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ON THE MOVEMENT OF GRANULAR MATERIALS IN BINS AND HOPPERS PART I: TWO-DIMENSIONAL PROBLEMS

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The paper is devoted to commemoration
of the contribution of Professor Jerzy Litwiniszyn
to the development of the mechanics of granular media

A simple numerical procedure for determining the field of displacements of granular media in bins and hoppers is proposed. The procedure is based on the concept of Jerzy Litwiniszyn, who treated the gravity flow of granular media as a stochastic process. He and his co-workers presented analytical solutions to a certain number of simple problems concerning deformations of terrain caused by underground mining operations. The numerical procedure proposed in this paper is much more effective in solving numerous problems of practical significance.

1. Introduction

In numerous practical problems, displacements of particles of granular materials are analysed with the use of the method of slip-lines. This method proved to be effective allowing to find theoretical solutions to many problems (see e.g. [1, 2]). However, as indicated by J. Litwiniszyn in his early works [3–5], the theory of limit states and the method of slip-lines not always corresponds to real fields of displacements in granular materials. For example in the sand, in granular materials with relatively large grains contained in bins, or even in some cases of tectonic movements of earth crust when it is composed of separated blocks of rock, the continuity of displacements often cannot be satisfied, because particular elements of such medium move to same degree independently. The same remark may concern the movements of earth masses caused by underground mining works. Such discontinuity of displacements in granular media is especially evident when the movements are caused by the gravity forces (gravity flow).

According to LITWINISZYN's approach [5], the movement of a granular medium is characterized by the mass character of random changes in contacts between the particles of the medium. Consequently the displacements of particles are random. Hence, it is reasonable to analyse the movements of a mass of a granular medium as a random process.

As an introduction to the LITWINISZYN's methodology let us use a well-known demonstrating device (see e.g. [6]), in which small metal balls falling down from a container strike numerous regularly located pins and are randomly directed to the right or to the left with the same probability equal to 0.5. Finally they fall at random into one of separate small containers at the bottom of the device. The distribution of the number of balls in consecutive containers is close to the normal distribution. This is demonstrated in Fig. 1, where the pins are shown as small circles, and the field under consideration is divided into a set of rectangular cells. The ball starting from level I will be displaced to a cell on any

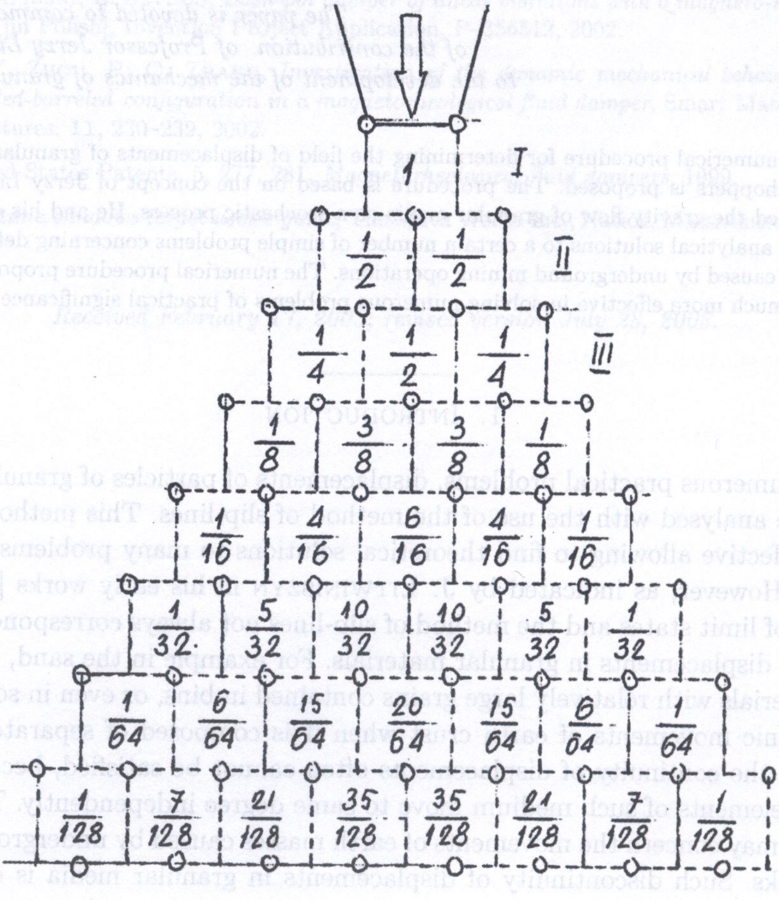

FIG. 1.

other level below. The probability that it falls into the left or into the right-hand cell on level II is equal to 0.5. The probabilities for cells on level III are 1/4, 1/2, 1/4 and similarly for other levels, as shown in the figure.

