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ALGORITHMS OF THE METHOD OF STATICALLY ADMISSIBLE DISCONTINUOUS STRESS FIELDS (SADSF) – PART II

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Summary of the whole paper: By now, the SADSF method is practically the only tool of shape design of complex machine elements that provides an effective solution even to the problems of 3D distribution of the material, and at the same time it is still enough user-friendly to be useful for engineers. This unique property of the method is due to the existence of its simple application version. When using it, a design engineer does not need to solve by oneself any statically admissible field – which could be very difficult – but obtains such a solution by assembling various ready-made particular solutions. The latter are in general obtained by means of individual and complex analyses and provided to a designer in a form of libraries.

The algorithms presented in this paper break up with the individual approach to a particular field. The algorithms are the first ones of general character, as they apply to the fundamental problems of the method. The algorithms enable solving practically any boundary problem that one encounters in constructing 2D statically admissible, discontinuous stress fields, first of all the limit fields. In the presented approach, one deals first with the fields arising around isolated nodes of stress discontinuity lines (Parts II and III), then integrates these fields into 2D complex fields (Part IV).

The software, created on the basis of the algorithms, among other things, allows one to find all the existing solutions of the discontinuity line systems and present them in a graphical form. It gives the possibility of analysing, updating and correcting these systems. In this way, it overcomes the greatest difficulty of the SADSF method following from the fact that the systems of discontinuity lines are not known a priori, and appropriate relationships are not known either, so that they could be found only in an arduous way by postulating the line systems, and verifying them.

Application version of the SADSF method is not described in this paper; however, a reference is given to inform the reader where it can be found.

PART II

THE ALGORITHMS FOR SOLVING LIMIT STRESS FIELDS AROUND ISOLATED NODES OF STRESS DISCONTINUITY LINES

Summary of Part II: In the paper, the author introduces the sets of conditions that create the algorithms of the functions on which one defines the boundary problems met in the search for discontinuous limit fields existing around isolated nodes. Among those, there are functions describing states of stress in the component homogeneous regions, the parameters of lines that separate these regions, and, first of all, the formulae for determining the domains based on the general conditions of existence. These formulae play a key role in numerical implementations of the method.

The fields satisfying the Huber–Mises yield condition are of primary choice however, the derived relationships have a general meaning. To emphasise this fact one presents not only the areas of existence valid for the Huber–Mises condition, but also the areas obtained for several other yield conditions applicable to plastically homogeneous materials. The knowledge of the areas opens the possibility of developing the method of search for the fields that obey these conditions, and for algorithmizing this method. This could be applied even for the fields that are characterised by arbitrary, admissible states of stress.

One also presents, basing on a mathematically complete set of conditions, typical formulations of problems concerning the fields around the nodes. One discusses the balance between the set of conditions and the unknowns, as well as the transformations into global systems connected with complex fields.

One consequently applies parametrisation of the yield conditions, which not only reduces the number of unknowns and leads to simple, recursive forms of the formulae, but, first of all, makes it possible to find the formulae for generation of domains, without which numerical solution of the fields and algorithmization of the method would not be possible at all.

Key words: shape design, limit analysis, numerical methods.

6. Convergent system of stress discontinuity lines on a plane

6.1. *Introductory remarks*

In the accepted hierarchy, the fields arising around the nodes of stress discontinuity lines are treated as fundamental component units of complex fields. However, in the problem considered here, these will be also treated as isolated fragments of such fields, which means that the analysis of field interaction around the neighbouring nodes will be omitted.

In construction of the formulae presented in this paper, one refers to the works [2] that give grounds for the formulae, and then present particular forms of them – the forms valid only for the Huber–Mises condition. In this paper, this condition is also treated as the preferred one, however, the formulae derived here have a general character, independent of the assumed yield condition.

The considerations concerning segments of a straight line can be, in a natural way, adapted to the elements ds of curves [3]. The presented analyses of fields around the nodes can then be referred also to infinitesimally small neighbourhoods of nodes of curves.

The symbols and denotations accepted here are almost identical with those used in the software implementation created on the basis of fragments of the presented algorithms.

6.2. *Description of the field around the node in a global system*

Figure 4 shows a general sketch of the field created around an isolated node of stress discontinuity line. It is assumed that its component homogeneous regions are numbered by consecutive natural numbers, $\alpha = 1..N$, in ascending order, anticlockwise. In each region, there exists a planar state of stress, defined either by the components $\overset{(\alpha)}{\sigma}_{ij}$ $(i, j = 1, 2)$, or the principal stresses and the principal stress angle $\overset{(\alpha)}{\sigma}_i$, $\overset{(\alpha)}{\phi}$ $(i = 1, 2)$. When one assumes the limit state and parametrisation of the yield condition, these are defined by: $\overset{(\alpha)}{\omega}$, $\overset{(\alpha)}{\phi}$ ([2]).

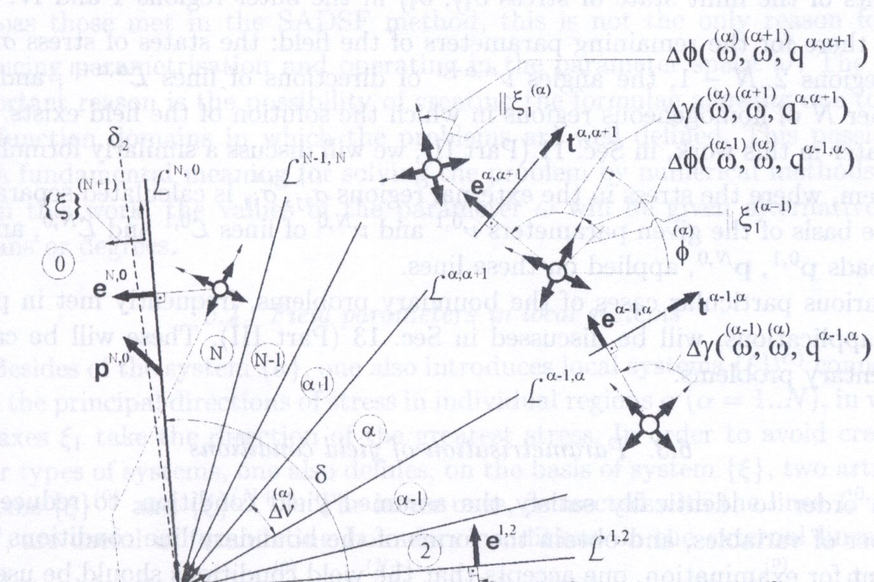

FIG. 4. General diagram of convergent stress discontinuity line system.

The external fields are traditionally denoted by index 0. Accordingly, the internal lines of the field that separate the previous region α from the next one $\alpha + 1$ are denoted by $\mathcal{L}^{\alpha,\alpha+1}$, while the external lines are denoted by $\mathcal{L}^{0,1}$ and $\mathcal{L}^{N,0}$ respectively. It is assumed that external loads $\mathbf{p}^{0,1}$ and $\mathbf{p}^{N,0}$ are applied on these lines. The case when loads $\mathbf{p}^{\alpha,\alpha+1}$ are applied along the internal stress discontinuity lines is considered separately in Part III.

The whole field is defined in an orthogonal system of co-ordinates $\{a\}$ called the global system, in contrast to the local systems $\{\xi\}^{(\alpha)}$ (introduced in further part of this paper), which are connected with the principal directions of stress in each of the homogeneous regions. The versors $\mathbf{t}^{\alpha,\beta}$ of discontinuity lines $\mathcal{L}^{\alpha,\beta}$ that separate the adjacent regions α and β are defined by the angles $\nu^{\alpha,\beta}$ measured with reference to the axis a_1 of the system $\{a\}$. Notice that for internal lines there is $\beta = \alpha + 1$, while for external lines the angles are $\nu^{0,1}$ and $\nu^{N,0}$, respectively.

