a high *swelling pressure* whilst the fine macromolecular mesh ensures a low hydraulic permeability. These two characteristics allied to the high water content of cartilage combine to make the tissue suitable as a load- bearing material with a low coefficient of friction.

There is evidence that isolated glycosaminoglycans such as chondroitin and dermatan sulphates do interact with collagen although hyaluronic and keratan sulphate do not.

Because of the quantitative agreement between the total glycosaminoglycan (GAG) content (determined by chemical analysis) and that corresponding to the concentration of negatively charged fixed groups (as obtained by physical methods), it was posible to adopt the latter parameter as a direct measure of the GAG content.

1.2.4. Chondrocytes

Up to 10 percent of volume concentration in cartilage is occupied by the cells called chondrocytes, cf. [3]. These are essential for the physiology of joint and for the development, existence and renewal of the cartilage matrix. Mechanical load is an important factor influencing the chondrocyte metabolism, cf. [27].

A histological photomicrograph section of normal rabbit articular cartilage is shown in Fig. 2. The typical layered arrangenent of chondrocyte cells, with a narrow band of oblong chondrocytes near the articular surface is observed. A similar layered structure is observed for collagen fibres distribution, cf. Fig. 5.

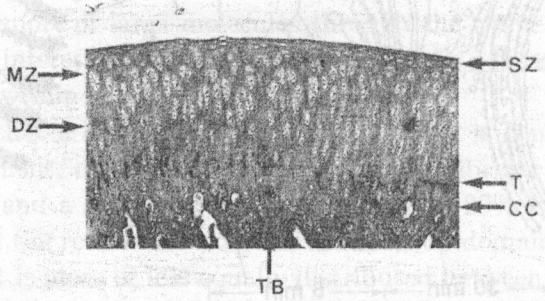

FIG. 2. Histological section of normal rabbit articular cartilage stained with Massons trichrome, ×370. The surface of the tissue is smooth. The cells in the surface zone (SZ) have a flattened shape. In the middle zone (MZ) the cells are larger and ovoid in shape. In the deeper zone (DZ) the cells are built in columns; CC – calcified cartilage, TB – trabecular bone; T – the tidemark (demarcation) between the calcified cartilage and uncalcified cartilage; after [28].

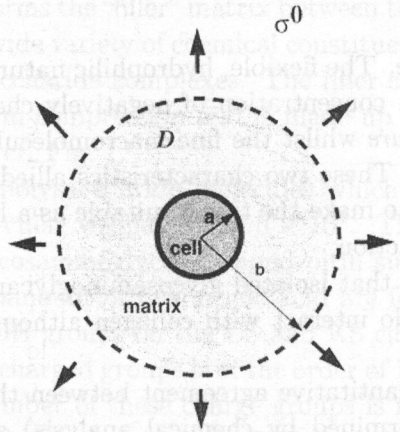

FIG. 3. Chondrocyte embedded in the matrix, after [33].

The elastic modulus of these cells is smaller by three orders and its permeability is greater by five orders than those of the extracellular matrix, cf. [29 – 32].

Because of such essential difference in elasticities and permeabilities between chondrocytes and the surrounding extracellular matrix, the deformation of cells is different from this of the surrounding matrix. In the primary investigations, the cell – matrix interaction was simulated by an isolated cell embedded in an infinite cartilage matrix. Recently, in order to take into account the effect of

spatial distribution of chondrocytes, a sort of homogenization method was used, cf. [33, 34, 35].

The analysis of the flow in a cartilage under compression shows that the synthesis of proteoglycans is localized in places where the velocity of flow is large, cf. [35].

1.2.5. Fixed charge density

The fixed charge density (FCD) is defined as the concentration of fixed groups in milliequivalents per gram of wet tissue, see Fig. 4.

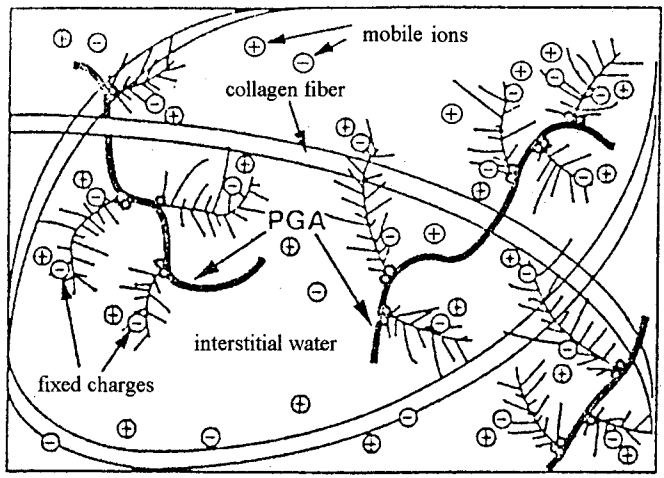

FIG. 4. Structure of the charged hydrated extracelleur matrix of cartilage, after [36].

Electrolytes – water solutions of acids, bases or salts – are electrically neutral in bulk; they manifest however electric charge separation in the vicinity of contact surfaces where double layer arises as a result of lowering of symmetry of the system. The appearance of charges leads to a series of elektrokinetic phenomena, especially important for metabolism in living tissues and for transformations of inanimate nature.

Since the matrix of cartilage (proteoglycans – water gel) contains negatively charged groups – the sulphate and carboxylate groups of chondroitin and keratan sulphates – which are not free to move, a Donnan potential is set up across the interface when cartilage is immersed in an electrolyte solution. The distribution of mobile ions between cartilage and the external solution obeys the Donnan equilibrium equation

$$\left(\frac{a_{\text{cation}}}{\tilde{a}_{\text{cation}}}\right) z_{\text{anion}} = \left(\frac{\tilde{a}_{\text{anion}}}{a_{\text{anion}}}\right) z_{\text{cation}} , \tag{1.1}$$

where a denotes the activity of the ion in solution, $\tilde{a}$ is the activity of the ion in the cartilage, and z stands for the valency of the ion [37].
Obviously we can also write

$$\left(\frac{a^{(+)}}{\tilde{a}^{(+)}}\right) z^{(-)} = \left(\frac{\tilde{a}^{(-)}}{a^{(-)}}\right) z^{(+)}. \tag{1.2}$$

The activities are related to the concentration m expressed on a molal basis (moles of solute per volume of water) by the following equations

$$a^{(+)} = \gamma^{(+)}\, m^{(+)}, \quad a^{(-)} = \gamma^{(-)}\, m^{(-)}, \tag{1.3}$$

where $\gamma^{(+)}$, $\gamma^{(-)}$, are the activity coefficients of positive and negative ions, respectively.
Introducing into the Donnan equation the quantities γ and m we get

$$\frac{z^{(-)}}{z^{(+)}} \frac{\gamma^{(+)}\,\gamma^{(-)}}{\tilde{\gamma}^{(+)}\tilde{\gamma}^{(-)}} = \frac{\tilde{m}^{(+)}\,\tilde{m}^{(-)}}{m^{(+)}\,m^{(-)}}. \tag{1.4}$$

In the external solution $z^{(-)} = z^{(+)}$ and $m^{(-)} = m^{(+)} = m$. Hence we can write

$$\frac{\tilde{\gamma}^{(+)}\tilde{\gamma}^{(-)}}{\gamma^{(+)}\,\gamma^{(-)}} = \frac{m^2}{\tilde{m}^{(+)}\,\tilde{m}^{(-)}}. \tag{1.5}$$

The deviation of the term on left-hand side of Eq. (1.5) from unity describes to what extent the behaviour of the electrolyte in the tissue departs from its behaviour in the solution.