Let us consider three "cells" A, B, C separated from the model (Fig. 2 – cf. [5]), and, moreover, let us assume that the probability distribution in the model may be described by a continuous function $P(x,y)$, if the distances between the pins are tending to zero ($a \to 0$, $b \to 0$).

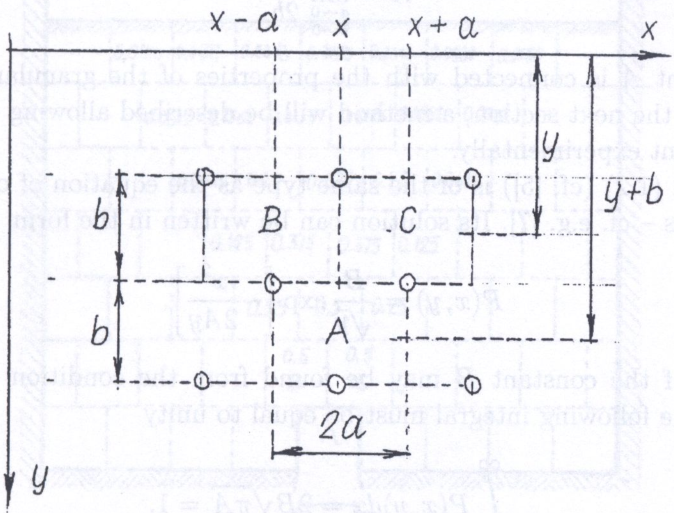

FIG. 2.

Let the probabilities in cells B and C be denoted as

$$P(x-a,y) \quad \text{and} \quad P(x+a,y)$$

respectively. Hence, we may express the momentary probability of migration of balls to the cell A in the form

(1.1) $$P(x,y+b) = \frac{1}{2}P(x-a,y) + \frac{1}{2}P(x+a,y).$$

Then, using Taylor's expansion of all terms of Eq. (1.1) we can write

$$\begin{aligned}
P(x,y+b) - P(x,y) &= b\frac{\partial P(x,y)}{\partial y} + \frac{1}{2}b^2\frac{\partial^2 P(x,y)}{\partial y^2} + \dots,\\
P(x-a,y) - P(x,y) &= -a\frac{\partial P(x,y)}{\partial x} + \frac{1}{2}a^2\frac{\partial^2 P(x,y)}{\partial x^2} + \dots,\\
P(x+a,y) - P(x,y) &= a\frac{\partial P(x,y)}{\partial x} + \frac{1}{2}a^2\frac{\partial^2 P(x,y)}{\partial x^2} + \dots .
\end{aligned}$$

Introducing these expressions to Eq. (1.1) and taking into account only the first terms, we obtain the differential equation

$$\frac{\partial P(x,y)}{\partial y} - A\frac{\partial^2 P(x,y)}{\partial x^2} = 0, \tag{1.2}$$

where

$$A = \lim_{\substack{a\to 0\\ b\to 0}} \frac{a^2}{2b}. \tag{1.3}$$

Coefficient A is connected with the properties of the granular material in question. In the next section, a method will be described allowing to determine this coefficient experimentally.

Equation (1.2) (cf. [5]) is of the same type as the equation of conduction of heat in solids – cf. e.g. [7]. Its solution can be written in the form

$$P(x,y) = \frac{B}{\sqrt{y}} \exp\left[-\frac{x^2}{2Ay}\right]. \tag{1.4}$$

The value of the constant B may be found from the condition that for any $y = \text{const}$ the following integral must be equal to unity

$$\int\limits_{-\infty}^{\infty} P(x,y)dx = 2B\sqrt{\pi A} = 1.$$

Hence

$$B = \frac{1}{2\sqrt{\pi A}}.$$

Thus solution (1.4) can be written in the final form

$$P(x,y) = \frac{1}{2\sqrt{\pi A y}} \exp\left[-\frac{x^2}{2Ay}\right]. \tag{1.5}$$

Alternatively we can write

$$P(x,y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma^2}\right], \tag{1.6}$$

where

$$\sigma = \sqrt{2Ay} \tag{1.7}$$

Relation (1.7) may be used for experimental determination of the parameter A. This problem will be discussed in the next section.