In one of the most typical cases of the problem formulated for the fields surrounding the nodes, mentioned in Part I of this work, the data are the components of the limit state of stress $\overset{(1)}{\sigma}_{ij}$, $\overset{(N)}{\sigma}_{ij}$ in the outer regions 1 and N. One seeks there for the remaining parameters of the field: the states of stress $\overset{(\alpha)}{\sigma}_{ij}$ in the regions $2..N-1$, the angles $\nu^{\alpha,\alpha+1}$ of directions of lines $\mathcal{L}^{\alpha,\alpha+1}$, and the number N of homogeneous regions in which the solution of the field exists.

Later in this work, in Sec. 11 (Part II), we will discuss a similarly formulated problem, where the stress in the external regions $\overset{(1)}{\sigma}_{ij}$, $\overset{(N)}{\sigma}_{ij}$ is calculated separately on the basis of the given parameters $\nu^{0,1}$ and $\nu^{N,0}$ of lines $\mathcal{L}^{0,1}$ and $\mathcal{L}^{N,0}$, and of the loads $\mathbf{p}^{0,1}$, $\mathbf{p}^{N,0}$, applied on these lines.

Various particular cases of the boundary problems, frequently met in practical applications, will be discussed in Sec. 13 (Part III). These will be called elementary problems.

6.3. Parametrisation of yield conditions

In order to identically satisfy the assumed yield condition, to reduce the number of variables, and obtain the forms of the boundary line conditions convenient for examination, one accepts that the yield conditions should be used in a parametrised form. For the Huber–Mises condition, treated in this work as the preferable one, the author found it convenient to apply an analytical parametrisation formerly used by NADAI ([1]):

$$(6.1)\qquad \overset{(\alpha)}{\sigma}_1 = k\left(\sqrt{3}\cos\overset{(\alpha)}{\omega} + \sin\overset{(\alpha)}{\omega}\right), \quad \overset{(\alpha)}{\sigma}_2 = k\left(\sqrt{3}\cos\overset{(\alpha)}{\omega} - \sin\overset{(\alpha)}{\omega}\right) \quad \left(k = \frac{\sigma_{\mathrm{pl}}}{\sqrt{3}}\right).$$

In the space of principal stresses, the parameter ω can be interpreted as a certain angle measured clockwise from the straight line $\sigma_1 = \sigma_2$ (see Fig. 7a), although, in the case of Huber–Mises condition, this angle has not any geometrical interpretation.

In cases of other yield conditions, the parameter ω should be treated in a similar way (Figs. 8a to 10a), and the parametrisation may be assumed in a more general form:

$$(6.2)\qquad \overset{(\alpha)}{\sigma}_i = \sigma_{\mathrm{pl}}\, f_i(\overset{(\alpha)}{\omega}) \qquad (i = 1, 2).$$

One can go even further. Having additionally introduced stress multipliers $\overset{(\alpha)}{m}$ that take values from the interval [0,1], one consequently assumes parametrisation in the form:

$$\overset{(\alpha)}{\sigma}_i = \overset{(\alpha)}{m}\,\sigma_{\mathrm{pl}}\, f_i(\overset{(\alpha)}{\omega}) \qquad (i = 1, 2), \tag{6.3}$$

which makes it possible to take into account also the discontinuity lines that separate arbitrary states of stress, not necessarily the limit states.

Although the possibility of reducing the number of variables and the number of conditions is very important for solving complicated nonlinear problems such as those met in the SADSF method, this is not the only reason for introducing parametrisation and operating in the parameter space $\overset{(\alpha)}{\omega}$. The most important reason is the possibility of creating the formulae allowing one to find the function domains in which the problems are well defined. This possibility has a fundamental meaning for solving the problem by numerical methods.

In this work, the values of the parameter ω will be given alternatively in radians or degrees.

6.4. Field parameters in local systems

Besides of the system $\{a\}$, one also introduces local systems $\{\xi\}^{(\alpha)}$ connected with the principal directions of stress in individual regions α ($\alpha = 1..N$), in which the axes ξ_1 take the direction of the greatest stress. In order to avoid creating other types of systems, one also defines, on the basis of system $\{\xi\}$, two artificial systems $\{\xi\}^{(0)}$ and $\{\xi\}^{(N+1)}$. The latter ones, connected with the lines $\mathcal{L}^{0,1}$ and $\mathcal{L}^{N,0}$, are useful in formulating boundary conditions on the external lines, and are defined by the angles $\overset{(0)}{\phi}$ and $\overset{(N+1)}{\phi}$, and by the relationships: $\overset{(0)}{\phi} = \nu^{0,1} - \pi$, $\overset{(N+1)}{\phi} = \nu^{N,0}$ (see Fig. 4).

One can formulate the problems either in the arbitrary systems $\{a\}$ or in particular systems $\{\xi\}$, and both possibilities are equally justifiable. However, the choice of system is important as far as the method of solving and the level of difficulties are concerned. In particular, the functions expressing differences between the angles of principal stresses $\Delta\phi = \overset{(\alpha+1)}{\phi} - \overset{(\alpha)}{\phi}$, and the functions defining the unit vector $\mathbf{e}^{\alpha,\alpha+1}$ normal to the line $\mathcal{L}^{\alpha,\alpha+1}$, can be formulated in the local systems $\{\xi\}^{(\alpha)}$ in the simplest way.

Unlike the field parameters given in the system $\{a\}$, the parameters $\Delta\phi$, $\Delta\gamma$, $\Delta\nu$, defined in the systems $\{\xi\}^{(\alpha)}$, actually express differences, and therefore they are preceded by the symbol Δ. This fact has not only a formal meaning. As it turns out, the mentioned parameters can be expressed only by the components of stress states in the adjacent regions. According to this property, one can easily create recursive formulas, convenient for the algorithmic approach.

6.5. *Families and subfamilies of stress discontinuity lines*

In the case when the limit state of stress is given in the region α ($\overset{(\alpha)}{\omega}$, $\overset{(\alpha)}{\phi}$, see Fig. 5), and additionally the stress parameter $\overset{(\alpha+1)}{\omega}$ is set up in the region $\alpha+1$, one can obtain two families ($Q^{\alpha,\alpha+1}=1,2$) of directions of lines $\mathcal{L}^{\alpha,\alpha+1}$, and two values $\pm\Delta\phi$ associated with these lines. One can join a pair of two opposite versors $\mathbf{e}^{\alpha,\alpha+1}$ to each of these directions. In consequence, there are four ($q^{\alpha,\alpha+1}=1,2,3,4$) half-lines that originate from the node. The parameters $q^{\alpha,\alpha+1}$ are called the parameters of subfamilies of the lines $\mathcal{L}^{\alpha,\alpha+1}$. It is assumed that the subfamilies $q^{\alpha,\alpha+1}=1,3$ are assigned to the family $Q^{\alpha,\alpha+1}=1$, while the subfamilies $q^{\alpha,\alpha+1}=2,4$ are assigned to the family $Q^{\alpha,\alpha+1}=2$.