1.2.6. Osmotic pressure

Due to the presence of the negatively charged fixed groups in cartilage, the distribution of mobile ionic species is not equal between cartilage and synovial fluid. In accordance with the Donnan equilibrium there are more ions in the cartilage than in the synovial fluid. The component of the swelling pressure of cartilage which is due to this excess of ions is referred to as the ionic or the Donnan contribution to the swelling pressure and can be calculated from

$$\Delta\pi = \tilde{\pi} - \pi, \tag{1.6}$$

where $\tilde{\pi}$ and π are the Donnan osmotic pressures of the inner and outer phases respectively. They are given by the expressions

$$\pi = RT\phi\left(m^{(+)} + m^{(-)}\right), \qquad \tilde{\pi} = RT\phi\left(\tilde{m}^{(+)} + \tilde{m}^{(-)}\right), \tag{1.7}$$

where ϕ is the osmotic coefficient of the electrolyte salt (*e.g.* NaCl) in the synovial fluid and $\tilde{\phi}$ is the osmotic coefficient of the electrolyte salt (*e.g.* NaCl) in the cartilage.

The osmotic pressure is one of the factors ensuring the strength resistance of cartilage and this pressure increases with fixed charge density. Under the load, the number of fixed charge being preserved, the decrease of the volume induces an increase of the charge density.

The elastic modulus of articular cartilage depends on the species of counterions. If c_1 and c_2 are Na^+ and Ca^{2+} concentrations then the formula proposed is as follows, cf. [39 - 44],

$$\lambda + 2\,\mu = \frac{1}{c_1 + c_2}\,[c_1 H_1(c_1) + c_2 H_2(c_2)] \tag{1.8}$$

where λ and μ are the Lamé coefficients and and H_1 and H_2 are in general functions of c_1 and c_2.

We propose the following generalization of the above relation

$$a^{\varepsilon}_{ijmn} = \frac{n^{(+)\varepsilon}}{n^{(+)\varepsilon} + n^{(2+)\varepsilon}} h^{(+)\varepsilon}_{ijmn} + \frac{n^{(2+)\varepsilon}}{n^{(+)\varepsilon} + n^{(2+)\varepsilon}} h^{(2+)\varepsilon}_{ijmn}\,. \tag{1.9}$$

1.3. Cartilage as periodic porous medium

The layered structure of cartilage is visible to the naked eye, and the surface (10 – 20% of the total thickness), middle (40 – 60% of the total thickness), and deep zones can be distinguished, see Fig. 2.

The arrangement of the cells changes throughout the zones and different zones have different metabolic activities. The arrangement of the cells is reflected in the organization of the dense collagen network of the extracellular matrix of cartilage. The collagen and proteoglycan content in tissue also varies with depth from the articulating surface. The collagen content is the highest in the surface zone, and there is a decrease of approximately 15% in the middle and deep zones. The proteoglycan content is the lowest at the surface and rising 15% in the middle and deep zones.

The collagen fibre orientation changes also throughout the zones. In the superficial zone, fine collagen fibrils are organized parallel to the articular surface. In the middle zone the collagen fibres appear to be randomly arranged with a slight preference of 45% orientation. In the deep zone, the fibres forme large fibre bundles organized perpendicularly to the bottom surface and inserted into the calcified cartilage, providing a strong anchoring system for the tissue on the subchondral bone, cf. Fig. 5.

In a first approximation the cartilage is treated as a porous medium with nonuniformly microperiodic structure (since its properties depend on the distance from the cartilage surface). All solid parts of the cartilage are assumed to constitute the skeleton (matrix) in the sense of porous media.

2. Mathematical description of charged hydrated tissue

2.1. Notations and basic relations

Let $\Omega \subset I\!R^3$ be a bounded sufficiently regular domain. This domain consists of two parts $\Omega = \Omega_S \cup \Omega_L$, $\overline{\Omega} = \overline{\Omega}_S \cup \overline{\Omega}_L$. We admit that Ω_S is made of piezoelectric material (the skeleton) while Ω_L is filled with a conductive fluid. The interface solid-liquid is denoted by Γ; $\partial\Omega_S$ ($\partial\Omega_L$) is the boundary of Ω_S (Ω_L). We have $\partial\Omega_S \cup \partial\Omega_L = \partial\Omega$.

2.2. Skeleton

The material of the skeleton is described by the linear relations between the stress S_{ij}^S, strain e_{ij}, electric induction D_i^S and the electric field E_i as follows

$$S_{ij}^S = a_{ijmn}e_{mn} - \pi_{kij}E_k\,, \qquad D_i^S = \pi_{imn}e_{mn} + \epsilon_{ik}^S E_k\,, \tag{2.1}$$

where (a_{ijmn}) is the elasticity tensor, (π_{kij}) denotes the piezoelectric tensor and (ϵ_{ik}^S) is the tensor of dielectric coefficients, cf. [45]. Moreover, the strain tensor is given by

$$e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\partial_{x_j}u_i + \partial_{x_i}u_j\right). \tag{2.2}$$

The electric field E_i is given by gradient of the electric potential Φ

$$E_i = -\partial_{x_i}\Phi. \tag{2.3}$$

The processes considered are slow, near the equilibrium and are described by the equations

$$\mathrm{div}\mathbf{S}^S + \mathbf{f}^S = 0, \qquad \mathrm{div}\mathbf{D}^S = 0, \tag{2.4}$$

where $\mathbf{f}^S$ denotes the body force of nonelectric origin.

2.3. Electrolyte

2.3.1. Currents in electrolyte

Here we apply Planck's theory of electrolytes, cf. [46 – 48]. The number of positive and negative carriers in the unit volume is $n^{(+)}$ and $n^{(-)}$, respectively.

The resulting number of carriers is $n = n^{(+)} + n^{(-)}$. If $\rho^{(+)}$ and $\rho^{(-)}$ are the charges of individual carriers then charges *per* unit volume are equal to

$$q^{(+)} = \rho^{(+)} n^{(+)}, \qquad q^{(-)} = \rho^{(-)} n^{(-)}. \tag{2.5}$$

The *net* charge *per* unit volume (the charge density) is equal to $q = q^{(+)} + q^{(-)}$. Note that for $\rho^{(+)} = -\rho^{(-)} = \rho, q = \rho(n^{(+)} - n^{(-)})$. For ions with valency $z, \rho = ze$, in particular, for valency $1, \rho = e$, where e is the value of elementary charge, $e > 0$; $e = 1.602177 \cdot 10^{-19}$C.

Under the electrical potential Ψ_{eq} the number of carriers in equilibrium is given by the Boltzmann distribution

$$n_{eq}^{(+)} = n_0^{(+)} e^{-\beta\rho^{(+)}\Psi_{eq}}, \qquad n_{eq}^{(-)} = n_0^{(-)} e^{-\beta\rho^{(-)}\Psi_{eq}}, \tag{2.6}$$

where $n_0^{(+)} = n_0^{(-)} \equiv \frac{1}{2} n_0$ and $\beta = 1/(k_B T)$. The Boltzmann constant is equal to $k_B = 1.38066 \cdot 10^{-23}$ J/K. Hence

$$n_{eq}^{(+)} + n_{eq}^{(-)} = n_0 \mathrm{ch}(\beta\rho\Psi_{eq}), \qquad n_{eq}^{(+)} - n_{eq}^{(-)} = -n_0 \mathrm{sh}(\beta\rho\Psi_{eq}).$$

Consequently, $q_{eq} = -\rho n_0 \mathrm{sh}(\beta\rho\Psi_{eq})$.

