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Elastoplastic material law in 6-parameter nonlinear shell theory

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ABSTRACT: We develop the elastoplastic constitutive relations for nonlinear exact 6-parameter shell theory. A J_2-type theory with strain hardening is formulated that takes into account asymmetric membrane strain measures. The incremental equations are solved using implicit Euler scheme with closest point projection algorithm. The presented test example shows the correctness of the proposed approach. Influence of micropolar material parameters on numerical results is studied.

1 INTRODUCTION

This paper presents elastoplastic constitutive equations within the framework of a nonlinear 6-parameter shell theory, see for instance Chróścielewski & Witkowski (2006), Konopińska & Pietraszkiewicz (2007) or Pietraszkiewicz & Eremeyev (2009) and the references given there. In this formulation the kinematical relations of Cosserat type emerge naturally as the work-conjugate objects to stress measures in the underlying weak formulation. Consequently, the 6th degree of freedom appears naturally at each node of the resulting Finite Element Formulation (FEM).

For the purpose of this paper, the asymmetric strain components are collected in the vector ε (spatial representation)

$$\begin{aligned}\boldsymbol{\varepsilon} &= \{\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{12}\ \varepsilon_{21} | \varepsilon_1\ \varepsilon_2 || \kappa_{11}\ \kappa_{22}\ \kappa_{12}\ \kappa_{21} | \kappa_1\ \kappa_2\}^T = \\ &= \{\boldsymbol{\varepsilon}_m | \boldsymbol{\varepsilon}_s || \boldsymbol{\varepsilon}_b | \boldsymbol{\varepsilon}_d\}^T\end{aligned} \tag{1}$$

where labels m, s, b, d denote the membrane, shear, bending and drilling parts respectively. To Eq. (1) we adjoin the vector of shell stress and couple resultants

$$\begin{aligned}\mathbf{s} &= \{N^{11} N^{22} N^{12} N^{21} | Q^1 Q^2 || M^{11} M^{22} M^{12} M^{21} | M^1 M^2\}^T = \\ &= \{\mathbf{s}_m | \mathbf{s}_s || \mathbf{s}_b | \mathbf{s}_d\}^T\end{aligned} \tag{2}$$

through the constitutive equation

$$\mathbf{s} = \mathbf{C}\boldsymbol{\varepsilon} \tag{3}$$

The form of **C**, which is the main goal of the present paper, and the definitions of shell resultants are presented in Section 2 where we take into account an analogue to the first order shear deformation theory (FOSD, e.g. Kreja & Schmidt 2006).

2 MATERIAL LAW

2.1 *Plane stress formulation*

We assume the Cosserat plane stress state in each layer, with stress and strain components assembled in vectors (cf. for instance de Borst 1991):

$$\boldsymbol{\sigma} = \left\{\frac{\boldsymbol{\sigma}_m}{\boldsymbol{\sigma}_d}\right\} = \left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ \sigma_{21}\\ \hline \sigma_1\\ \sigma_2\end{array}\right\} = \left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ \sigma_{21}\\ \hline m_1/l\\ m_2/l\end{array}\right\},\quad \mathbf{e} = \left\{\frac{\mathbf{e}_m}{\mathbf{e}_d}\right\} = \left\{\begin{array}{c}e_{11}\\ e_{22}\\ e_{12}\\ e_{21}\\ \hline e_1\\ e_2\end{array}\right\} = \left\{\begin{array}{c}e_{11}\\ e_{22}\\ e_{12}\\ e_{21}\\ \hline \kappa_1\cdot l\\ \kappa_2\cdot l\end{array}\right\}. \tag{4}$$

