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ROLLING CONTACT OF LONG ELASTIC CYLINDERS WITH SURFACE ROUGHNESS DESCRIBED BY A TWO PARAMETER MODEL

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Rolling contact problem for rough cylinders is considered. A new model of the surface roughness is proposed. The problem is reduced to the system of singular integral equations which is solved numerically.

Key words: rolling contact, surface roughness, integral equation.

1. Introduction

The stationary tractive rolling of two long elastic cylinders is considered, Fig. 1. Classical formulation of this kind of problems [1] neglects the surface roughness existing in real engineering bodies. Recently [2], Pauk and Zastrau proposed a phenomenological model of the boundary roughness, which is suitable for the consideration of rolling contact problems.

In this paper, a structural model of the surface roughness is proposed (Sec. 2), on the basis of which the system of integral equations for the rolling contact problem is derived (Sec. 3). In Sec. 4, the numerical scheme for the solution

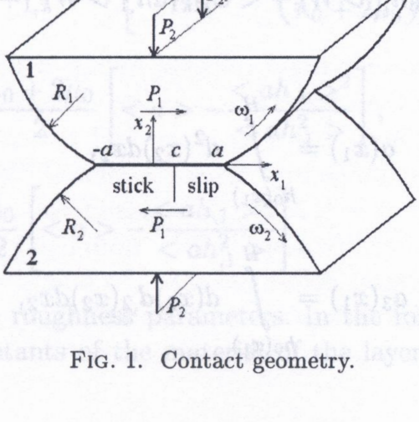

Fig. 1. Contact geometry.

of the obtained integral equations is described and some numerical results are presented.

2. Model of surface roughness

When two elastic rough bodies are in the rolling contact, the total displacements include bulk parts and some additional ones due to the deformation of the boundary roughness. To describe these additional displacements we will use the approach proposed in [3] which is based on certain concepts applied in the investigation of periodic composite materials [4].

We consider the plane deformation of a subsurface layer which occupies the region

$$\Omega = \{(x_1, x_2) : |x_1| < \infty, \quad h_0(x_1) < x_2 < H\}, \tag{2.1}$$

where H is the mean thickness of the layer, $h_0(x_1)$ is the l-periodic function describing the distribution of geometrical imperfection (roughness) over the upper surface of the layer. This surface is subjected in the area $|x_1| < a$ to the action of the tangential $p_1(x_1)$ and normal $p_2(x_1)$ traction. The material properties of the layer under consideration are described by the elasticity tensor C_{ijkl} which can also be considered as a l-periodic function.

Omitting here the modelling procedure (it is similar to that performed in [3]), we obtain the general form of the governing equations

$$R_i - S_{i,1} = p_i \qquad H_i = 0, \tag{2.2}$$

where

$$R_i = < C_{i2k2} a_{33} > W_k + < C_{i2k1} a_3 > W_{k,1} + < C_{i2k1} a_3 h_{,1} > V_k, \tag{2.3}$$

$$S_i = < C_{i1k2} a_3 > W_k + < C_{i1k1} a > W_{k,1} + < C_{i1k1} a h_{,1} > V_k, \tag{2.4}$$

$$H_i = < C_{i1k2} a_3 h_{,1} > W_k + < C_{i1k1} a h_{,1} > W_{k,1} + < C_{i1k1} a h_{,1}^2 > V_k, \tag{2.5}$$

and

$$a(x_1) = \int_{h_0(x_1)}^{H} d^2(x_2) dx_2, \tag{2.6}$$

$$a_3(x_1) = \int_{h_0(x_1)}^{H} d(x_2) d_{,2}(x_2) dx_2, \tag{2.7}$$

$$a_{33}(x_1) = \int_{h_0(x_1)}^{H} d_{,2}^2(x_2)dx_2. \tag{2.8}$$

Here $W_i(x_1)$, $V_i(x_1)$ are displacements of the upper boundary of the considered layer and some unknown functions called the correctors [4]; $d(x_2)$ is an arbitrary function defined on $(0,H)$ satisfying conditions $d(0) = 1, d(H) = 0, d_{,2}(x_2) < 0$; $h(x_1)$ is the given l-periodic function known as micro-shape function [4]. Moreover, the averaging operator is defined as

$$< f > = \frac{1}{l}\int_0^l f(x_1)dx_1. \tag{2.9}$$

In formulas (2.2)–(2.5) and in subsequence the subscripts i, j, k,... run over 1,2 and the summation convention holds. The symbol $f_{,i}$ means the derivative of the function $f(x_1, x_2)$ with respect to the variable x_i.