2. The method of finite cells

In the papers mentioned above, J. Litwiniszyn ingeniously analysed the inverse problem in which the cavities in a bulk of a loose material move randomly upwards from the bottom. To illustrate his idea, let us consider a two-dimensional problem of a relatively wide container with an outlet at the middle of the bottom. Figure 3 shows the assumed initial system of finite cells.

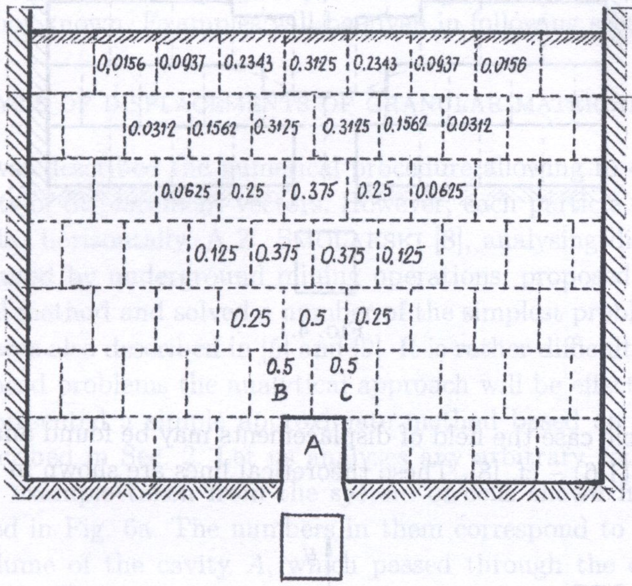

Fig. 3.

A portion of the loose medium has just now left cell A leaving an empty space in it. The cavity in A formed in such a manner migrates upwards. We assume, as in the inverse problem shown in Fig. 1, that each time a portion of that cavity moves upwards, the probability of migrating into the right or into the left-hand cell lying just above is equal to 0.5. It means that at the beginning of the migration process, one half of the initial cavity A moves to the cell B and the other half is shifted to the cell C. If the volume of each cell is assumed to be a unit volume, the numbers in consecutive cells indicate how large portion of the initial unit volume A passed through the cell during the migration process. Since after migration each portion of empty space must be filled by the granular medium falling downwards, these numbers correspond to the average vertical displacement of the medium in particular cells. These vertical displacements are represented in Fig. 4. Thus the approximate field of displacements is shown by the family of stepwise lines.

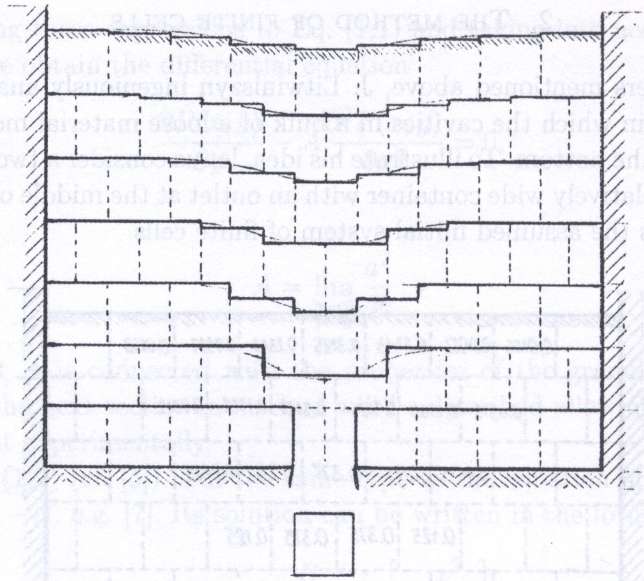

FIG. 4.

In this simple case the field of displacements may be found analytically with the use of Eq. (1.6) – cf. [8]. These theoretical lines are shown in Fig. 5.

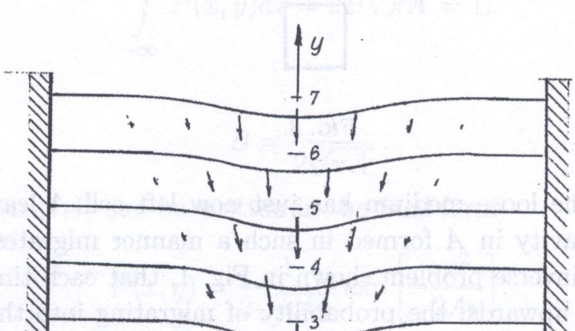

FIG. 5.