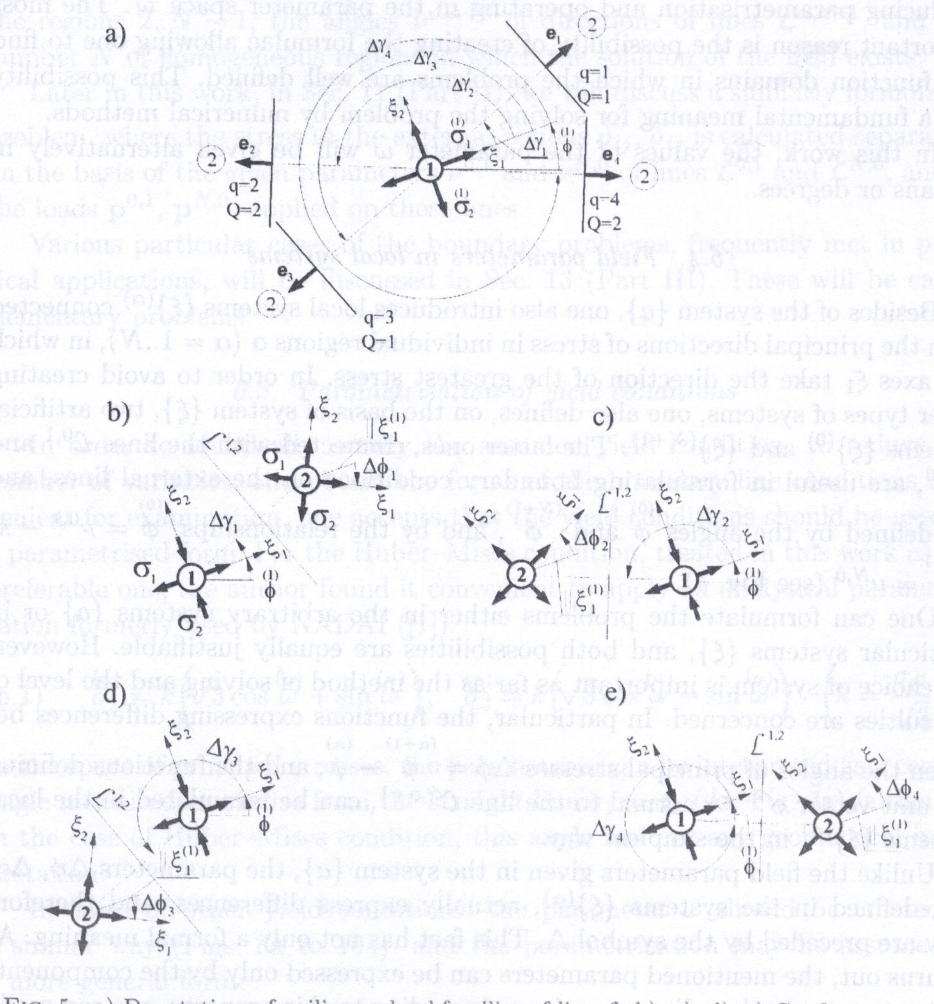

FIG. 5. a) Denotations, families and subfamilies of line $\mathcal{L}$; b), c), d), e) Configurations of homogeneous regions and stress discontinuity lines on the physical plane.

7. Recursive formulae

Assuming that operations are performed on the systems $\{\xi\}^{(\alpha)}$ and making use of parametrisation (6.2), one can derive the function $\Delta\phi$ from the equality condition of existence (see Eq. $(1.6)_2$ in Part I). The algorithm for determining this function on a physical plane is given by the following set of equations (see also Fig. 4):

$$(7.1)\qquad \Delta\phi\Big(\overset{(\alpha)}{\omega},\overset{(\alpha+1)}{\omega},Q^{\alpha,\alpha+1}\Big)\equiv\overset{(\alpha+1)}{\phi}-\overset{(\alpha)}{\phi}$$
$$=(-1)^{(Q^{\alpha,\alpha+1}+1)}\operatorname{sign}\Big(\overset{(\alpha+1)}{\omega}-\overset{(\alpha)}{\omega}\Big)\Delta\hat{\phi}\Big(\overset{(\alpha)}{\omega},\overset{(\alpha+1)}{\omega}\Big),$$
$$\Big(\Delta\phi\in\Big[-\frac{\pi}{2},\frac{\pi}{2}\Big],\quad \alpha=1..N-1\Big),$$

where:

$$(7.2)\quad \Delta\hat{\phi}(\overset{(\alpha)}{\omega},\overset{(\alpha+1)}{\omega})=\arcsin\left(\sqrt{\frac{\Big(\overset{(\alpha)}{\sigma}_1-\overset{(\alpha+1)}{\sigma}_1\Big)\Big(\overset{(\alpha)}{\sigma}_2-\overset{(\alpha+1)}{\sigma}_2\Big)}{\Big(\overset{(\alpha)}{\sigma}_1-\overset{(\alpha)}{\sigma}_2\Big)\Big(\overset{(\alpha+1)}{\sigma}_1-\overset{(\alpha+1)}{\sigma}_2\Big)}}\right)\Big(\Delta\hat{\phi}\in\Big[0,\frac{\pi}{2}\Big]\Big),$$

and, when the limit field is considered:

$$\overset{(\alpha)}{\sigma}_i=\sigma_{\rm pl}f_i(\overset{(\alpha)}{\omega}),\qquad \overset{(\alpha+1)}{\sigma}_i=\sigma_{\rm pl}f_i(\overset{(\alpha+1)}{\omega})\qquad (i=1,2).$$

The symbol $\Delta\hat{\phi}$ is used to denote the function expressed here only by the parameters $\overset{(\alpha)}{\omega}$, $\overset{(\alpha+1)}{\omega}$, and uniquely determined in the interval $\Big[0,\frac{\pi}{2}\Big]$, which is defined in order to separate the variables. One can substitute the parameters of the families Q into Eq. (7.1), or, alternatively, substitute there the parameters of the subfamilies q.

Consequently, on the basis of equilibrium conditions (see Eq. (1.2) in Part I), and assuming that the equality condition of existence (Eq. $(1.6)_2$ in Part I) is fulfilled, one can derive angular parameters $\Delta\gamma$ determining the versors $\mathbf{e}$ normal to lines $\mathcal{L}$. Making use of parametrisation (6.2) in the system $\{\xi\}^{(\alpha)}$, one obtains (see Fig. 4):

$$(7.3)\quad \Delta\gamma\Big(\overset{(\alpha)}{\omega},\overset{(\alpha+1)}{\omega},q^{\alpha,\alpha+1}\Big)\equiv\nu^{\alpha,\alpha+1}-\overset{(\alpha)}{\phi}+\frac{\pi}{2}$$
$$=s_q\pi+(-1)^{(q^{\alpha,\alpha+1}+1)}\Delta\hat{\gamma}\Big(\overset{(\alpha)}{\omega},\overset{(\alpha+1)}{\omega}\Big)$$
$$\Big(\Delta\gamma\in[0,2\pi];\ \alpha=1..N;\ \overset{(\alpha)}{\sigma}\neq\overset{(\alpha+1)}{\sigma}\Big),$$

where s – selector defined in the following way: $\mathbf{s} \overset{\text{def}}{\equiv} [0,1,1,2]$; $\Delta\hat{\gamma}\Big(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\Big)$ – term of function $\Delta\gamma\Big(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1}\Big)$; the function $\Delta\hat{\gamma}\Big(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\Big)$ is determined exclusively by $\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}$, and is weighted in the interval $\Big[0, \dfrac{\pi}{2}\Big]$ in which it is unique. The function has the form:

$$\Delta\hat{\gamma}\Big(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\Big) = \arcsin\left(\sqrt{\frac{\Big(\overset{(\alpha)}{\sigma_1} - \overset{(\alpha+1)}{\sigma_2}\Big)\Big(\overset{(\alpha)}{\sigma_1} - \overset{(\alpha+1)}{\sigma_1}\Big)}{\Big[\Big(\overset{(\alpha)}{\sigma_1} + \overset{(\alpha)}{\sigma_2}\Big) - \Big(\overset{(\alpha+1)}{\sigma_1} + \overset{(\alpha+1)}{\sigma_2}\Big)\Big]\Big(\overset{(\alpha)}{\sigma_1} - \overset{(\alpha)}{\sigma_2}\Big)}}\right), \quad \overset{(\alpha)}{\sigma} \neq \overset{(\alpha+1)}{\sigma}, \tag{7.4}$$

and

$$\overset{(\alpha)}{\sigma_i} = \sigma_{\rm pl}\, f_i\Big(\overset{(\alpha)}{\omega}\Big), \qquad \overset{(\alpha+1)}{\sigma_i} = \sigma_{\rm pl}\, f_i\Big(\overset{(\alpha+1)}{\omega}\Big), \qquad (i = 1,2).$$

As it can be seen, although the above relationships are possibly the simplest ones that can be derived here, they are still quite complicated. In this situation, one gives up beforehand the attempt to transform them, and leaves them in their original form – such as that given by their algorithms. The simplicity of description is achieved, however, at the cost of complexity of the numerical implementation.