Components for vector **e** are calculated from quantities in Eq. (1) as:

$$\mathbf{e}_m = \boldsymbol{\varepsilon}_m + \zeta\boldsymbol{\varepsilon}_b,\quad \mathbf{e}_d = \boldsymbol{\varepsilon}_d \cdot l, \tag{5}$$

where ζ means coordinate through thickness of a shell h which varies from $-h^-$ to h^+. The l-parameter, due to its dimension of length, is called characteristic length. The elastic operator

$$\mathbf{C}_l^e = \left[\begin{array}{cccc|cc} Gc_1 & Gc_2 & 0 & 0 & 0 & 0\\ Gc_2 & Gc_1 & 0 & 0 & 0 & 0\\ 0 & 0 & G+G_c & G-G_c & 0 & 0\\ 0 & 0 & G-G_c & G+G_c & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 2G & 0\\ 0 & 0 & 0 & 0 & 0 & 2G\end{array}\right], \tag{6}$$

where:

$$c_1 = 2(1+\nu)/(1-\nu^2) \text{ and } c_2 = \nu c_1, \tag{7}$$

where G_c is an material parameter of Cosserat continuum (cf. for instance Nowacki 1968). The elastic constants G and ν have the classical meaning of the shear modulus and Poisson's ratio. G, ν, l and G_c that are needed to describe the elastic behavior of an isotropic Cosserat continuum under planar deformations. The coefficient 2 has been introduced in the terms $\mathbf{C}^e_{l,55}$ and $\mathbf{C}^e_{l,66}$ in order to arrive at a convenient form of the elastoplastic constitutive equations. We observe that all stiffness moduli that enter the elastic stress-strain matrix $\mathbf{C}^e_l$ have the same dimension. This is attributable to the fact that all components of the strain vector $\mathbf{e}$ and the stress vector $\boldsymbol{\sigma}$ have the same dimensions. Therefore, the constitutive matrix follows in the differential form

$$d\boldsymbol{\sigma}=\mathbf{C}_l d\mathbf{e}\,,\quad \begin{Bmatrix} d\boldsymbol{\sigma}_m \\ d\boldsymbol{\sigma}_d \end{Bmatrix}=\begin{bmatrix} \mathbf{C}_{mm} & \mathbf{C}_{md} \\ \mathbf{C}_{dm} & \mathbf{C}_{dd} \end{bmatrix}\begin{Bmatrix} d\mathbf{e}_m \\ d\mathbf{e}_d \end{Bmatrix}, \tag{8}$$

where

$$\mathbf{C}_l=\begin{cases} \mathbf{C}^e_l & \text{Eq. (6) in elastic state} \\ \mathbf{C}^{ep}_l & \text{Eq. (26) in plastic state} \end{cases} \tag{9}$$

2.2 *Stress and couple resultant formulation*

First, we express the membrane $\mathbf{s}_m$, shear $\mathbf{s}_s$, bending $\mathbf{s}_b$ and drilling $\mathbf{s}_d$ resultants from Eq. (2) in terms of stresses Eq. (8). The derivation follows standard steps for a Reissner-Mindlin type shell theory. The stress and couple resultants Eq. (2) are formally defined as integrals over the shell thickness $h=h^+ + h^-$ from appropriate stresses. Using Eqs. (1) and (2) we have:

$$\begin{aligned}\mathbf{s}_m &= \int_{-h^-}^{+h^+} \boldsymbol{\sigma}_m \mu\, d\zeta = \int_{-h^-}^{+h^+} [\mathbf{C}_{mm}\underbrace{(\boldsymbol{\varepsilon}_m+\zeta\boldsymbol{\varepsilon}_b)}_{\mathbf{e}_m}+\mathbf{C}_{md}\underbrace{l\cdot\boldsymbol{\varepsilon}_d}_{\mathbf{e}_d}]\mu\, d\zeta \\ &= \mathbf{A}_{mm}\boldsymbol{\varepsilon}_m+\mathbf{B}_{mb}\boldsymbol{\varepsilon}_b+\mathbf{C}_{md}\boldsymbol{\varepsilon}_d\end{aligned} \tag{10}$$