Suppose now that such lines have been determined experimentally in a simple testing device in which granular material in question is located in a sufficiently wide box between two glass plates. After measuring the parameters of the experimental lines of deformation, one can calculate the standard deviation, and subsequently the value of the parameter A according to formula (1.7). This value can be used for constructing the system of rectangular cells with dimensions a and b satisfying relation $b = a^2/2A$. Such systems may be used for numerical analysis of more advanced problems of practical significance, for which analytical solutions are not known. Examples will be given in following sections.

3. Analysis of displacements of granular materials in bins

In Sec. 2 was described the numerical procedure allowing to calculate vertical components of displacement vectors. However, each particle of the medium is displaced also horizontally. A.Z. Smolarski [8], analysing displacements of the terrain caused by underground mining operations, proposed a rather complex analytical method and solved a number of the simplest problems only. This methodology was also described in [5] and [9]. It is rather difficult to expect that in more advanced problems the analytical approach will be effective.

Below is presented a simple approximate method based on the finite cells technique described in Sec. 2. Let us analyses any arbitrary set of three adjacent cells, for example taken from the system of cells shown in Fig. 3. They are represented in Fig. 6a. The numbers in them correspond to the fraction of the initial volume of the cavity A, which passed through the cell during the migration towards the free surface of the bulk of the medium. According to the finite cells methodology, only one half of these fractions migrates from each cell A and B to the cell C. It is assumed that this migration takes place along the respective lines A–C or B–C joining central points of the cells. Directions and magnitudes of these migrating portions of the cavity may be represented by vectors $\mathbf{w}_{BC}$ and $\mathbf{w}_{AC}$ as shown in Fig. 6b. They may be treated as components of the resulting vector $\mathbf{w}_{\text{cav}}$ representing the direction and the magnitude of the averaged momentary flux of the cavity into cell C during the migration process. The opposite vector $\mathbf{w}_{\text{mat}}$ may be treated as representation of the flux of the mass of granular medium filling the space left by cavities moving upwards.

In order to calculate the magnitude of the averaged displacement vector $\mathbf{u}$ of the particles of the medium, it is assumed that its direction coincides with the direction of the vector $\mathbf{w}_{\text{mat}}$. To make this procedure consistent with that described in Sec. 2, it is assumed that the vertical component of the displacement vector $\mathbf{u}$ is equal to the vertical displacement of the respective sector of the stepwise deformed boundary between the rows of cells (cf. Fig. 4). Using this approximate numerical procedure, the vectors of displacements have been calculated for the problem shown in Figs. 3 and 4. Results are shown in Fig. 5.

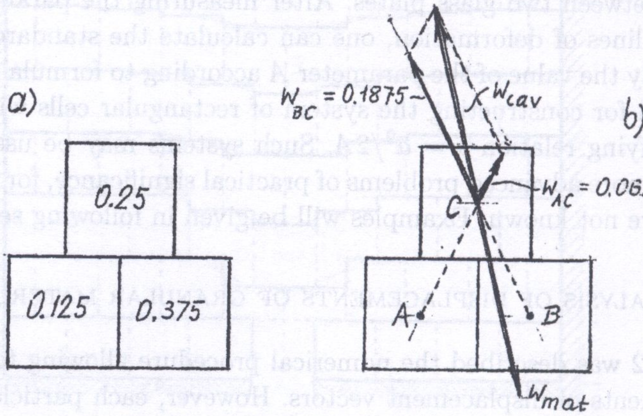

FIG. 6.

4. INFLUENCE OF A RIGID WALL

Suppose now that the bulk of granular material is bounded from one side by a rigid wall. Referring to the basic problem shown in Fig. 1 let us suppose that the wall is located to the left of the initial cell located on level I – Fig. 7. The balls falling down from the cells directly in contact with the wall cannot move to the left. Thus the probability that they move to the right is equal to one. For

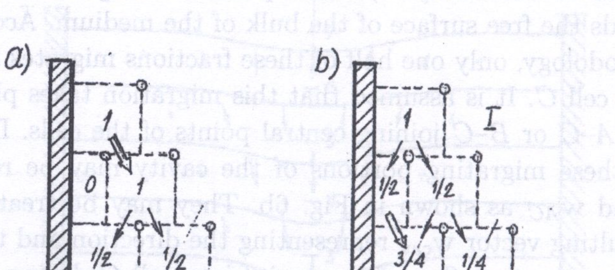

FIG. 7.

small cut in halves cells, two algorithms may be assumed. Either they are assumed to be non-existing (Fig. 7a), or they are treated as regular cells (Fig. 7b). The differences between final resu1ts of calculations according to the first or the second variant are small, especially for increasing number of levels in the system of cells.