The electrical current in the electrolyte is created by fluxes of positive and negative ions. The density of those fluxes is denoted by $\mathbf{j}^{(+)}$ and $\mathbf{j}^{(-)}$.

The mean velocities of carriers are proportional to the electrical field. The proportionality coefficients between the velocities and forces acting on carriers, *i.e. mobilities*, are denoted by $\mathbf{b}^{(+)}$ and $\mathbf{b}^{(-)}$ for charges $\rho^{(+)}$ and $\rho^{(-)}$, respectively. Velocities of carriers are therefore equal to

$$v_i^{(+)} = b_{ij}^{(+)} f_j^{(+)}, \qquad v_i^{(-)} = b_{ij}^{(-)} f_j^{(-)}.$$

For the special case of the force generated by an electric field we have

$$v_i^{(+)} = \rho^{(+)} b_{ij}^{(+)} E_i, \qquad v_i^{(-)} = \rho^{(-)} b_{ij}^{(-)} E_i.$$

The densities of electrical currents are then given by

$$j_i^{E(+)} \equiv q^{(+)} v_i^{(+)} = \rho^{(+)} n^{(+)} v_i^{(+)} = [\rho^{(+)}]^2 n^{(+)} b_{ij}^{(+)} E_j,$$

$$j_i^{E(-)} \equiv q^{(-)} v_i^{(-)} = \rho^{(-)} n^{(-)} v_i^{(-)} = [\rho^{(-)}]^2 n^{(-)} b_{ij}^{(-)} E_j.$$

The density of resulting electrical current is $j_i^E \equiv q^{(+)} v_i^{(+)} + q^{(-)} v_i^{(-)}$ or

$$j_i^E = \{[\rho^{(+)}]^2 n^{(+)} b_{ij}^{(+)} + [\rho^{(-)}]^2 n^{(-)} b_{ij}^{(-)}\} E_j.$$

Thus for $\rho^{(+)} = -\rho^{(-)} = \rho$ we have

$$j_i^E = \rho^2[n^{(+)}b_{ij}^{(+)} + n^{(-)}b_{ij}^{(-)}]E_j.$$

Comparing this result with the general relationship between current and the electrical field given by $j_i^E = \sigma_{ij}E_j$, we get

$$\sigma_{ij} = \rho^2(n^{(+)}b_{ij}^{(+)} + n^{(-)}b_{ij}^{(-)}). \tag{2.7}$$

In the presence of nonhomogeneities of certain fields like the charge density, the additional diffusional current $\mathbf{j}^{D(\pm)}$ is observed

$$j_i^{D(+)} = -\rho^{(+)}d_{ij}^{(+)}\partial_{x_j}n^{(+)}, \qquad j_i^{D(-)} = -\rho^{(-)}d_{ij}^{(-)}\partial_{x_j}n^{(-)}.$$

Density of the resulting diffusive electrical current is then equal to

$$j_i^D = -\rho^{(+)}d_{ij}^{(+)}\partial_{x_j}n^{(+)} - \rho^{(-)}d_{ij}^{(-)}\partial_{x_j}n^{(-)}.$$

In general, the diffusion coefficients of both charges are not equal, $\mathbf{d}^{(+)} \neq \mathbf{d}^{(-)}$.

Adding both components of the current we get

$$j_i^{ED(+)} = [\rho^{(+)}]^2 n^{(+)}b_{ij}^{(+)}E_j - \rho^{(+)}d_{ij}^{(+)}\partial_{x_j}n^{(+)}, \tag{2.8}$$

$$j_i^{ED(-)} = [\rho^{(-)}]^2 n^{(-)}b_{ij}^{(-)}E_j - \rho^{(-)}d_{ij}^{(-)}\partial_{x_j}n^{(-)}. \tag{2.9}$$

Particularly, for $\rho^{(+)} = -\rho^{(-)} = \rho$ we have

$$j_i^{ED} = \rho^2[n^{(+)}b_{ij}^{(+)} + n^{(-)}b_{ij}^{(-)}]E_j - \rho[d_{ij}^{(+)}\partial_{x_j}n^{(+)} - d_{ij}^{(-)}\partial_{x_j}n^{(-)}).$$

Note that in equilibrium the currents $\mathbf{j}^{ED(\pm)}, i = 1, 2, 3$, vanish and the Nernst-Einstein relation follows

$$b_{ij}^{(\pm)} = \beta d_{ij}^{(\pm)}. \tag{2.10}$$

If an electrolyte is not at rest but flows with a velocity $\mathbf{v}$ then the velocities $\mathbf{v}^{(\pm)}$ should be replaced by

$$v_i^{(\pm)} + v_i$$

and for currents $\mathbf{j}^{E(\pm)}$ we obtain

$$j_i^{E(+)} = q^{(+)}(v_i^{(+)} + v_i) = [\rho^{(+)}]^2 n^{(+)}b_{ij}^{(+)}E_j + \rho^{(+)}n^{(+)}v_i,$$

$$j_i^{E(-)} = q^{(-)}(v_i^{(-)} + v_i) = [\rho^{(-)}]^2 n^{(-)}b_{ij}^{(-)}E_j + \rho^{(-)}n^{(-)}v_i.$$

Then

$$j_i^E = \rho^2[n^{(+)}b_{ij}^{(+)} + n^{(-)}b_{ij}^{(-)}]E_j + \rho(n^{(+)} - n^{(-)})v_i.$$

In a similar manner for currents $\mathbf{j}^{ED(\pm)}$ we obtain

$$j_i^{(+)} = [\rho^{(+)}]^2 n^{(+)} b_{ij}^{(+)} E_j - \rho^{(+)} d_{ij}^{(+)} \partial_{x_j} n^{(+)} + \rho^{(+)} n^{(+)} v_i, \tag{2.11}$$

$$j_i^{(-)} = [\rho^{(-)}]^2 n^{(-)} b_{ij}^{(-)} E_j - \rho^{(-)} d_{ij}^{(-)} \partial_{x_j} n^{(-)} + \rho^{(-)} n^{(-)} v_i, \tag{2.12}$$

and

$$j_i = j_i^{(+)} + j_i^{(-)} = \rho^2 [n^{(+)} b_{ij}^{(+)} + n^{(-)} b_{ij}^{(-)}] E_j$$
$$-\rho [d_{ij}^{(+)} \partial_{x_j} n^{(+)} - d_{ij}^{(-)} \partial_{x_j} n^{(-)}] + \rho (n^{(+)} - n^{(-)}) v_i.$$

The last term, known as the convection term, vanishes for $n^{(+)} = n^{(-)}$. If we set

$$\rho k_i = j_i^{(+)} - j_i^{(-)},$$

$$B_{ij} = \frac{1}{2}(b_{ij}^{(+)} + b_{ij}^{(-)}), \qquad D_{ij} = \frac{1}{2}(d_{ij}^{(+)} + d_{ij}^{(-)}),$$

$$b_{ij} = \frac{1}{2}(b_{ij}^{(+)} - b_{ij}^{(-)}) \qquad d_{ij} = \frac{1}{2}(d_{ij}^{(+)} - d_{ij}^{(-)}),$$

then

$$j_i = \rho(\rho n B_{ij} + q b_{ij}) E_j - (\rho d_{ij} \partial_{x_j} n + D_{ij} \partial_{x_j} q) + q v_i, \tag{2.13}$$

$$k_i = (\rho n b_{ij} + q B_{ij}) E_i - (D_{ij} \partial_{x_j} n + \frac{1}{\rho} d_{ij} \partial_{x_j} q) + n v_i. \tag{2.14}$$