$$\begin{aligned}\mathbf{s}_b &= \int_{-h^-}^{+h^+} \boldsymbol{\sigma}_m \zeta \mu\, d\zeta \\ &= \int_{-h^-}^{+h^+} [\mathbf{C}_{mm}(\zeta\boldsymbol{\varepsilon}_m+\zeta^2\boldsymbol{\varepsilon}_b)+\zeta l\cdot\mathbf{C}_{mb}\boldsymbol{\varepsilon}_d]\mu\, d\zeta\,, \\ &= \mathbf{B}_{bm}\boldsymbol{\varepsilon}_m+\mathbf{E}_{bb}\boldsymbol{\varepsilon}_b+\mathbf{F}_{bd}\boldsymbol{\varepsilon}_d\end{aligned} \tag{11}$$

$$\begin{aligned}\mathbf{s}_d &= \int_{-h^-}^{+h^+} l\cdot\boldsymbol{\sigma}_d \mu\, d\zeta \\ &= \int_{-h^-}^{+h^+} [l\cdot\mathbf{C}_{dm}(\boldsymbol{\varepsilon}_m+\zeta\boldsymbol{\varepsilon}_b)+l\cdot\mathbf{C}_{dd}\cdot l\cdot\boldsymbol{\varepsilon}_d]\mu\, d\zeta\,, \\ &= \mathbf{C}_{dm}\boldsymbol{\varepsilon}_m+\mathbf{F}_{db}\boldsymbol{\varepsilon}_b+\mathbf{H}_{dd}\boldsymbol{\varepsilon}_d\end{aligned} \tag{12}$$

where μ is the determinant of the shifter tensor (e.g. Konopińska & Pietraszkiewicz 2007). Here, it is assumed that the approximation $\mu \approx 1$ holds, which is justifiable for thin shells ($h \ll L$, L – a typical dimension) with a small curvature $h \ll R_{\min}$. In the present approach transverse shear terms $\mathbf{s}_s$ are treated as elastic, which may be assumed true for thin shells. Finally, Eq. (3) is written as

$$\begin{Bmatrix} \mathbf{s}_m \\ \mathbf{s}_s \\ \mathbf{s}_b \\ \mathbf{s}_d \end{Bmatrix}=\begin{bmatrix} \mathbf{A}_{mm} & \mathbf{0} & \mathbf{B}_{mb} & \mathbf{C}_{md} \\ \mathbf{0} & \mathbf{D}_{ss} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^T_{mb} & \mathbf{0} & \mathbf{E}_{bb} & \mathbf{F}_{bd} \\ \mathbf{C}^T_{md} & \mathbf{0} & \mathbf{F}^T_{bd} & \mathbf{H}_{dd} \end{bmatrix}\begin{Bmatrix} \boldsymbol{\varepsilon}_m \\ \boldsymbol{\varepsilon}_s \\ \boldsymbol{\varepsilon}_b \\ \boldsymbol{\varepsilon}_d \end{Bmatrix}. \tag{13}$$

Note that Eq (13) shows a new definition of $\mathbf{C}$ in comparison with that used in e.g. Chróścielewski & Witkowski (2006).

3 COSSERAT ELASTOPLASTICITY

In the paper we use J_2-flow theory, where the yield function f is written as

$$f=\sqrt{3J_2}-\bar{\sigma}(\gamma)\le 0\,, \tag{14}$$

where J_2 denotes the second invariant of deviatoric stress tensor and $\bar{\sigma}$ is yield stress which here is assumed a function of hardening parameter γ solely. In plane stress micropolar continuum J_2 can be generalized as (cf. for instance de Borst 1991)

$$\begin{aligned}J_2 &= \tfrac{1}{3}\sigma_{11}^2+\tfrac{1}{3}\sigma_{22}^2-\tfrac{1}{3}\sigma_{11}\sigma_{22}+a_1\sigma_{12}^2+ \\ &\quad +2a_2\sigma_{12}\sigma_{21}+a_1\sigma_{21}^2+a_3\left((m_1/l)^2+(m_2/l)^2\right)\end{aligned}, \tag{15}$$