In the following considerations, only the headers of definitions of the functions $\Delta\phi$ and $\Delta\gamma$ will be used, as these are not too complicated. However, in order to properly use them, one should collect some additional information:

1. The formulae (7.1) as well as (7.3) do not exhibit any symmetry about $\overset{(\alpha)}{\omega}$, $\overset{(\alpha+1)}{\omega}$, and then are valid in the system $\{\xi\}^{(\alpha)}$ connected with the state of stress in the region α. If the system $\{\xi\}$ was related to the state of stress in the region α+1, then the value of $\Delta\gamma$ would be different, and the index of subfamily q would be different, too (see Fig. 5). In order to emphasise this fact, the parameters of q and Q are, in the whole work, provided with additional indices:
   $q^{\alpha,\alpha+1}$, $Q^{\alpha,\alpha+1}$ – in the case of formulae valid in $\{\xi\}^{(\alpha)}$
   or
   $q^{\alpha+1,\alpha}$, $(Q^{\alpha+1,\alpha})$ – when the formulae are valid in $\{\xi\}^{(\alpha+1)}$.
   Similarly, it is convenient to use $\mathbf{e}^{\alpha,\alpha+1}$ as a denotation of the unit vector normal to line $\mathcal{L}^{\alpha,\alpha+1}$ directed towards outside of the region α. Then, it holds that $\mathbf{e}^{\alpha,\alpha+1} = -\,\mathbf{e}^{\alpha+1,\alpha}$
2. It can be shown that the relationships (7.1) and (7.3) keep the property that the values of parameters $\overset{(\alpha)}{\omega}$, $\overset{(\alpha+1)}{\omega}$ belong only to the interval $[0, \pi]$. Then, in order to express all the states of principal stresses, one must allow for changing the indices of the principal stresses.

3. The formulae (7.1) and (7.3) have been constructed in a partly artificial way. These are based on the equality condition of existence of the lines, and on the equilibrium condition, both defined on the sets of stress parameters. On the other hand, the values of $\Delta\phi$ and $\Delta\gamma$, calculated from these equations, are specified on a physical plane. By establishing the uniqueness of assignment of the indices to the function values, one could successfully select the terms connected with the index (Q, q) of the solution root, and, consequently, obtain such a notation of both functions in which all the base variables $\left\{ \overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1} \right\}$, belonging to a set that is not any more reducible, are explicitly specified.
4. The forms of functions (7.2) and (7.4), defined directly on $\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}$, could so far be obtained only for the Huber–Mises condition parametrised according to (6.1).

8. Loaded lines

Let us first consider a particular case, when the external line $\mathcal{L}^{0,1}$ is externally loaded, and when the Huber–Mises condition, parametrised according to (6.1), is satisfied in the adjacent homogeneous region 1.

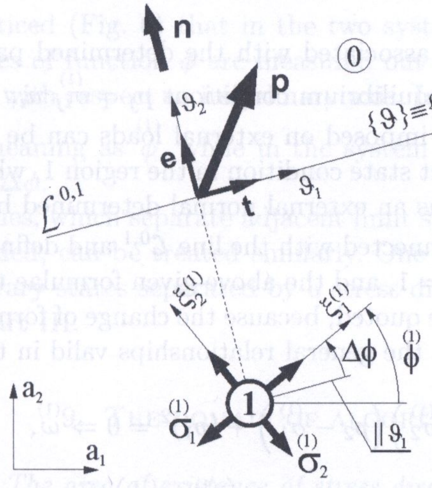

Fig. 6.

Substituting (6.1) into the equilibrium conditions $\mathrm{p}_j = \overset{(1)}{\sigma}_{ij}\, n_i$ one initially gets the equation:

$$\cos^2\omega - \frac{\sqrt{3}}{2}\left(p_1 n_1 + p_2 n_2\right)\cos\omega + \frac{1}{4}\left(p_1^2 + p_2^2 - 1\right) = 0,$$

and then obtains a set of formulae being the solution to the system $p_j = \overset{(1)}{\sigma}_{ij}\, n_i$, valid in the system $\{a\}$:

$$\Delta = \frac{3}{4}(p_1 n_1 + p_2 n_2)^2 - (p_1^2 + p_2^2) + 1 \geq 0,$$

(8.1)
$$\cos\omega = \frac{\frac{\sqrt{3}}{2}(p_1 n_1 + p_2 n_2) + (-1)^{Q+1}\sqrt{\Delta}}{2} \Rightarrow \omega \quad \Big(\omega \equiv \overset{(1)}{\omega} \in \big[0, \pi\big]\Big),$$

$$\cos 2\phi = \frac{p_1 n_1 - p_2 n_2 - \sqrt{3}(n_1^2 - n_2^2)\cos\omega}{\sin\omega} \Rightarrow \phi \quad \Big(\phi \equiv \overset{(1)}{\phi} \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]\Big).$$

where:

p_i – components of stress vector $\mathbf{p}^{0,1}$ applied to the line $\mathcal{L}^{0,1}$ (dimensionless, related to $k = \sigma_{\mathrm{pl}}\sqrt{3}$);

n_i – components of unit vector $\mathbf{n}$ normal to $\mathcal{L}^{0,1}$ directed outside of the region 1;

$Q = 1, 2$ – indices of the root ω having a similar meaning as the indices of families of lines $\mathcal{L}$.

It is worth mentioning that:

- in technical terms, calculation of the value of ω from $(8.1)_2$ with fixed Q gives unique results, because the parameter ω takes values from the interval $[0, \pi]$;
- the choice of root ϕ associated with the determined parameter ω can be made by inspecting equilibrium conditions $p_j = \overset{(1)}{\sigma}_{ij}\, n_i$;
- the limitation $(8.1)_1$ imposed on external loads can be interpreted as the realisation of the limit state condition in the region 1, whose edge is loaded with stress $\mathbf{p}$, and has an external normal determined by vector $\mathbf{n}$.

In the system $\{\xi\}^{(0)}$ connected with the line $\mathcal{L}^{0,1}$ and defined as shown in the figure, there is $n_1 = 0$, $n_2 = 1$, and the above-given formulae take a more simple form. The latter will not be quoted, because the change of form is not significant. Instead, we should present the general relationships valid in this system:

(8.2)
$$\Big(p_2 - \overset{(1)}{\sigma}_2\Big)\Big(p_2 - \overset{(1)}{\sigma}_1\Big) + (p_1)^2 = 0 \Rightarrow \overset{(1)}{\omega},$$

$$\cos 2\Delta\phi = \frac{2p_2 - \Big(\overset{(1)}{\sigma}_1 + \overset{(1)}{\sigma}_2\Big)}{\overset{(1)}{\sigma}_1 - \overset{(1)}{\sigma}_2} \Rightarrow \Delta\phi.$$

These are expressed by principal stresses and components of the stress vector, and then they remain independent of the assumed yield condition. Such a relationship can only be established after the condition is imposed, preferably in the parametrised form (6.2). This will lead to the reduction of the number of variables, and will make it possible to determine first $\overset{(1)}{\omega}$, and then $\Delta\phi$. In such a generalised case, the inequality condition, related to $(8.1)_1$, must be formulated individually for each yield condition on the basis of the conditions of existence of roots $\overset{(1)}{\omega}$ of Eqs. (8.2).