5. Examples of application

As a typical example let us analyse displacements in a relatively high bin with the relation of dimensions shown in Fig. 8. This problem and the other one discussed in this section will be treated as a two-dimensional problem with the

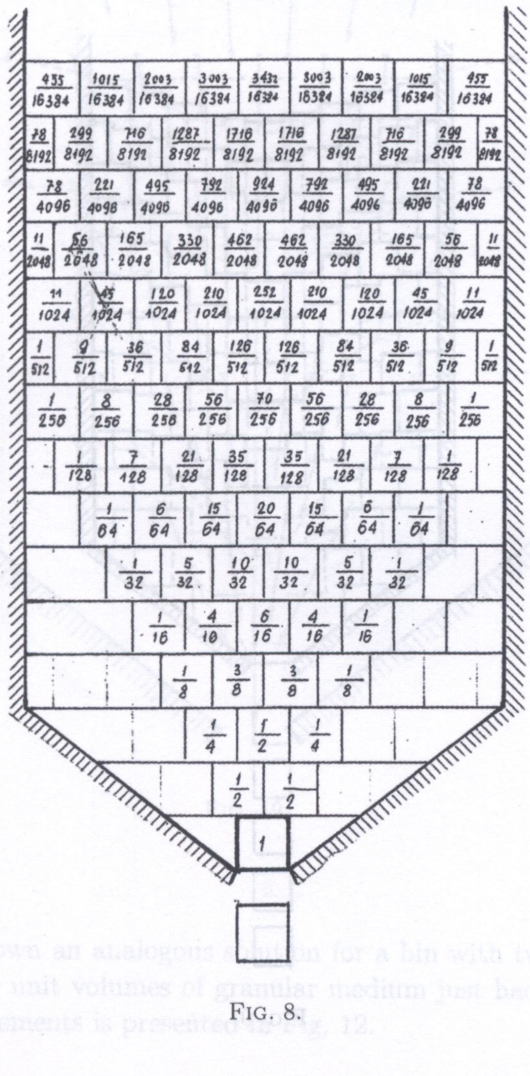

Fig. 8.

use of the finite cells method described in the previous sections. For the problem in question the assumed system of cells is shown in the figure. It is assumed that the portion of granular material filling the bin just left the lowest cell at the bottom. The numbers in particular cells indicate how large is the fraction of so formed cavity of unit volume passing through the cell during the process of migration upwards. For cells adjacent to the rigid wall the scheme shown in Fig. 7b has been assumed.

Figure 9 presents the deformed boundaries between the horizontal rows of cells calculated for the case when five unit volumes of the medium left the outlet at the bottom. The vectors of displacements calculated according to the approximate procedure described in Sec. 3 are shown in Fig. 10. It is seen that the granular medium flows mainly in the central part of the bin and that a depression

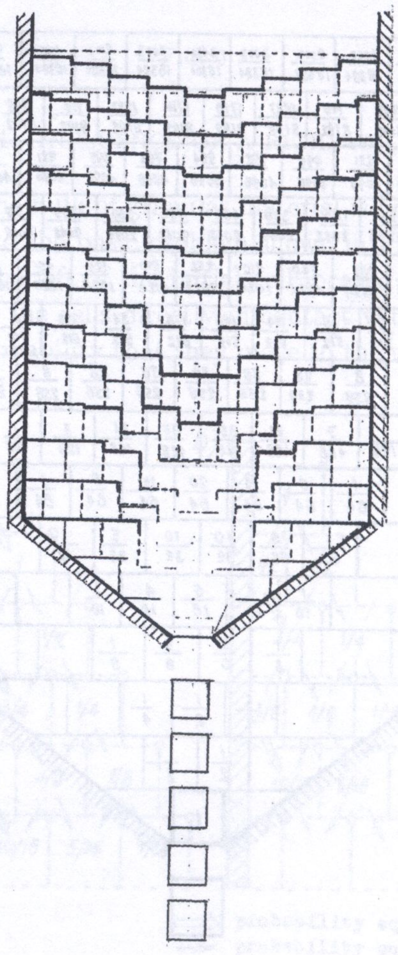

FIG. 9.

is formed at the center of the top of the filling. This theoretical result is confirmed by experimental investigations – see e.g. [10, 11].