where a_1, a_2, a_3 are material parameters with constraint $a_1+a_2=1/2$ (de Borst 1991). Associated plastic flow rule is used in the conventional form

$$d\mathbf{e}^p=d\lambda\, \partial f/\partial\boldsymbol{\sigma}\,. \tag{16}$$

Hardening parameter is obtained from strain-hardening hypothesis

$$d\gamma=\sqrt{\tfrac{2}{3}d\mathbf{e}^p d\mathbf{e}^p}\,, \tag{17}$$

where $d\mathbf{e}^p$ is infinitesimal increment of plastic strains. For uniaxial stressing $d\gamma=d\varepsilon^p_{xx}$, so γ can be called equivalent plastic strain and denoted as $\gamma=\varepsilon^p_{eff}$. In micropolar continuum $d\gamma$ can be generalized as

$$\begin{aligned}d\gamma=\Big[&\tfrac{2}{3}\left((de^p_{11})^2+(de^p_{22})^2\right)+b_1(de^p_{12})^2+2b_2 de^p_{12}de^p_{21} \\ &+b_1(de^p_{21})^2+b_3\left((d\kappa^p_1)^2+(d\kappa^p_2)^2\right)\Big]^{1/2}\end{aligned}, \tag{18}$$

where b_1, b_2, b_3 are material parameters with constraint (de Borst 1991) $b_1+b_2=2/3$. Our choice is $a_1=1/4$, $a_2=1/4$, $a_3=1/2$, $b_1=1/3$, $b_2=1/3$, $b_3=2/3$ (de Borst 1991), which also preserves $d\lambda=d\gamma$.

4 AN ALGORITHM FOR ELASTOPLASTICITY

We use the closest point projection algorithm (e.g. Simo & Hughes 1998). In this method, constraint (14) is enforced at the end of step and stress $\boldsymbol{\sigma}_n$ and strain increment $\Delta\mathbf{e}=\mathbf{e}_{n+1}-\mathbf{e}_n$ are assumed to be known. In the above and the following equations $(.)_n$ denotes variables from the last computed step and $(.)_{n+1}$ denotes variables from the end of the loading step. Increment $\Delta\mathbf{e}$ is decomposed into elastic and plastic parts

$$\Delta\mathbf{e}=\Delta\mathbf{e}^e+\Delta\mathbf{e}^p\,. \tag{19}$$

Plastic increment $\Delta \mathbf{e}^p$ is obtained from the associated flow rule (16)

$$\Delta \mathbf{e}^p = \Delta\lambda\, \mathbf{a}_{n+1}, \quad \mathbf{a}_{n+1} = \partial f / \partial \boldsymbol{\sigma}\big|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}_{n+1}}. \tag{20}$$

The stress at the end of a step depends only on the elastic part of strain increment

$$\boldsymbol{\sigma}_{n+1} = \mathbf{C}_l^e \mathbf{e}_{n+1}^e = \mathbf{C}_l^e \left(\mathbf{e}_{n+1} - \mathbf{e}_{n+1}^p\right). \tag{21}$$

Introduction of test stress $\sigma_{n+1}^{trial} = \sigma_n + \mathbf{C}_l^e \Delta \mathbf{e}$ into Eqs. (20), (21) gives

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{trial} - \Delta\lambda\, \mathbf{C}^e \mathbf{a}_{n+1}. \tag{22}$$

At the end of a step constraint

$$f_{n+1} = f\left(\boldsymbol{\sigma}_{n+1}\right) = 0 \tag{23}$$

is enforced. Eqs. (22), (23) gives a set of nonlinear equations (indices $n+1$ are omitted)

$$\begin{cases} f = f(\boldsymbol{\sigma}) = 0 \\ \mathbf{r} = -\boldsymbol{\sigma} + \boldsymbol{\sigma}^{trial} - \Delta\lambda \mathbf{C}^e \mathbf{a} = \mathbf{0} \end{cases}. \tag{24}$$