After some calculations and additional assumptions on the function $h(x_1)$ and $d(x_2)$, we obtain the final form of differential equations for unknown tangential $W_1(x_1)$ and normal $W_2(x_1)$ displacements of the rough boundary

$$\begin{aligned} k^{(1)}W_1(x_1) - 2t^{(1)}W_{1,11}(x_1) &= p_1(x_1), \\ k^{(2)}W_2(x_1) - 2t^{(2)}W_{2,11}(x_1) &= p_2(x_1), \end{aligned} \tag{2.10}$$

where the constant coefficients in these differential equations have the following forms

$$k^{(1)} = \mu_0\left[< a_{33} > - \frac{< a_3 h_{,1} >^2}{< a h_{,1}^2 >}\right], \tag{2.11}$$

$$k^{(2)} = (\lambda_0 + 2\mu_0)\left[< a_{33} > - \left(\frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0}\right)^2 \frac{< a_3 h_1 >^2}{< a h_1^2 >}\right], \tag{2.12}$$

$$t^{(1)} = \frac{\lambda_0 + 2\mu_0}{2}\left[< a > - \frac{< a h_{,1} >^2}{< a h_{,1}^2 >}\right], \tag{2.13}$$

$$t^{(2)} = \frac{\mu_0}{2}\left[< a > - \frac{< a h_{,1} >^2}{< a h_{,1}^2 >}\right] \tag{2.14}$$

and can be called the roughness parameters. In the formulas (2.11)–(2.14) λ_0 and μ_0 are Lamé constants of the material of the layer which is now assumed to be isotropic and homogeneous. Note that a phenomenological model of the boundary roughness proposed in [2] can be obtained directly from (2.10) by setting $t^{(1)} = t^{(2)} = 0$.

The equations (2.10) have the solutions bounded for $|x_1| \to \infty$ in the forms

$$(2.15) \quad W_i(x_1) = \frac{1}{4t^{(i)}} \int_{-a}^{a} p_i(s) \exp(-\alpha^{(i)} |s - x_1|) \mathrm{sgn}(x_1 - s) ds, \qquad i = 1, 2,$$

where

$$\alpha^{(i)} = \sqrt{k^{(i)}/2t^{(i)}}, \qquad i = 1, 2.$$

The formulas (2.15) describe the relations between the normal and the tangential traction and the corresponding deformation of periodically distributed roughness.

3. System of integral equations

The boundary conditions of the plane rolling contact under assumption of quasi-identical materials of rollers have the following forms [5]

$$(3.1) \qquad V_{,1}^{(1)}(x_1) + V_{,1}^{(2)}(x_1) = -\frac{x_1}{R}, \qquad -a < x_1 < a,$$

$$(3.2) \qquad v(x_1) = 0, \qquad -a < x_1 < c,$$

$$(3.3) \qquad |p_1(x_1)| < f\,|p_2(x_1)|, \qquad -a < x_1 < c,$$

$$(3.4) \qquad v(x_1) > 0, \qquad c < x_1 < a,$$

$$(3.5) \qquad |p_1(x_1)| = f\,|p_2(x_1)|, \qquad c < x_1 < a,$$

where

$$(3.6) \qquad v(x_1) = \xi_1 - [U_{,1}^{(1)}(x_1) - U_{,1}^{(2)}(x_1)], \quad -a < x_1 < a$$

is the relative velocity between the surfaces of rolling cylinders; f is the Coulomb friction coefficient; ξ_1 is the creep ratio; c is the unknown point separating the stick ($-a < x < c$) and slip ($c < x < a$) zones; and $R = R_1 R_2/(R_1 + R_2)$.

Normal $V^{(\alpha)}(x_1)$ and tangential displacements $U^{(\alpha)}(x_1)$ of the surfaces of two contacting cylinders ($\alpha = 1, 2$) are the sums of two terms: first ones, due to the roughness deformation, are described by formulas (2.15) and second parts are results of the bulk elastic deformation of the bodies. The latter ones under the Hertz assumptions for rollers can be obtained as solutions of the elasticity equations [5]. After some manipulations we obtain

$$(3.7)\quad V_{,1}^{(1)}(x_1)+V_{,1}^{(2)}(x_1)=\frac{2(1-\nu)}{\pi\mu}\int_{-a}^{a}\frac{p_2(s)ds}{s-x_1}+\int_{-a}^{a}p_2(s)K_2(s-x)ds,$$