The formulae (8.2) can be obtained almost immediately by simple transformation of the expressions for stress components $\overset{(1)}{\sigma}_{ij}$ defined in the system $\{\xi\}^{(0)}$ (see Fig. 6):

$$p_1 = \overset{(1)}{\sigma}_{12} = \frac{\left(\overset{(1)}{\sigma}_1 - \overset{(1)}{\sigma}_2\right)}{2}\sin 2\Delta\phi, \quad p_2 = \overset{(1)}{\sigma}_{22} = \frac{\left(\overset{(1)}{\sigma}_1 + \overset{(1)}{\sigma}_2\right)}{2} - \frac{\left(\overset{(1)}{\sigma}_1 - \overset{(1)}{\sigma}_2\right)}{2}\cos 2\Delta\phi.$$

The system of relationships (8.1) will be treated as definitions which are not to be transformed any further – similarly as all the functions of complicated form, mentioned in previous examples. Symbolic notation of the function's headers that specify all the function variables is sufficient for the use in references used in the algorithms. According to (8.1), one can see that the notation may have the form:

$$(8.3) \qquad \Delta(\mathbf{p},\mathbf{n}) \geq 0, \qquad \overset{(1)}{\omega} = \bar{\omega}(\mathbf{p},\mathbf{n},Q), \qquad \overset{(1)}{\phi} = \bar{\phi}(\mathbf{p},\mathbf{n},\overset{(1)}{\omega},Q).$$

It must be noticed (Fig. 6) that in the two systems of co-ordinates, $\{\xi\}^{(0)}$ and $\{a\}$, the values of function ϕ are measured out in different ways – in each particular system with respect to its primary axis. In the system $\{a\}$, the angle ϕ has the same meaning as $\overset{(1)}{\phi}$, while in the system $\{\xi\}^{(0)}$ its interpretation is similar to that of $\Delta\phi$.

The cases of lines, which separate adjacent limit states, and at the same time are externally loaded, can be treated similarly. One can also treat them as the cases of two arbitrary states separated by a stress discontinuity line, which will be described in Part III.

9. The domain of algorithm

9.1. The area of existence of stress discontinuity line

The domain of functions (7.1), (7.3) is defined by the following system of inequality conditions (see formulae (1.6) in Part I):

$$(9.1) \qquad 0 \leq \frac{\left(\overset{(\alpha)}{\sigma}_1 - \overset{(\alpha+1)}{\sigma}_2\right)\left(\overset{(\alpha)}{\sigma}_1 - \overset{(\alpha+1)}{\sigma}_1\right)}{\left[\left(\overset{(\alpha)}{\sigma}_1 + \overset{(\alpha)}{\sigma}_2\right) - \left(\overset{(\alpha+1)}{\sigma}_1 + \overset{(\alpha+1)}{\sigma}_2\right)\right]\left(\overset{(\alpha)}{\sigma}_1 - \overset{(\alpha)}{\sigma}_2\right)} \leq 1,$$

which can be conveniently expressed by the parameters $\overset{(\alpha)}{\omega}$, $\overset{(\alpha+1)}{\omega}$ (6.2), and which, in the products:

$$(9.2) \qquad \Omega \equiv \left\{ \left(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega} \right) : \overset{(\alpha)}{\omega} \in [0, \pi], \overset{(\alpha+1)}{\omega} \in [0, \pi] \right\},$$

determine the admissible areas Λ (see Figs. 7..10). If the pair of numbers $\overset{(\alpha)}{\omega}$, $\overset{(\alpha+1)}{\omega}$, being the image $\mathbf{P}^{\alpha,\alpha+1}$ of the line $\mathcal{L}^{\alpha\alpha+1}$ in Λ, satisfies the condition:

$$(9.3) \qquad \left\{ \overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega} \right\} \in \Lambda,$$

then the functions (7.1), (7.3) are defined in their domain.

The conditions (9.3) are formulated separately for each stress discontinuity line. These conditions have the highest priority, because they define the domain of both functions. One can use them as the conditions of existence of both $\Delta\hat{\gamma}$ ($\Delta\gamma$) and $\Delta\,\hat{\phi}$ ($\Delta\phi$).

Having the knowledge about the admissible area Λ, one can effectively take control over the contents of variables in the domain, and prevent *a priori* the attempts of performing illegal numerical operations that usually lead to a breakdown of the calculation process. This ability has then a fundamental meaning for solving the problem by numerical methods.

Some of such areas Λ, obtained for different yield conditions, are shown in Figs. 7–10. It is worth noticing that the functions $\Delta\,\hat{\phi}$, $\Delta\hat{\gamma}$ actually have almost the same domains in all that cases.

Then, if the conditions of existence of $\Delta\,\hat{\phi}$ are fulfilled, the conditions of existence of $\Delta\hat{\gamma}$ are satisfied as well, although the points of indeterminacy, or sets of points of indeterminacy, generally are not identical. There might be cases when $\Delta\,\hat{\phi}$ exists, but $\Delta\hat{\gamma}$ is undetermined, and *vice versa.* We have to leave out

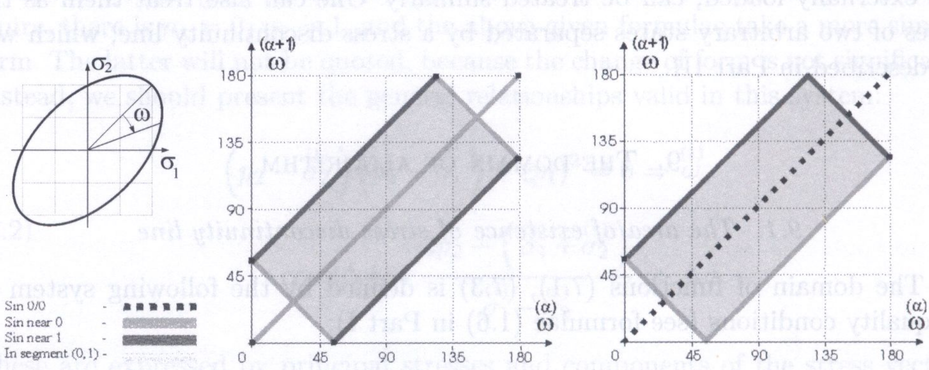

FIG. 7. The areas of existence of line $\mathcal{L}$ for Huber–Mises yield condition and plane stress; a) limit curve, b) area determined from the condition of existence $\delta\phi$, c) area determined from the condition of existence $\delta\hat{\gamma}$.

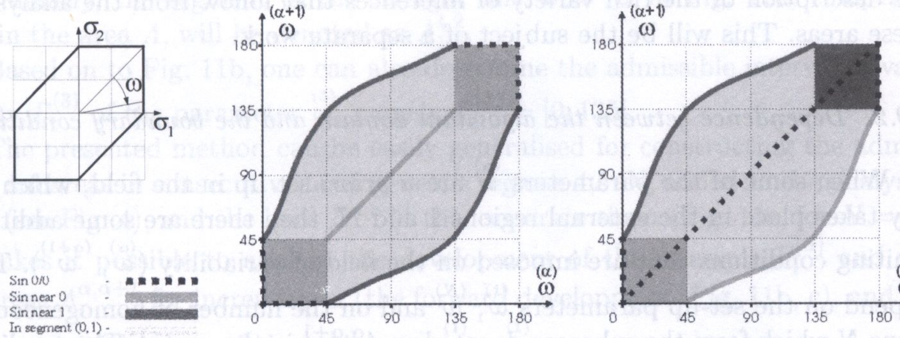

FIG. 8. The areas of existence of line $\mathcal{L}$ for Tresca yield condition and plane stress; a) limit curve, b) area determined from the condition of existence $\delta\hat{\phi}$, c) area determined from the condition of existence $\delta\hat{\gamma}$.