It can be solved with iterative Newton-Raphson method (e.g. Szymkiewicz 2010). Linearization of (24) gives a set of linear equations

$$\begin{cases} f^{(k)} + \left(\mathbf{a}^{(k)}\right)^T \Delta\boldsymbol{\sigma}^{(k)} + \left(\partial f / \partial \Delta\lambda\right)^{(k)} \delta\lambda^{(k)} = 0 \\ \mathbf{r}^{(k)} + \left(\partial \mathbf{r} / \partial \boldsymbol{\sigma}\right)^{(k)} \Delta\boldsymbol{\sigma}^{(k)} + \left(\partial \mathbf{r} / \partial \Delta\lambda\right)^{(k)} \delta\lambda^{(k)} = 0 \end{cases}, \tag{25}$$

from which $\Delta\lambda^{(k+1)} = \Delta\lambda^{(k)} + \delta\lambda^{(k)}$ and $\sigma^{(k+1)} = \sigma^{(k)} + \Delta\sigma^{(k)}$ can be calculated. Indices k, $k+1$ refer to the known and the searched iteration in Newton-Raphson method. The required initial values are $\Delta\lambda^{(0)} = \mathbf{0}$ and $\sigma^{(0)} = \sigma^{trial}$. In calculations consistent elastoplastic modulus is used (de Borst 1991), given as

$$d\boldsymbol{\sigma} = \left(\mathbf{H} - \frac{\mathbf{H}\mathbf{a}\mathbf{a}^T\mathbf{H}}{\left(h + \mathbf{a}^T\mathbf{H}\mathbf{a}\right)}\right) d\mathbf{e} = \mathbf{C}_l^{ep} d\mathbf{e}, \tag{26}$$

where

$$\mathbf{H} = \left[\left[\mathbf{C}^e\right]^{-1} - \Delta\lambda \frac{\partial \mathbf{a}}{\partial \boldsymbol{\sigma}}\right]^{-1} \text{ and } h = \frac{d\bar{\sigma}}{d\gamma}. \tag{27}$$

5 NUMERICAL EXAMPLE

In a numerical example we consider a channel-section beam fully clamped at one end and loaded with vertical force at the other end (Fig. 1). It was presented in Chróścielewski et al. (1992) and later studied in e.g. Chróścielewski et al. (2004), Wiśniewski & Turska (2012) with purely elastic material and for instance in Eberlein & Wriggers (1999) with elastoplastic material. Tan & Vu-Quoc (2005) analysed both material

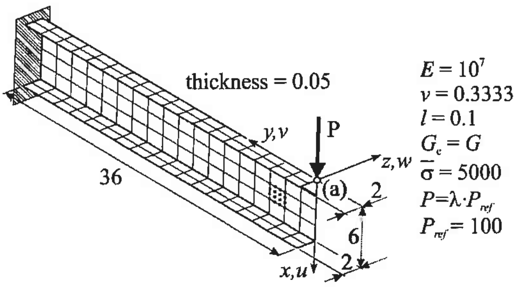

Figure 1. Channel-section beam. Geometry and material.

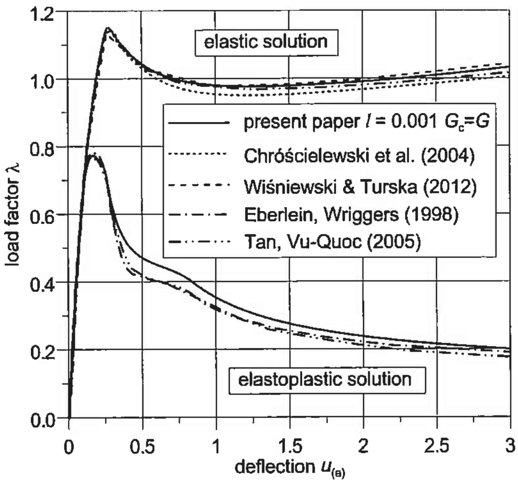

Figure 2. Nonlinear solutions.

constitutive laws. For computations the discretization of $(2+4+2) \times 18$ 16-node CAM (Chróścielewski et al. 1992) finite shell element with full integration in its surface is chosen. Eq. (10)–(12) are integrated with 7-point Gauss quadrature. Figure 2 shows the comparison of nonlinear solutions with purely elastic and perfectly elastoplastic material. Curves from present analysis and reference papers coincide, which proves that theory presented in present paper is reliable in examples including intersections of shell branches.