$$(3.8)\quad v(x_1)=\xi_1-\frac{2(1-\nu)}{\pi\mu}\int_{-a}^{a}\frac{p_1(s)ds}{s-x_1}-\int_{-a}^{a}p_1(s)K_1(s-x_1)ds,$$

where

$$(3.9)\quad K_1(z)=-\left[\frac{\exp(-\alpha_1^{(1)}|z|)}{4t_1^{(1)}}+\frac{\exp(-\alpha_2^{(1)}|z|)}{4t_2^{(1)}}\right]\operatorname{sgn}(z),$$

$$(3.10)\quad K_2(z)=-\left[\frac{\exp(-\alpha_1^{(2)}|z|)}{4t_1^{(2)}}+\frac{\exp(-\alpha_2^{(2)}|z|)}{4t_2^{(2)}}\right]\operatorname{sgn}(z).$$

Here ν, μ are respectively, Poisson's ratio and shear modulus of the material of cylinders; $\alpha_\beta^{(i)}=\sqrt{k_\beta^{(i)}/2t_\beta^{(i)}}$ and $k_\beta^{(i)}$, $t_\beta^{(i)}$ are the roughness parameters of two rollers ($\beta=1,2$), defined by (2.11)–(2.14).

Satisfying the boundary conditions (3.1), (3.2) with the help of (3.7), (3.8) we arrive at the integral equations

$$(3.11)\quad \frac{2(1-\nu)}{\pi\mu}\int_{-a}^{a}\frac{p_2(s)ds}{s-x_1}+\int_{-a}^{a}p_2(s)K_2(s-x_1)ds=-\frac{x_1}{R},\qquad -a<x_1<a,$$

$$(3.12)\quad \frac{2(1-\nu)}{\pi\mu}\int_{-a}^{a}\frac{p_1(s)ds}{s-x_1}+\int_{-a}^{a}p_1(s)K_1(s-x_1)ds=\xi_1,\qquad -a<x_1<c,$$

which have to be completed by the equilibrium conditions

$$(3.13)\quad \int_{-a}^{a}p_1(s)ds=P_1,$$

$$(3.14)\quad \int_{-a}^{a}p_2(s)ds=P_2,$$

where P_2 is the unit normal load and P_1 is the unit shearing load transmitted by the rollers.

To satisfy the remaining boundary conditions (3.3)–(3.5) the function $p_1(x_1)$ is assumed in the form

$$p_1(x_1) = f p_2(x_1) + \begin{cases} q_0(x_1), & -a < x_1 < c, \\ 0, & c < x_1 < a, \end{cases} \tag{3.15}$$

where $q_0(x_1)$ is an unknown corrective traction.

After substituting the formula (3.15) into (3.12) and (3.13) and after some calculations, we arrive at the integral equations

$$\frac{2(1-\nu)}{\pi\mu} \int\limits_{-a}^{c} \frac{q_0(s)ds}{s - x_1} + \int\limits_{-a}^{c} q_0(s) K_1(s - x_1) ds = \xi_1 - f G[x_1, p_2(x_1)], \qquad -a < x_1 < c, \tag{3.16}$$

$$\int\limits_{-a}^{c} q_0(s) ds = P_1 - f P_2. \tag{3.17}$$

where we have denoted

$$G[x_1, p_2(x_1)] = \frac{x_1}{R} + \int\limits_{-a}^{a} p_2(s)[K_2(s - x_1) - K_1(s - x_1)] ds. \tag{3.18}$$

The equations (3.11), (3.14) are sufficient to find the distribution of the normal traction $p_2(x_1)$ and after this it is possible to solve the tangential problem (3.16), (3.17) and determine the corrective traction $q_0(x_1)$.