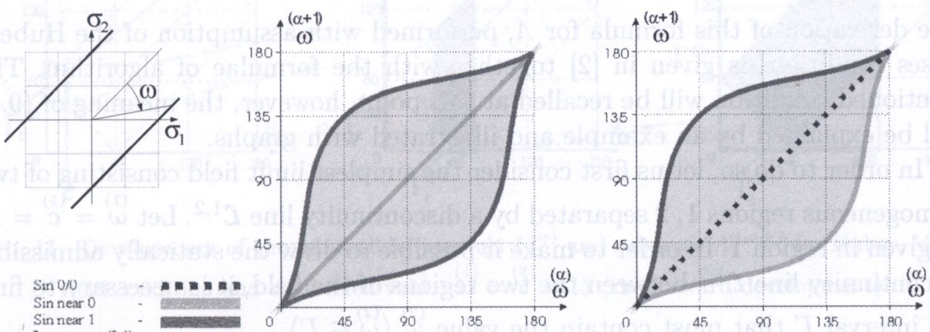

FIG. 9. The areas of existence of line $\mathcal{L}$ for Tresca yield condition and plane strain; a) limit curve, b) area determined from the condition of existence $\delta\hat{\phi}$, c) area determined from the condition of existence $\delta\hat{\gamma}$.

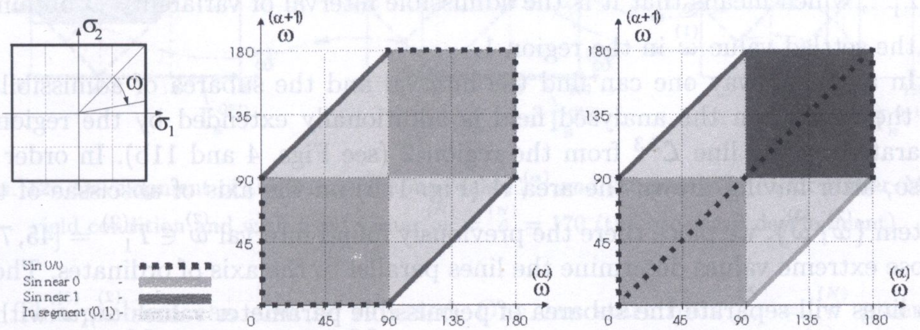

FIG. 10. The areas of existence of line $\mathcal{L}$ for the yield condition of maximum principal stress and plane stress; a) limit curve, b) area determined from the condition of existence $\delta\hat{\phi}$, c) area determined from the condition of existence $\delta\hat{\gamma}$.

the description of the rich variety of inferences that follow from the analyses of these areas. This will be the subject of a separate work.

9.2. Dependence between the algorithm domain and the boundary conditions

When some of the parameters $\overset{(\alpha)}{\omega}$ are *a priori* set up in the field, which usually takes place in the external regions 1 and N, then there are some additional limiting conditions that are imposed on the field of variability $(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega})$. These depend on the set-up parameters $\overset{(1)}{\omega}, \overset{(N)}{\omega}$ and on the number of homogeneous regions N which form the subareas denoted as $\Lambda_{1,N}^{\alpha,\alpha+1}$ in the area Λ. The conditions of existence of functions (7.1), (7.3) in the domains take then the form:

$$\{\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\} \in \Lambda_{1,N}^{\alpha,\alpha+1} \subset \Lambda. \tag{9.4}$$

The derivation of this formula for Λ, performed with assumption of the Huber–Mises condition, is given in [2] together with the formulae of algorithm. The mentioned condition will be recalled at this point, however, the meaning of (9.4) will be explained by an example and illustrated with graphs.

In order to do so, let us first consider the simplest limit field consisting of two homogeneous regions 1, 2 separated by a discontinuity line $\mathcal{L}^{1,2}$. Let $\overset{(1)}{\omega} = \overset{(1)}{c} = 15$ be given in region 1. In order to make it possible to draw the statically admissible discontinuity line $\mathcal{L}^{1,2}$ between the two regions of the field, it is necessary to find the interval Γ that must contain the value $\overset{(2)}{\omega}$ $(\overset{(2)}{\omega} \in \Gamma)$.

Taking into consideration the segment of the straight line $\overset{(1)}{\omega} = 15$ enclosed within Λ in Fig. 11a, we immediately find that the interval $\overset{(2)}{\omega}$ we sought for is as follows: $\overset{(2)}{\omega} \in \Gamma = [45, 75]$ (heavy dashed lines in Figs. 11 and 12). We denote it by $\Gamma_1^{(2)}$, which means that it is the admissible interval of variability $\overset{(2)}{\omega}$ obtained for the settled value $\overset{(1)}{\omega}$ in the region 1.

In a similar way one can find the interval and the subarea of admissibility for the case when the analysed field is additionally extended by the region 3 separated by the line $\mathcal{L}^{2,3}$ from the region 2 (see Figs. 4 and 11b). In order to do so, after having drawn the area Λ (Fig. 11b) on the axis of abscissae of the system $\{\overset{(2)}{\omega}, \overset{(3)}{\omega}\}$, we place there the previously found interval $\overset{(2)}{\omega} \in \Gamma_1^{(2)} = [45, 75]$, whose extreme values determine the lines parallel to the axis of ordinates. These two lines will separate the subarea of permissible parameter values $\overset{(2)}{\omega}, \overset{(3)}{\omega}$ within the area Λ. The subarea will be denoted as $\Lambda_1^{2,3}$. The indices $\Lambda_1^{2,3}$ inform that this is the subarea of permissible parameter values $\overset{(2)}{\omega}, \overset{(3)}{\omega}$ determined with the fixed value $\overset{(1)}{\omega} = \overset{(1)}{c}$ (see Fig. 11a, b).

Similarly, the segment of the straight line $\overset{(1)}{\omega} = \overset{(1)}{c}$ in Fig. 11a, comprised within the area Λ, will be denoted as $\Lambda_1^{1,2}$.

Based on to Fig. 11b, one can also determine the admissible interval of variability $\Gamma_1^{(3)}$ of the parameter $\overset{(3)}{\omega}$, namely $\Gamma_1^{(3)} = [0, 135]$.

The presented method can be easily generalised for constructing the admissible subareas in consecutive homogeneous regions 4,5..N, added to the analysed field (see Fig. 4), and the lines $\mathcal{L}^{\alpha,\alpha+1}$ that separate them (Fig. 11c, for $N = 4$). It makes it possible to analyse the development of the intervals $\Gamma_1^{(\alpha)}$ and the subareas $\Lambda_1^{\alpha,\alpha+1}$ for increasing α (the forward development, Fig. 11b, c), and for different initial values of the parameter $\overset{(1)}{\omega} = \overset{(1)}{c}$

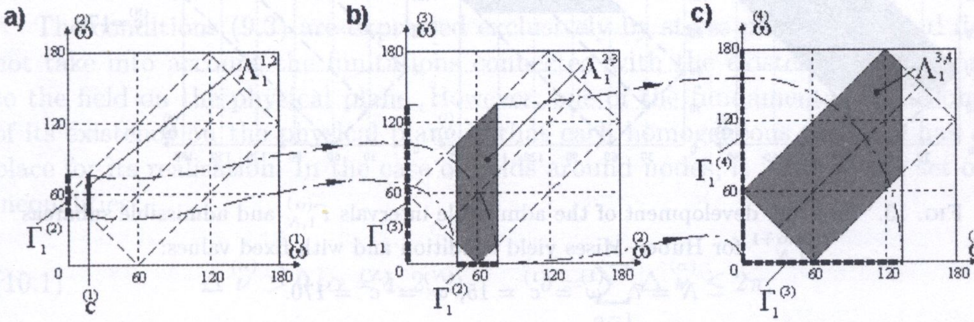

FIG. 11. Development of the admissible intervals $\Gamma_1^{(\alpha)}$ and subareas $\Lambda_1^{\alpha,\alpha+1}$ for Huber–Mises yield condition and with fixed value: $\overset{(1)}{\omega} = \overset{(2)}{c} = 15$ (the forward development).