In our previous papers [13,16] the simplified equation was used, with the total current denoted by $\mathbf{J}$,

$$J_i = -q b_{ij} \partial_{x_j} \Phi - d_{ij} \partial_{x_j} q + q v_i \qquad \text{in} \quad \Omega_L .$$

The current of positive (negative) charges without convection can be written in the form

$$j_i^{ED(\pm)} = \rho^{(\pm)} [b_{ij}^{(\pm)} q^{(\pm)} E_j - d_{ij}^{(\pm)} \partial_{x_j} n^{(\pm)}].$$

Thus the velocity of the positive charge is equal to

$$v_i^{(+)} = \frac{1}{q^{(+)}} j_i^{ED(+)} = \rho^{(+)} b_{ij}^{(+)} E_j - \frac{1}{n^{(+)}} d_{ij}^{(+)} \partial_{x_j} n^{(+)}].$$

The force acting on the one positive charge is given by

$$f_i^{(+)} = \overline{b}_{ij}^{(+)} v_j^{(+)} = \rho^{(+)} E_i - \frac{1}{n^{(+)}} d_{jk}^{(+)} \partial_{x_j} n^{(+)},$$

where the matrix $\overline{\mathbf{b}}^{(+)}$ is the inverse of $\mathbf{b}^{(+)}$, $\overline{b}_{ik}^{(+)} b_{kj}^{(+)} = \delta_{ij}$. Consequently, the force acting on the positive charge in the unit volume consists of 2 forces, electrical and diffusional, and is given by the expression:

$$f_i^{ED(+)} = n^{(+)} f_i^{(+)} = q^{(+)} E_i - \overline{b}_{ij}^{(+)} d_{jk}^{(+)} \partial_{x_k} n^{(+)}.$$

Similar relationship holds for the force $\mathbf{f}^{ED(-)}$ acting on negative charges. Thus, from the Nernst-Einstein relation (2.10) we have

$$f_i^{ED(+)} = q^{(+)} E_i - \frac{1}{\beta} \partial_{x_i} n^{(+)}, \qquad f_i^{ED(-)} = q^{(-)} E_i - \frac{1}{\beta} \partial_{x_i} n^{(-)}.$$

The resulting force exerted on the unit volume is equal to the sum of forces exerted on positive and negative charges in this volume. Therefore

$$f_i^{ED} = (q^{(+)} + q^{(-)}) E_i - \frac{1}{\beta} \partial_{x_i} (n^{(+)} + n^{(-)}) = q E_i - \frac{1}{\beta} \partial_{x_i} n. \tag{2.15}$$

Obviously, in equilibrium, where $n^{(+)}$ and $n^{(-)}$ are given by the Boltzmann distribution, we have $\mathbf{f}^{ED(+)} = \mathbf{f}^{ED(-)} = 0$.

2.3.2. Electrolyte description

The electrolyte is treated as an incompressible fluid with the viscosity tensor η_{ijmn}. The linear relation between the stress S_{ij}^L, pressure p and liquid strain rate $e_{ij}(\mathbf{v})$ is assumed as

$$S_{ij}^L = -p\delta_{ij} + \eta_{ijmn} e_{mn}(\mathbf{v}). \tag{2.16}$$

According to (2.11) and (2.12) the relation between the electric current $\mathbf{J}$, electric field $\mathbf{E}$, liquid velocity $\mathbf{v}$ and electric charge q has the following form:

$$J_i^{(+)} \equiv j_i^{ED(+)} = \sigma_{ij}^{(+)} E_j - d_{ij}^{(+)} \partial_{x_j} q^{(+)} + q^{(+)} v_i, \tag{2.17}$$

$$J_i^{(-)} \equiv j_i^{ED(-)} = \sigma_{ij}^{(-)} E_j - d_{ij}^{(-)} \partial_{x_j} q^{(-)} + q^{(-)} v_i, \tag{2.18}$$

where

$$\sigma_{ij}^{(+)} = b_{ij}^{(+)} q^{(+)}, \qquad \sigma_{ij}^{(-)} = b_{ij}^{(-)} q^{(-)}$$

are the electrical conductivities of both currents,

$$b_{ij}^{(+)} = \rho^{(+)} B_{ij}^{(+)}, \qquad b_{ij}^{(-)} = \rho^{(-)} B_{ij}^{(-)}$$

and $\mathbf{B}^{(+)}$ and $\mathbf{B}^{(-)}$ are the mobilities of free charges while $\mathbf{d}^{(+)}$ and $\mathbf{d}^{(-)}$ are the coefficients of diffusion. If the dielectric tensor of the liquid is denoted by $\boldsymbol{\epsilon}^L$, then

$$D_i^L = \epsilon_{ik}^L E_k. \tag{2.19}$$

For an isotropic incompressible liquid these tensors are of the form

$$\eta_{ijmn} = \eta\left(\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in} - \frac{2}{3}\delta_{ij}\delta_{mn}\right),$$

$$\sigma_{ij}^{(\pm)} = \sigma^{(\pm)}\delta_{ij}, \qquad b_{ij}^{(\pm)} = b^{(\pm)}\delta_{ij} \tag{2.20}$$

$$d_{ij}^{(\pm)} = d^{(\pm)}\delta_{ij}, \qquad \epsilon_{ik}^{L} = \epsilon^{L}\delta_{ij},$$

where η, $\sigma^{(\pm)}$, $b^{(\pm)}$, $d^{(\pm)}$ and ϵ^L are material coefficients.

The flow of liquid is stationary, and the body force is given by sum $(\mathbf{f}^L+\mathbf{f}^{ED})$; cf. Eq. (2.15). Thus

$$\mathrm{div}\mathbf{S}^L + \mathbf{f}^L + q\,\mathbf{E} - k\nabla q = 0. \tag{2.21}$$

In the last equation the term $\mathbf{f}^L$ denotes the body force of nonelectric origin (cf. [13]); the coefficient $k = \dfrac{1}{\beta} = \kappa_B T/e$.

The incompressiblity condition reads

$$\mathrm{div}\mathbf{v} = 0 \qquad \text{in } \Omega^L. \tag{2.22}$$

The vector D_i satisfies the Gauss equation [46]

$$\mathrm{div}\mathbf{D} = q \qquad \text{in } \Omega^L \tag{2.23}$$

and the stationarity implies, cf. [13, 49],

$$\mathrm{div}\mathbf{J}^{(+)} = \mathrm{div}\mathbf{J}^{(-)} = 0 \qquad \text{in } \Omega^L. \tag{2.24}$$

2.4. Solid-liquid interface

The conditions at the solid-fluid interface Γ are assumed to be given by

$$\begin{aligned} &[\![S_{ij}]\!]n_j = 0, \qquad [\![\Phi]\!] = 0, \qquad [\![D_i]\!]n_i = \gamma, \\ &v_i = 0, \qquad J_i^{(+)}n_i = 0, \qquad J_i^{(-)}n_i = 0 \end{aligned} \tag{2.25}$$

where γ is the density of electric charges on the surface of skeleton [48, 49]. We observe that the papers [50, 51] refer to the bone tissue. In (2.25) the symbol $[\![\cdot]\!]$ stands for the jump on Γ, e.g.

$$[\![S_{ij}]\!]n_j = S_{ij}^L n_j - S_{ij}^S n_j, \qquad [\![\Phi]\!] = \Phi_L - \Phi_S,$$

with Φ_S and Φ_L denoting the values of potential Φ on both sides of Γ.