Introducing the dimensionless variables

$$r = x_1/a, \qquad \eta = s/a$$

in the equations (3.11), (3.14) and

$$t = (x_1 + r_1)/r_0, \qquad \tau = (s + r_1)/r_0, \qquad r_0 = (a + c)/2, \qquad r_0 = (a - c)/2$$

in the equations (3.16), (3.17) as well as the dimensionless parameters and functions

$$\xi = \frac{a\mu\xi_1}{2(1-\nu)P_2}, \quad c_0 = \frac{c}{a}, \quad Q = \frac{P_1}{P_2}, \quad p(r) = \frac{a p_2(x_1)}{P_2}, \quad q(t) = \frac{a p_1(x_1)}{P_2},$$

we obtain the dimensionless form of the governing integral equations

$$\frac{1}{\pi} \int\limits_{-1}^{1} \frac{p(\eta) d\eta}{\eta - r} + \int\limits_{-1}^{1} p(\eta) K_2^*(\eta - r) d\eta = \frac{2}{\pi} \frac{P_H}{P_2} \frac{a^2}{a_H^2} r, \qquad -1 < r < 1, \tag{3.19}$$

(3.20) $$\int_{-1}^{1} p(r)dr = 1,$$

(3.21) $$\frac{1}{\pi}\int_{-1}^{1}\frac{q(\tau)d\tau}{\tau - t} + \int_{-1}^{1} q(\tau)K_1^*(\tau - t)d\tau = \xi - fG^*[t, p(t)], \qquad -1 < t < 1,$$

(3.22) $$\frac{1+c_0}{2}\int_{-1}^{1} q(t)dt = Q - f,$$

where

(3.23) $$K_1^*(z) = -\left[T_1^{(2.1)}\exp(-T_1^{(2.1)}|z|) + T_2^{(2.1)}\exp(-A_2^{(2.1)}|z|)\right]\operatorname{sgn}(z),$$

(3.24) $$K_2^*(z) = -\left[T_1^{(2.2)}\exp(-T_1^{(2.2)}|z|) + T_2^{(2.2)}\exp(-A_2^{(2.2)}|z|)\right]\operatorname{sgn}(z),$$

(3.25) $$G[t, p(t)] = \frac{2}{\pi}\frac{P_H}{P_2}\frac{a^2}{a_H^2}t^* + \int_{-1}^{1} p(\eta)[K_2^*(\eta - t^*) - K_1^*(\eta - t^*)]d\eta,$$

$$t^* = [(1+c_0)t + (1-c_0)]/2,$$

(3.26) $$T_\beta^{(i)} = \frac{a\mu}{8(1-\nu)t_\beta^{(i)}}, \quad A_\beta^{(i)} = a\alpha_\beta^{(i)}, \quad i, \beta = 1, 2.$$

Here a_H and P_H are, respectively, the contact area size and the normal load in the Hertz problem [5]

(3.27) $$a_H^2 = \frac{4(1-\nu)RP_H}{\pi\mu}.$$

In the further analysis the contact area size in the considered problem is assumed to be equal to that in the Hertz problem, i.e. $a/a_H = 1$, but the ratio P_H/P_2 has to be determined.

4. Numerical solution of integral equations

The integral equations (3.19), (3.21) are of Cauchy type. From the physical reasons the unknown functions $p(r)$ and $q(t)$ are looked for in the class of bounded functions. They are presented in the following forms:

(4.1) $$p(r) = \varphi(r)\sqrt{1-r^2}, \qquad q(t) = \psi(t)\sqrt{1-t^2},$$

where $\varphi(r)$ and $\psi(t)$ are new unknown regular functions.

Applying the Gauss–Chebyshev quadrature [6], the system of singular integral equations (3.19)–(3.22) is reduced to the linear algebraic equations

$$(4.2)\qquad \gamma_{0n} + \frac{1}{\pi}\sum_{k=1}^{n}\frac{w_k\varphi(\eta_k)}{\eta_k - s_m} + \sum_{k=1}^{n} w_k\varphi(\eta_k)K_2^*(\eta_k - s_m) + \frac{2}{\pi}\frac{P_H}{P_2}s_m = 0,$$

$$m = 1,...,n+1,$$

$$(4.3)\qquad \sum_{k=1}^{n} w_k\varphi(\eta_k) = 1,$$

$$(4.4)\qquad \frac{1}{\pi}\sum_{k=1}^{n}\frac{w_k\psi(\eta_k)}{\eta_k - s_m} + \sum_{k=1}^{n} w_k\psi(\eta_k)K_1^*(\eta_k - s_m) - \xi = fG^*\left[s_m, \varphi(s_m)\right],$$

$$m = 1,...,n+1,$$

$$(4.5)\qquad \frac{1+c_0}{2}\sum_{k=1}^{n} w_k\psi(\eta_k) = Q - f,$$

where the collocation points s_m, integration points η_k and the weight coefficients w_k are given by formulas [6]

$$(4.6)\qquad \eta_k = \cos\frac{k\pi}{n+1}, \qquad w_k = \frac{\pi}{n+1}\sin^2\frac{k\pi}{n+1}, \qquad k = 1,...,n,$$