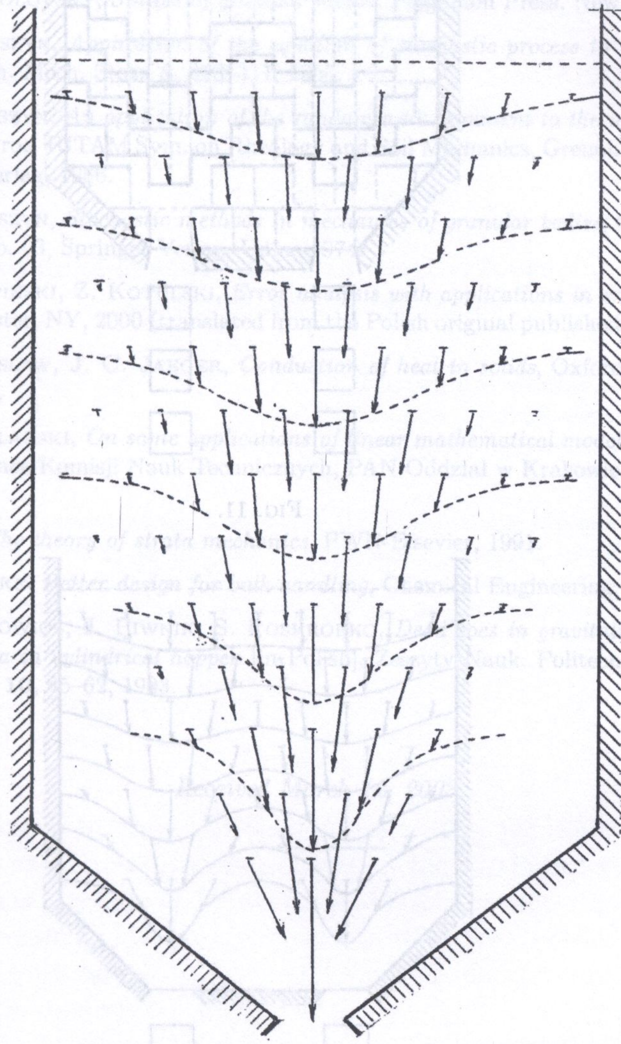

FIG. 10.

In Fig. 11 is shown an analogous solution for a bin with two outlets. It was assumed that three unit volumes of granular medium just had left each outlet. The field of displacements is presented in Fig. 12.

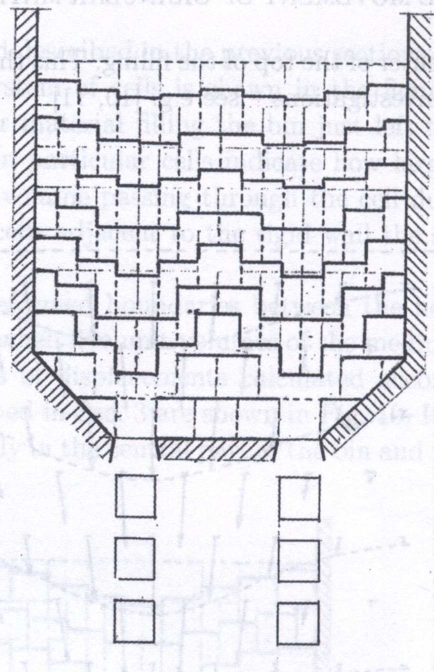

Fig. 11.

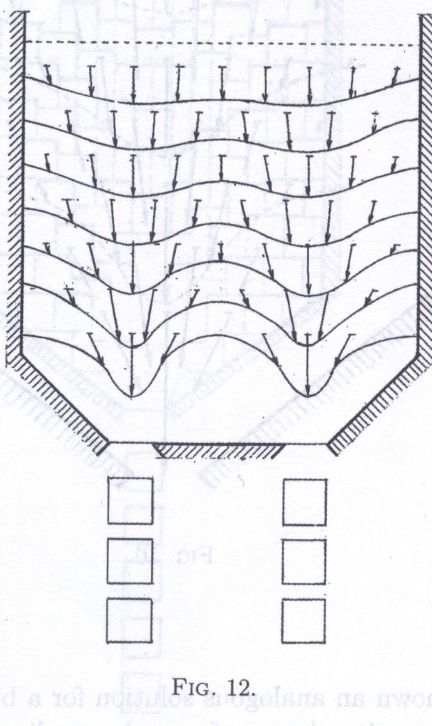

Fig. 12.

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