The coefficients a_{ijmn}, ϵ_{ij}, η_{ijmn}, b_{ij} and d_{ij} satisfy the usual symmetry and positivity conditions, for instance

$$\exists\, c > 0 \qquad \epsilon_{ij}\eta_i\eta_j \geq c|\eta|^2 \quad \forall\, \eta \in I\!R^3\,, \tag{2.26}$$
$$d_{ij}\eta_i\eta_j \geq c|\eta|^2 \quad \forall\, \eta \in I\!R^3\,.$$

The conditions to be imposed on the elastic, dielectric and piezoelectric coefficients have been discussed in [10].

3. Equations of microperiodic porous media

Table 1 in Subsec. 1.1 allows to infer that the cartilage exibits a hierarchical structure. The different levels are interconnected and the full description of the process should consider these mutual links. However, in this approach we consider the description on one level only.

Consider now a porous medium with an εY-periodic microstructure. Here ε is a small positive parameter, $\varepsilon = l/L$, and l, L are characteristic lengths at the micro- and macro-scale, cf. Fig. 5 in Sec. 5. The basic cell Y consists of two parts, Y_S and Y_L with $\overline{Y} = \overline{Y}_S \cup \overline{Y}_L$.

For a fixed $\varepsilon > 0$ all the relevant quantities have now the superscript ε. From Eqs. (2.1) – (2.4) and (2.16) – (2.25) we obtain the set of equations for the fields $\mathbf{u}^\varepsilon$, $\mathbf{v}^\varepsilon$, Φ^ε, p^ε and q^ε

- in Ω_S^ε

$$\partial_{x_j} S_{ij}^\varepsilon = 0\,, \qquad \partial_{x_i} D_i^\varepsilon = -q^{F\varepsilon} + q^\varepsilon \tag{3.1}$$

- in Ω_L^ε

$$\partial_{x_j} S_{ij}^\varepsilon + f_i^g - q^\varepsilon \partial_{x_i}\Phi^\varepsilon - k\partial_{x_i} q^\varepsilon = 0\,,$$
$$\partial_{x_i} v_i^\varepsilon = 0\,, \qquad \partial_{x_i} D_i^\varepsilon = -q^\varepsilon \tag{3.2}$$

- in $\Omega_S^\varepsilon \cup \Omega_L^\varepsilon$

$$q^{(+)\varepsilon} = en^{(+)\varepsilon}\,, \quad q^{(2+)\varepsilon} = 2en^{(2+)\varepsilon}\,, \quad q^{(-)\varepsilon} = -en^{(-)\varepsilon}\,,$$
$$q^\varepsilon = q^{(+)\varepsilon} + q^{(2+)\varepsilon} + q^{(-)\varepsilon}\,, \tag{3.3}$$
$$\partial_{x_i} J_i^{(+)\varepsilon} = 0\,, \quad \partial_{x_i} J_i^{(2+)\varepsilon} = 0\,, \quad \partial_{x_i} J_i^{(-)\varepsilon} = 0\,.$$

Here S_{ij}^{ε}, D_i^{ε}, and $J_i^{(+)\varepsilon}$ $J_i^{(2+)\varepsilon}$ and $J_i^{(-)\varepsilon}$ are given by

$$S_{ij}^{\varepsilon} = \begin{cases} a_{ijmn}^{\varepsilon}\partial_{x_n}u_m^{\varepsilon} + \pi_{kij}^{\varepsilon}\partial_{x_k}\Phi^{\varepsilon} & \text{in } \Omega_S^{\varepsilon}, \\ -p^{\varepsilon}\delta_{ij} + \varepsilon^2\eta_{ijmn}^{\varepsilon}\partial_{x_n}v_m^{\varepsilon} & \text{in } \Omega_L^{\varepsilon}; \end{cases}$$

(3.4)

$$D_i^{\varepsilon} = \begin{cases} \pi_{imn}^{\varepsilon}\partial_{x_n}u_m^{\varepsilon} - \epsilon_{ik}^{S\varepsilon}\partial_{x_k}\Phi^{\varepsilon} & \text{in } \Omega_S^{\varepsilon}, \\ -\epsilon_{ik}^{L\varepsilon}\partial_{x_k}\Phi^{\varepsilon} & \text{in } \Omega_L^{\varepsilon}, \end{cases}$$

$$J_i^{(+)\varepsilon} = -b_{ij}^{(+)\varepsilon}q^{(+)\varepsilon}\partial_{x_j}\Phi^{\varepsilon} + q^{(+)\varepsilon}v_i^{\varepsilon} - d_{ij}^{(+)\varepsilon}\partial_{x_j}n^{(+)\varepsilon}, \tag{3.5}$$

$$J_i^{(2+)\varepsilon} = -b_{ij}^{(2+)\varepsilon}q^{(2+)\varepsilon}\partial_{x_j}\Phi^{\varepsilon} + q^{(2+)\varepsilon}v_i^{\varepsilon} - d_{ij}^{(2+)\varepsilon}\partial_{x_j}n^{(2+)\varepsilon}, \tag{3.6}$$

$$J_i^{(-)\varepsilon} = -b_{ij}^{(-)\varepsilon}q^{(-)\varepsilon}\partial_{x_j}\Phi^{\varepsilon} + q^{(-)\varepsilon}v_i^{\varepsilon} - d_{ij}^{(-)}\partial_{x_j}n^{(-)\varepsilon}. \tag{3.7}$$

The conditions imposed on the solid-liquid interface, now denoted by Γ^{ε}, read,

$$\begin{aligned} &[\![S_{ij}^{\varepsilon}]\!]n_j^{\varepsilon} = 0, && [\![\Phi^{\varepsilon}]\!] = 0, && [\![D_i^{\varepsilon}]\!]n_i = \gamma^{\varepsilon}, \\ &v_i^{\varepsilon} = 0, && J_i^{(+)\varepsilon}n_i = 0, && J_i^{(-)\varepsilon}n_i = 0, \end{aligned} \tag{3.8}$$

- in Ω_S^{ε}

$$\begin{aligned} \partial_{x_j}\left(a_{ijmn}^{\varepsilon}\partial_{x_n}u_m^{\varepsilon} + \pi_{kij}^{\varepsilon}\partial_{x_k}\Phi^{\varepsilon}\right) &= 0, \\ \partial_{x_i}\left(\pi_{imn}^{\varepsilon}\partial_{x_n}u_m^{\varepsilon} - \epsilon_{ik}^{S\varepsilon}\partial_{x_k}\Phi^{\varepsilon}\right) &= 0, \end{aligned} \tag{3.9}$$

- in Ω_L^{ε}

$$\begin{aligned} &\partial_{x_j}\left(-p^{\varepsilon}\delta_{ij} + \varepsilon^2\eta_{ijmn}^{\varepsilon}\partial_{x_n}v_m^{\varepsilon}\right) + f_i^g - q^{\varepsilon}\partial_{x_i}\Phi^{\varepsilon} - k\partial_{x_i}q^{\varepsilon} = 0, \\ &\partial_{x_i}v_i^{\varepsilon} = 0, \qquad \partial_{x_i}\left(\epsilon_{ik}^{L\varepsilon}\partial_{x_k}\Phi^{\varepsilon}\right) = -q^{\varepsilon}, \\ &q^{\varepsilon} = q^{(+)\varepsilon} + q^{(-)\varepsilon}, \\ &\partial_{x_i}J_i^{(+)\varepsilon} = 0, \qquad \partial_{x_i}J_i^{(-)\varepsilon} = 0. \end{aligned} \tag{3.10}$$

Note that in Eqs. $(3.2)_1$ and $(3.4)_2$, the following rescaling is introduced

$$\eta_{ijmn} \longmapsto \varepsilon^2\eta_{ijmn}. \tag{3.11}$$

According to the method of two-scale asymptotic expansions we assume the following *Ansatz*, cf. [52 – 58],

$$\Phi^\varepsilon = \Phi^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon F^{(1)}(\mathbf{x},\mathbf{y}) + \varepsilon^2\Phi^{(2)}(\mathbf{x},\mathbf{y}) + \dots, \qquad \mathbf{y} = \mathbf{x}/\varepsilon,$$

(3.12)

$$u_i^\varepsilon = u_i^{(0)}(\mathbf{x},\mathbf{y}) + \varepsilon u_i^{(1)}(\mathbf{x},\mathbf{y}) + \varepsilon^2 u_i^{(2)}(\mathbf{x},\mathbf{y}) + \dots, \qquad \mathbf{y} = \mathbf{x}/\varepsilon,$$

as well as analogous expansions for p^ε, q^ε, $\mathbf{v}^\varepsilon$ and $\mathbf{J}^{\pm\varepsilon}$.