Deformed configurations for elastic and elastoplastic material are shown in Figure 3. By a fully yielded integration point we mean such integration point on reference surface, where every point through the thickness is yielded. For comparison purposes initial undeformed configuration is plotted with dashed line in Fig. 3.

Influence of constitutive parameters l and G_c was also studied in linear analysis with $P_{\text{ref}} = 1$ (Fig. 4). The figure shows rapid decay of the control displacement values for $l \geq 0.1$. It can be also be observed that values of G_c play negligible role. This observation motivated us to study only the influence of characteristic length l in nonlinear elastoplastic solution (Fig. 5). In connection with Fig. 4 it is seen that some large values of l yield too stiff response of the structure, which is similar to the locking effect. Values of

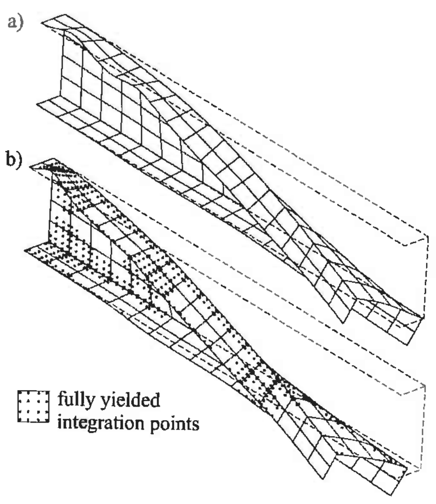

Figure 3. Deformed configurations for $u_{(a)} = 3$. a) pure elastic material, b) elastoplastic material.

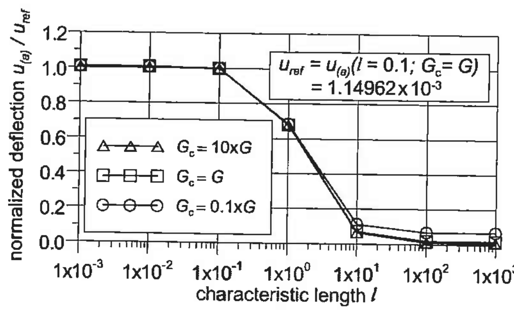

Figure 4. Influence of constitutive parameters on linear solution.

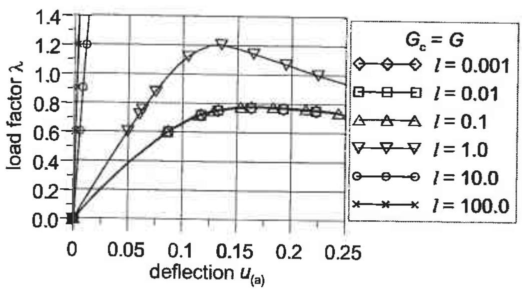

Figure 5. Influence of characteristic length l on nonlinear solution.

$l \leq 0.1$ result in responses with pronounced limit point however its locus depend on value of l. The smallest l values studied result in an almost indistinguishable response curve.

6 CONCLUSION

An original formulation of elastoplastic J_2-type constitutive relation for nonlinear 6-field shell theory with drilling dof has been presented. The required resultant quantities are obtained by through-the-thickness integration of Cosserat plane stress relation. As a consequence, the resulting constitutive relation is dependent on micropolar coefficients l and G_c. Some representative results show that the present formulation provides correct results when compared to the reference solutions.

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