$$(4.7)\qquad s_m = \cos\frac{(2m-1)\pi}{2n+2}, \qquad m = 1,...,n+1.$$

The regularized parameter γ_{0n} introduced in (4.2) provides the condition [6]

$$(4.8)\qquad \lim_{n\to\infty}\gamma_{0n} = 0.$$

The $(n{+}2)$ linear algebraic equations (4.2), (4.3) are sufficient for the determination of $(n{+}2)$ unknowns $\varphi(\eta_k)$, $k = 1,..,n$; γ_{0n}; P_H/P_2. Then it is possible to solve $(n+2)$ equations (4.4), (4.5) and determine $(n+2)$ unknowns $\psi(\eta_k)$, $k = 1,..,n$; ξ; c_0. These systems were solved by standard computational methods.

The input parameters for the calculation are: the friction coefficient f; dimensionless tangential load Q and the roughness parameters defined by the formulas (3.26). To reduce the number of input parameters, the surface of the second cylinder is assumed to be smooth: $T_2^{(i)} = A_2^{(i)} = 0$ $(i = 1, 2)$.

The most important characteristic in the rolling contact is the relation between the creep ξ and the tangential load Q. These relations, known as creep force-creep curves are presented in Fig. 2 for some values of the roughness parameters. The well-known analytical solution for the smooth cylinders [5] is drawn for comparison. We can conclude that the creep between rough rollers is higher than that between smooth bodies. This was property confirmed experimentally [7].

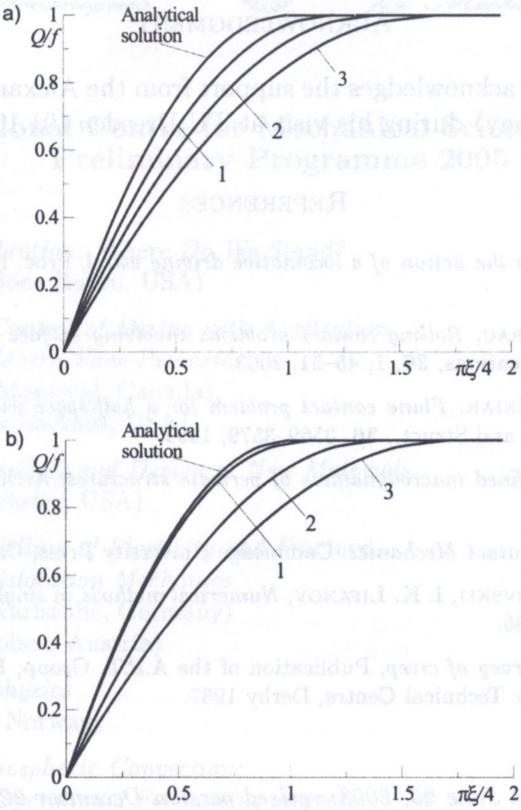

FIG. 2. a) Creep force-creep curves for $A_1^{(1)} = A_1^{(2)} = 0.5$ and different values of $T_1^{(1)}$, $T_1^{(2)}$: curve 1 - $T_1^{(1)} = T_1^{(2)} = 0$; curve 2 - $T_1^{(1)} = T_1^{(2)} = 0.5$; curve 3 - $T_1^{(1)} = T_1^{(2)} = 1$.
b) Creep force-creep curves for $T_1^{(1)} = T_1^{(2)} = 0.5$ and different values of $A_1^{(1)}$, $A_1^{(2)}$: curve 1 - $A_1^{(1)} = A_1^{(2)} = 0$; curve 2 - $A_1^{(1)} = A_1^{(2)} = 0.5$; curve 3 - $A_1^{(1)} = A_1^{(2)} = 1$.

5. Final remarks

The new model of the surface roughness is proposed and applied to the study of the rolling contact of rough bodies. To be able to use this model in the engineering practice we need to know the values of the roughness parameters (2.11)–(2.14) or the corresponding dimensionless values (3.26). From the relations (2.11)–(2.14) and (2.6)–(2.8), it is clear that the roughness parameters depend on the material properties λ_0, μ_0 of the subsurface layer, on the thickness of this layer, on the shape of surface imperfections as well as on some given functions $h(x_1)$ and $d(x_2)$. So, it is possible to calculate the roughness parameters for real engineering surfaces, characteristics of which can be found experimentally.

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