Assume also that in Eq. $(3.3)_3$ we have

$$\gamma^\varepsilon = \gamma(\mathbf{x},\mathbf{y}) = \gamma_k^{(0)}(\mathbf{y})\partial_{x_k}\Phi^{(0)}(\mathbf{x},\mathbf{y}) \quad \text{for} \quad \mathbf{y} \in \Gamma_Y, \tag{3.13}$$

where $\gamma_k^{(0)}(\mathbf{y})$ is a given function; particularly it may be constant.

Next, taking into account the relation

$$\partial_{x_i} f(\mathbf{x},\mathbf{y}) = \Big(\partial_{x_i} + \frac{1}{\varepsilon}\partial_{y_i}\Big) f(\mathbf{x},\mathbf{y}), \qquad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$$

and comparing terms with the same power of ε, we arrive at the homogenized set of equations.

According to the above given division of the basic cell Y we define two types of averages

$$\langle(\cdot)\rangle = \frac{1}{|Y|}\int_Y (\cdot)\, d\mathbf{y}, \qquad \langle(\cdot)\rangle_k = \frac{1}{|Y|}\int_{Y_k} (\cdot)\, d\mathbf{y}, \qquad k = S, L.$$

Note that $\partial Y_S = \Gamma_Y \cup P_S$ while the surface $\partial Y_L = \Gamma_Y \cup P_L$; Γ_Y is the contact surface solid-liquid whilst P_S and P_L are parts of the surfaces of the solid and liquid, respectively, coinciding with the boundary of Y. As we already know, the mechanical and nonmechanical properties of the cartilage depend on the location and vary with the depth from the articulating surface. To account for this effect we assume that

$$a_{ijmn}^\varepsilon = a_{ijmn}(\mathbf{x},\mathbf{x}/\varepsilon), \quad \pi_{kij}^\varepsilon = \pi_{kij}(\mathbf{x},\mathbf{x}/\varepsilon) \ \textit{etc.}$$

Thus the moduli $a_{ijmn}(\mathbf{x},\mathbf{y})$, $\pi_{kij}(\mathbf{x},\mathbf{y})$, etc., are Y-periodic in the second variable and also depend on the macroscopic variable $\mathbf{x}$. This leads to the so-called nonuniform homogenization, cf. [57, 58].

The basic equations given above will be treated by a homogenization method.

4. Macroscopic constitutive relations

In the considerations which follow we will omit the influence of electric field and keep our attention on mechanical properties only. Applying asymptotic

expansions to the constitutive relations $(3.4)_1$, $(3.5)_2$ and comparing the terms linked with ε_0 we get

$$S_{ij}^{(0)} = \begin{cases} a_{ijmn}\left(\partial_{x_n} u_m^{(0)} + \partial_{y_n} u_m^{(1)}\right) & \text{in } Y_S, \\ -p^{(0)}\delta_{ij} & \text{in } Y_L \end{cases} . \tag{4.1}$$

We observe that

$$\langle S_{ij}^{(0)} \rangle = \langle S_{ij}^{(0)} \rangle_S + \langle S_{ij}^{(0)} \rangle_L . \tag{4.2}$$

Using Eqs. (4.5) and (4.14) we get

$$\langle S_{ij}^{(0)} \rangle = a_{ijpq}^h \partial_{x_q} u_p^{(0)} + [c_{ij}^h - (1-f)\delta_{ij}]p^{(0)} , \tag{4.3}$$

where

$$a_{ijpq}^h = \langle a_{ijpq} + a_{ijmn}\partial_{y_n} A_m^{(pq)} \rangle_S , \quad c_{ij}^h = \langle a_{ijmn}\partial_{y_n} P_m \rangle_S . \tag{4.4}$$

We observe that according to our assumptions, the homogenized moduli a_{ijpq}^h and c_{ij}^h depend on the macroscopic variable $\mathbf{x}$. Here f denotes the volume fraction of the skeleton

$$f = \frac{|Y_S|}{|Y|} , \qquad 1 - f = \frac{|Y_L|}{|Y|} . \tag{4.5}$$

The Y-periodic functions $A_m^{(pq)}$, B_{mq}, etc., have to be determined from the following local equations posed on Y_S:

$$\partial_{y_j}\left(a_{ijpq} + a_{ijmn}\partial_{y_n} A_m^{(pq)}\right) = 0 , \quad \partial_{y_j}\left(a_{ijmn}\partial_{y_n} P_m\right) = 0 . \tag{4.6}$$

By using Eqs. (4.5) we get

$$\left(a_{ijpq} + a_{ijmn}\partial_{y_n} A_m^{(pq)}\right) n_j \Big|_S = 0 , \quad \left(a_{ijmn}\partial_{y_n} P_m\right) n_j \Big|_S = \delta_{ij} n_j \Big|_L . \tag{4.7}$$

5. Mechanical structure of cartilage

As we have seen in Sec. 1.3, from the mechanical point of view, the extracellular matrix of cartilage can be regarded as a fiber-reinforced composite solid consisting of a dense stable network of collagen fibers embedded in a very high concentration of proteoglycan gel, which itself is also a viscoelastic network. The content and structure of collagen and proteoglycan within the tissue vary with the depth below the articulating surface. The tissue can be regarded as having three separate structural zones. The collagen content is the highest in the surface zone (approximately 85% by dry weight), and it decreases to approximately

68% in the middle zone. In the superficial zone (10 to 20% of the total tissue thickness) fine collagen fibrils are approximately parallel to the articular surface, cf. Fig. 5.

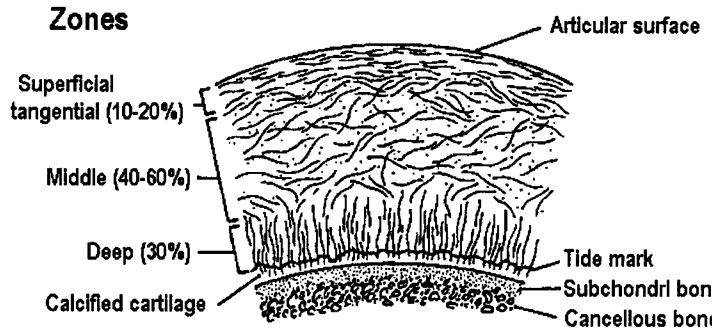

FIG. 5. Layered structure of cartilage collagen network showing three distinct regions, after [22].

The experimental evidence suggests that the fibrillar architecture incorporates the following structural principles:

(1) The overall arrangenent of fibrils is radial and they appear to be continuous for considerable distance through the cartilage depth. This provides the extremely high strength in the radial direction.