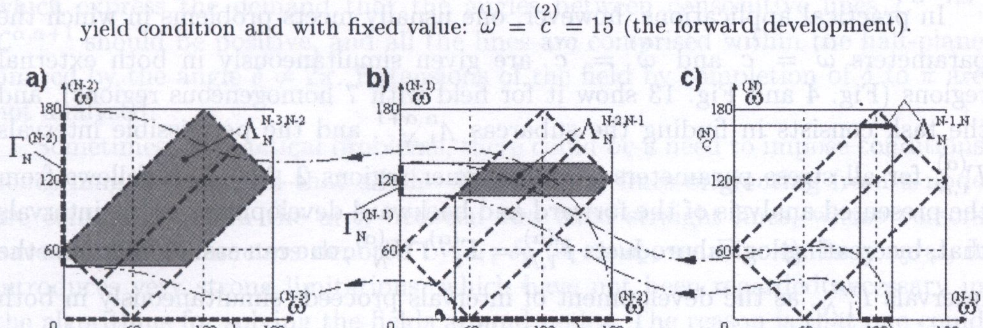

FIG. 12. Development of the admissible intervals $\Gamma_1^{(\alpha)}$ and subareas $\Lambda_1^{\alpha,\alpha+1}$ for Huber–Mises yield condition and with fixed value: $\overset{(N)}{\omega} = \overset{(N)}{c} = 170$ (the backward development).

This problem can also be reversed. One can start from $\overset{(N)}{\omega} = \overset{(N)}{c}$ given in the last homogeneous region N (Fig. 12c), and find intervals $\Gamma_N^{(\alpha)}$ and subareas $\Lambda_N^{\alpha,\alpha+1}$ for the consecutively decreasing α (the 'backward' development, Fig. 12c, b, a, one assumes: $\overset{(N)}{\omega} = \overset{(N)}{c} = 170$).

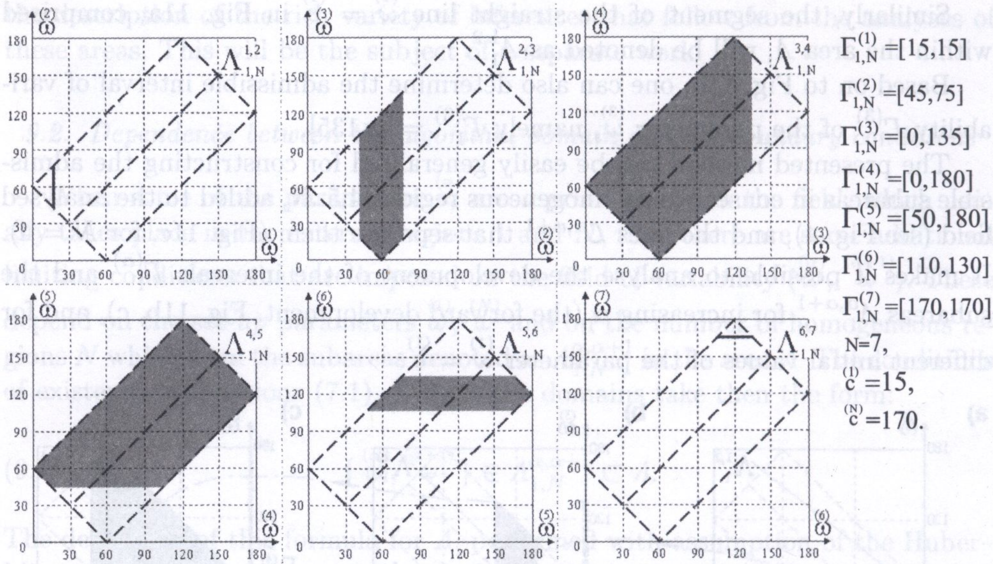

FIG. 13. Two-way development of the admissible intervals $\Gamma_{1,N}^{(\alpha)}$ and admissible subareas $\Lambda_{1,N}^{\alpha,\alpha+1}$ for Huber–Mises yield condition and with fixed values:

$$N=7,\ \overset{(1)}{\omega}=\overset{(1)}{c}=15,\ \overset{(N)}{\omega}=\overset{(N)}{c}=170.$$

In practical applications, however, one usually meets problems in which the parameters $\overset{(1)}{\omega}=\overset{(1)}{c}$ and $\overset{(N)}{\omega}=\overset{(N)}{c}$ are given simultaneously in both external regions (Fig. 4 and Fig. 13 show it for field with 7 homogeneous regions), and the task consists in finding the subareas $\Lambda_{1,N}^{\alpha,\alpha+1}$ and the permissible intervals $\Gamma_{1,N}^{(\alpha)}$ for all stress parameters ω in the inner regions $2..N-1$. It follows from the presented analysis of the forward and backward development of the intervals that, by creating logical products $\Gamma_{1,N}^{(\alpha)}=\Gamma_{1}^{(\alpha)}\cap\Gamma_{N}^{(\alpha)}$, one can easily determine the intervals $\Gamma_{1,N}^{(\alpha)}$, as the development of intervals proceeds simultaneously in both directions. Similarly, the subareas $\Lambda_{1,N}^{\alpha,\alpha+1}$ can be determined from the formula

$$\Lambda_{1,N}^{\alpha,\alpha+1}\equiv\Gamma_{1,N}^{(\alpha)}\times\Gamma_{1,N}^{(\alpha+1)}\cap\Lambda. \tag{9.5}$$

The analyses of the development of intervals and the subareas of existence, performed for different initial data $\overset{(1)}{\omega}$, $\overset{(N)}{\omega}$, bring out a multitude of inferences that can be hardly juxtaposed or even classified in a sensible way. It is not absolutely necessary, anyway. However, it is important that the subareas $\Lambda_{1,N}^{\alpha,\alpha+1}$ occupy only small segments of the area Λ, and sometimes can be reduced to isolated points, which are very difficult to find unless the formulae (9.4) are known.

Obviously, the solution of the field can only exist for the data $\overset{(1)}{\omega}, \overset{(N)}{\omega}$ if none of the intervals $I_{1,N}^{(\alpha)}$ (and consequently, none of the subareas $\Lambda_{1,N}^{\alpha,\alpha+1}$) is empty. It must be mentioned, however, that this is the domain of the algorithm, whose contents is only the necessary condition of existence of solution to the field on the physical plane.

A similar algorithm that allows for examining the evolution of subareas $\Lambda_{1,N}^{\alpha,\alpha+1}$ for the Tresca yield condition is presented in [4].

10. Geometrical and structural constrains

10.1. Conditions formulated on physical plane

The conditions (9.3) are expressed exclusively by stress parameters, and do not take into account the limitations connected with the existence of solution to the field on the physical plane. However, one of the fundamental conditions of its existence on the physical plane is that each homogeneous region α has a place for its realisation. In the case of fields around nodes, it leads to the set of inequalities

$$(10.1) \qquad \Delta \overset{(\alpha)}{\nu} > 0 \, (\alpha = 1, 2, ..) , \qquad \delta = \sum_{\alpha=1}^{N} \Delta \overset{(\alpha)}{\nu} \leq 2\pi,$$

which express the demand that the angles between consecutive lines $\mathcal{L}^{\alpha-1,\alpha}$, $\mathcal{L}^{\alpha,\alpha+1}$ should be positive, and all the lines are comprised within the half-plane limited by the angle $\delta \neq 2\pi$. Extensions of the field by completion of δ to π are not analysed.

Sometimes, in practical problems, there could be a need to impose conditions concerning the demand that all the discontinuity lines originating from a node are contained within the area between two fixed straight lines, whose versors have angular parameters denoted by δ_1, δ_2 ($\delta_1 > \delta_2$, see Fig. 4). In general, it introduces very strong limitations, which have not been regarded necessary in the algorithms for solving the fields around nodes. The reason is that one could almost immediately obtain a full spectrum of solutions, and then make a proper choice.