(2) The fibrils, as they pass radially through the cartilage depth, are repeatedly deflected sideways to form an arrangement of obliquely oriented segments in a crude zig-zag configuration. This secondary morphology accounts for the strain-locking effect observed when the matrix is stressed radially as in the transverse notch experiment.

(3) Some form of interfibril link should exist to give the overall radial array of fibrils structural cohesion in the transverse direction.

5.1. Cartilage as a honeycomb structure

The cartilage is an anisotropic (orthotropic) and inhomogeneous material. The biomechanical and ultrastructural studies provide evidence that the fibrillar architecture is a coherently ordered arrangement of fibrils developed from well defined structural relationships.

The arrangement of the cells changes throughout the zones and different zones have different metabolic activities. The arrangement of the cells is reflected in the organization of the dense collagen network of the extracellular matrix of cartilage. The collagen and proteoglycan content in the tissue also varies with depth below the articulating surface. The collagen content is the highest in the surface zone, and there is decrease of approximately 15% in the middle and deep zones. The

proteoglycan content is the lowest at the surface and rising 15% in the middle and deep zones.

BROOM and MARRA [59] proposed a network model of the cartilage. In the two-dimensional case the model was discussed by BROOM [60], cf. Fig. 6. As it can be seen, in this case the skeleton of the articular cartilage is modelled by a variable honeycomb structure.

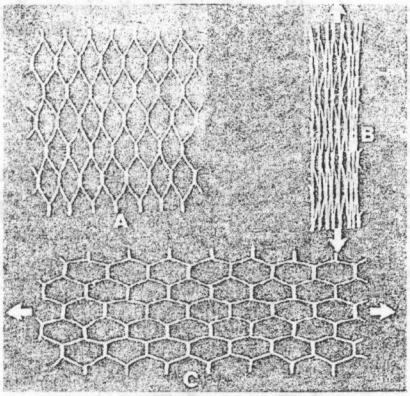

FIG. 6. A 2D wire mesh analogue of the interconnecting geometry of radial elements modelling the arrangements of fibrils in the matrix of articular cartilage. Note the rapid strain-locking in the radial direction (cf. A and B) versus the considerable extension permitted in the transverse direction (cf. A and C), after [60].

In this manner the inhomogeneity and anisotropy of the cartilage can be taken into account. To use the homogenization method, the element of the network is assumed in the form depicted in Fig. 7b. The angle α changes with depth in the cartilage in a given way. Thus we preserve the honeycomb structure of BROOM [60] and introduce a variation of the elementary cell with depth. To simplify the calculations the interaction, with electric field involved in Eq. $(4.4)_1$ is omitted.

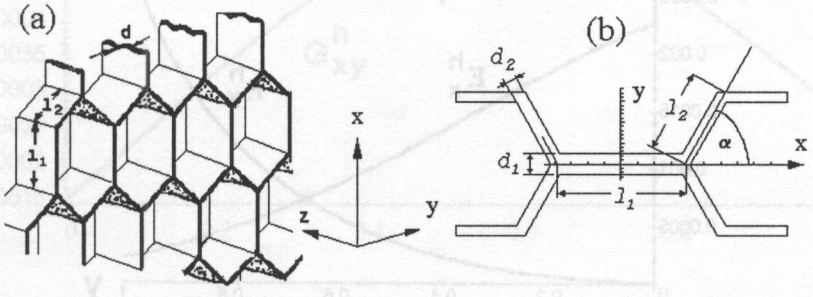

FIG. 7. The hexagonal network and its element used in the determination of effective (homogenized) elastic moduli.

The elastic moduli were calculated from Eq $(5.4)_1$

$$a^h_{ijpq} = \langle a_{ijpq} + a_{ijmn}\partial_{y_n} A^{(pq)}_m \rangle_S \,. \tag{5.1}$$

After lengthly calculation the effective technical moduli asume the following form, cf. [61, 62].

$$\frac{E^h_x}{E} = 8\,\frac{(\xi+\cos\alpha)^4}{(2+\xi)^3}\,\phi^3, \qquad \frac{E^h_y}{E} = 8\,\frac{(\xi+\cos\alpha)^2\,\sin^4\alpha}{\cos^2\alpha(2+\xi)^3}\,\phi^3,$$

$$\nu^h_{xy} = \frac{\cos\alpha(\xi+\cos\alpha)}{\sin^2\alpha}, \qquad \nu^h_{yx} = \frac{\sin^2\alpha}{\cos\alpha(\xi+\cos\alpha)},$$

$$\frac{G^h_{xz}}{\mu} = 2\,\frac{(1+\xi\,\cos\alpha)(\xi+\cos\alpha)}{\xi(2+\xi)^2}\phi, \qquad \frac{G^h_{yz}}{\mu} = 2\,\frac{\sin^2\alpha(\xi+\cos\alpha)}{(2+\xi)(1+\xi\cos\alpha)}\phi,$$

$$\frac{G^h_{xy}}{E} = 4\,\frac{\sin^2\alpha(\xi+\cos\alpha)^2(\xi\sin^2\alpha+\cos\alpha)(\xi+\cos\alpha+\xi\sin^2\alpha)}{(2+\xi)^3(2\xi+1)\xi^2}\phi^3,$$

where

$$\xi = \frac{l_1}{l_2}, \quad \phi = \frac{|Y_1|}{|Y|},$$

$$|Y_1| = 2\,l_2\,d\,(\xi+2), \quad |Y| = 4\,(l_2)^2\,\sin\alpha\,(\xi+\cos\alpha), \quad d \equiv d_1 = d_2.$$

The above formulae were used to describe the inhomogeneity of elastic properties of the cartilage. Some of the results are depicted in Figs. 8 – 11. They illustrate the change of the elastic moduli with depth z in the cartilage with the unit thickness. To perform the calculations we assumed that

$$\alpha = \frac{1}{10}\,\pi + \frac{3}{10}\,\pi\,z\,.$$

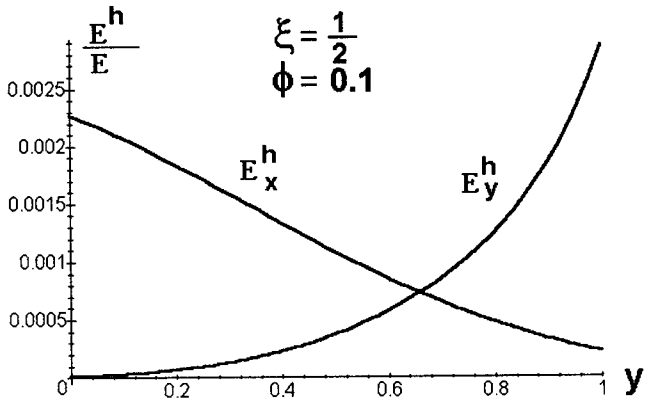

FIG. 8. The effective Young moduli across the depth of cartilage.

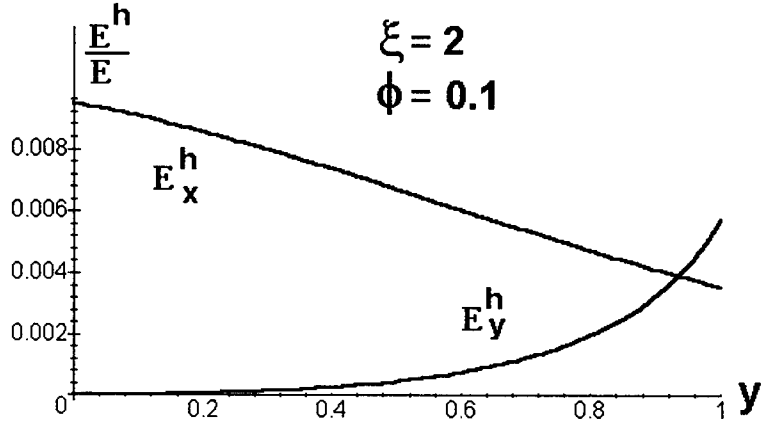

FIG. 9. The effective Young moduli across the depth of cartilage.

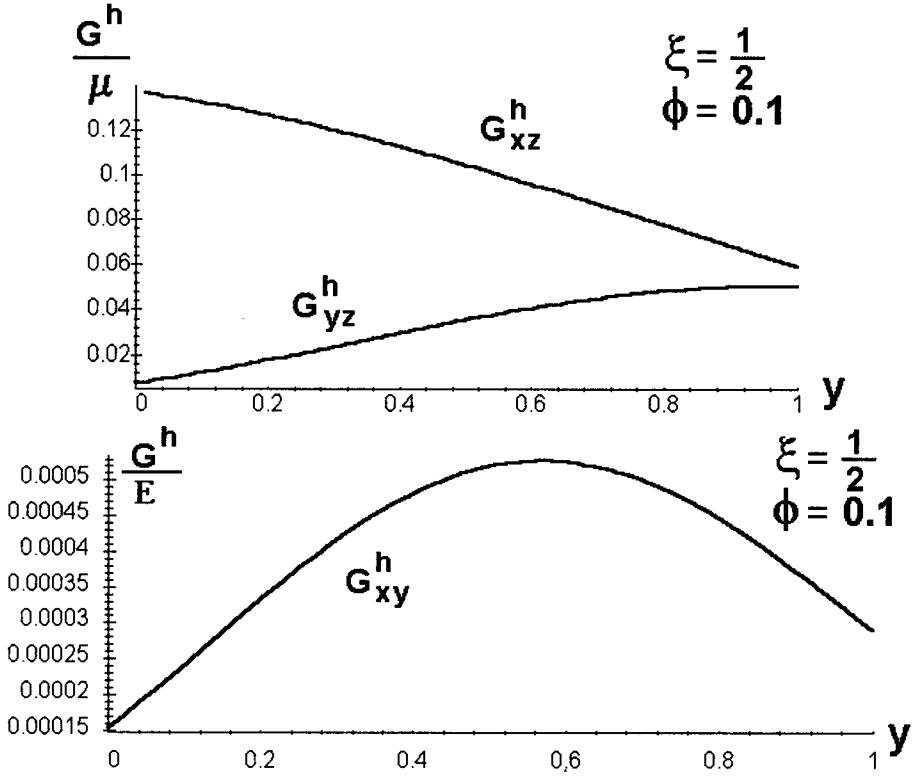

FIG. 10. Effective shear moduli across the depth of cartilage.

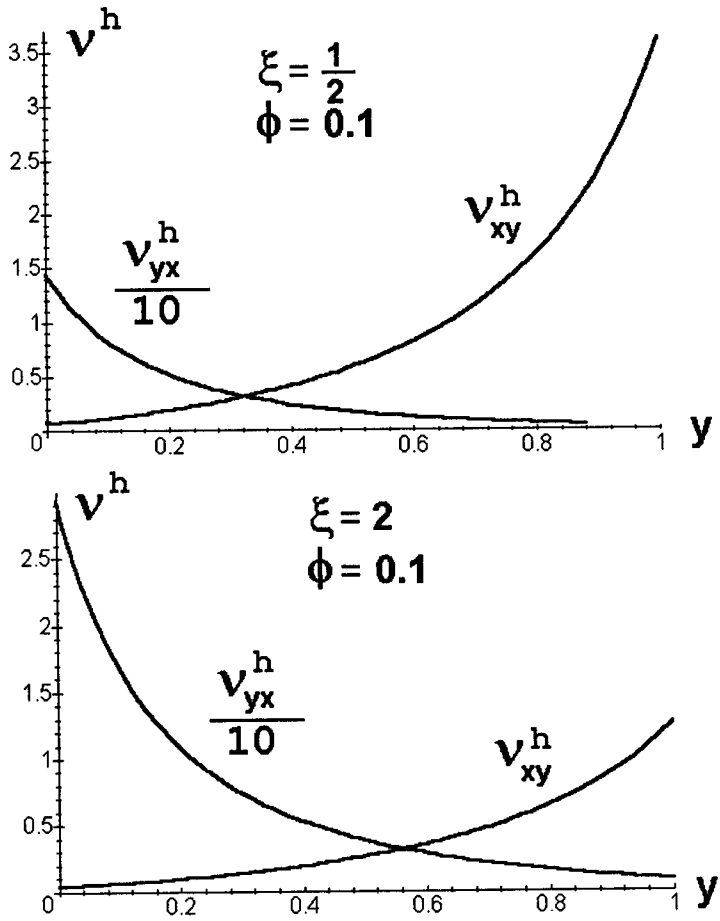

FIG. 11. The Poisson ratio across the depth of cartilage.

For comparison we present the following table:

Table 3. Tensile modulus (MPa) of articular cartilage.

	bovine glenoid	bovine humerus	human femoral groove	human femoral condyle
surface	5.9 (2.4)	13.4 (4.6)	13.9 (2.4)	7.8 (1.7)
middle	0.9 (0.5)	2.7 (1.6)	3.4 (1.4)	4.0 (1.1)
deep	0.2 (0.2)	1.7 (0.8)	1.0 (0.5)	

in which all samples were harvested in a direction parallel to the local split line direction, after AKIZUKI *et al.* and SETTON *et al.*, cf. [22]

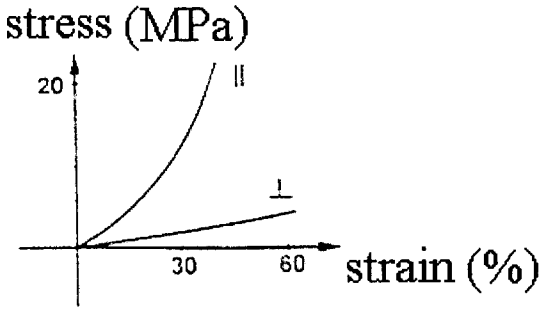

FIG. 12. The anisotropic tensile behaviour (stress in function of strain) of articular cartilage; $\parallel$ and $\perp$ denote parallel and perpendicular specimens, after [21].

Acceptable agreement between the results of our calculations and experimental data provided in Table 3 is observed.

According to Figs. 8 and 9, and Table 3, the Young modulus E_x decreases with the depth. Our model yields quantitative values of anisotropic macroscopic moduli. The anisotropic behaviour of the cartilage is well-known from experimental data, cf. Fig. 12 and FUNG [64].

6. FINAL REMARKS

The developed model of cartilage behaviour is confined to small deformations. Also, the well known phenomenon of cartilage swelling has not been examined. The nonlinearity of the cartilage skeleton can be modelled by applying the hyperelastic constitutive relationships recently proposed by JEMIOŁO and TELEGA [63, 65]. The swelling behaviour of cartilage within the framework developed in the present paper will be studied in the future. We are convinced that even the linear model proposed offers new possibilities for modelling the macroscopic behaviour of cartilage. The hierarchical structure of cartilage could be taken into account by employing the reiterated homogenization, cf. [57, 66]. The approach developed in the present paper was started with our short papers [18, 67].

ACKNOWLEDGEMENT

The authors were supported by the State Committee for Scientific Research (KBN Poland) through the grant No 8 T11F 017 18.

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Received December 18, 2000; revised March 29, 2001.