The conditions (10.1) are defined on the parameters $\Delta \overset{(\alpha)}{\nu}$ that can be expressed by functions (7.1), (7.3), and then, for $\alpha = 2, 3, .. (N-1)$, can be defined only by the parameters $\{\boldsymbol{\omega}, \mathbf{q}\}$ (see Fig. 4):

$$(10.2)_1 \quad \Delta \overset{(\alpha)}{\nu} \equiv \nu^{\alpha\alpha+1} - \nu^{\alpha-1,\alpha} \equiv \Delta\nu\Big(\overset{(\alpha-1)}{\omega}, \overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha-1,\alpha}, q^{\alpha,\alpha+1}\Big)$$

$$= \Delta\gamma\Big(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1}\Big) + \Delta\phi\Big(\overset{(\alpha-1)}{\omega}, \overset{(\alpha)}{\omega}, q^{\alpha-1,\alpha}\Big) - \Delta\gamma\Big(\overset{(\alpha-1)}{\omega}, \overset{(\alpha)}{\omega}, q^{\alpha-1,\alpha}\Big).$$

However, in the external regions limited externally by the lines of the type $\mathcal{L}^{0,1}$ and $\mathcal{L}^{N,0}$, where $\nu^{0,1}$, $\mathbf{p}^{0,1}$ and $\nu^{N,0}$, $\mathbf{p}^{N,0}$ are given *a priori*, one obtains:

$$(10.2)_2 \quad \Delta \overset{(1)}{\nu} = \nu^{1,2} - \nu^{0,1} = \left[\overset{(1)}{\phi} + \Delta\gamma\left(\overset{(1)}{\omega}, \overset{(2)}{\omega}, q^{1,2}\right) - \frac{\pi}{2}\right] - \nu^{0,1};$$

$$(10.2)_3 \quad \Delta \overset{(N)}{\nu} = \nu^{N,0} - \nu^{N-1,N} = \nu^{N,0} - \left[\overset{(N-1)}{\phi} + \Delta\gamma\left(\overset{(N-1)}{\omega}, \overset{(N)}{\omega}, q^{N-1,N}\right) - \frac{\pi}{2}\right].$$

10.2. Conditions of structure preservation

The conditions (10.1), and in particular $(10.1)_1$, have yet another important meaning. For the given boundary condition, the solution to the field exists, if it exists on the physical plane. In other words, there must exist each of its discontinuity lines and each region α of the homogeneous state of stress.

The conditions (10.1) ensure the fulfilment of the second demand. The first one, however, is satisfied when conditions (9.3) are fulfilled. The latter control the existence of the parameter $\Delta\gamma$ of vector $\mathbf{e}^{\alpha,\alpha+1}$ normal to $\mathcal{L}^{\alpha,\alpha+1}$, and consequently, the existence of the line $\mathcal{L}^{\alpha,\alpha+1}$ itself. These are then the same conditions (9.3) that have defined the domain of the algorithm in the space of stress parameters $\{\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\}$.

The system of connections of the component homogeneous regions, and the discontinuity lines that separate them, is called the field structure. However, one recognises that this notion pertains not only to the specificity and assignments of the component regions and lines, but, first of all, to the structure of the system of equations and inequalities that must be set up in order to solve a specific boundary problem. The structure determines then the number of the applied conditions and the number of unknowns that, at the initial moment of solving of each field, are not *a priori* known. Up to now, no formulae facilitating finding the structures have been created.

The conditions (10.1) and (9.3), in the sense discussed so far, can only be used to examine whether the initially assumed structure is preserved or not. For this reason, these are called structure preservation conditions.

In turn, the conditions for the contents of the field within the segment defined by δ_1, δ_2, have exclusively a geometrical sense. For this reason, these are called geometrical conditions for the solution existence.

Obviously, the conditions (10.1) and the geometrical conditions of existence, unlike the conditions (9.3), do not pertain to the domains of functions $\Delta\gamma$, $\Delta\phi$, which must be *a priori* defined. The possibility that (10.1) are not satisfied would not lead to the attempts of performing illegal numerical operations. Then, the conditions can be taken into account only at the end of solving of the boundary problem, and used to eliminate the roots that do not satisfy the conditions (10.1).

The presented conditions of structure preservation, as well as the geometrical conditions, have a particular form that is valid only for the fields arising around the nodes. For the complex fields, such as those discussed in Part IV, it is more convenient to formulate these conditions in a different way, while their sense remains the same. It should be mentioned, however, that the conditions (10.1) are defined there based on the co-ordinates of the field nodes (see formula (1.7) in Part I), so that they are called the geometrical conditions of structure preservation, in contrast to (9.3) which are called statical (or stress) conditions.

11. Formulation of problems for fields arising around nodes

11.1. System of conditions

In practical problems formulated for the fields around nodes, there are usually given two external lines of the field $\mathcal{L}^{0,1}$ and $\mathcal{L}^{N,0}$ ($\nu^{0,1}$, $\nu^{N,0}$) and the load applied to them represented by stress vectors $\mathbf{p}^{0,1}$, $\mathbf{p}^{N,0}$ that, in general, are different on both lines (see Fig. 4). Using (8.3) one can immediately calculate the stress parameters $(\overset{(1)}{\omega}, \overset{(1)}{\phi})$, $(\overset{(N)}{\omega}, \overset{(N)}{\phi})$, so that, for determining the states of stress in the inner regions and the parameters of the separating lines $\mathcal{L}^{\alpha,\alpha+1}$, one has the data:

$$\Big\{\overset{(1)}{\omega}, \overset{(N)}{\omega}, \overset{(1)}{\phi}, \overset{(N)}{\phi}, \nu^{0,1}, \nu^{N,0}\Big\},$$

while the number of regions N is not given beforehand. When this number is assumed arbitrarily, the solution of the field might not exist (and it does not exist generally).

To determine the unknown field parameters, we have at our disposal one equation defined on the physical plane:

$$\text{(11.1)} \qquad \chi \equiv \overset{(N)}{\phi} - \overset{(1)}{\phi} = \sum_{\alpha=1}^{N-1} \Delta\phi\Big(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, Q^{\alpha,\alpha+1}\Big), \qquad (\chi \in [-\pi, \pi]).$$

What is to be found, are the stress and geometrical parameters in all the inner regions with the assumed boundary conditions of type (9.4) given in the form:

$$\Big\{\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\Big\} \in \Lambda_{1,N}^{\alpha,\alpha+1},$$

structural limitations (10.1):

$$\Delta \overset{(\alpha)}{\nu} > 0 \quad (\alpha = 1, 2..N), \qquad \delta = \sum_{\alpha=1}^{N} \Delta \overset{(\alpha)}{\nu} \leq 2\pi,$$

(where $\Delta \overset{(\alpha)}{\nu}$ are defined by formulae (10.2)), and, in particular cases, also geometrical limitations.

The symbol χ in formula (11.1) denotes the angle between principal directions of stress in the external regions and N (see Fig. 4).

In order to construct correctly the algorithm solving the system of conditions (9.4), (10.1), (11.1), one should notice that the left-hand side of (11.1) is expressed by the angles $\overset{(1)}{\phi}$, $\overset{(N)}{\phi}$ which, according to the so far accepted assumptions, uniquely define orientation of the systems $\{\xi\}^{(1)}$, $\{\xi\}^{(N)}$. In effect, an overdetermination appears, which is due to determining one of the two possible positions of each system (that differ by the angle π), although both of them pertain to the same states of stress. In order to avoid this effect, it is enough to assume in applying formula (11.1) that $\overset{(1)}{\phi}$ is fixed, $\chi \equiv \overset{(N)}{\phi} - \overset{(1)}{\phi}$ takes values from the interval $[-\pi, \pi]$, and then substitute into the left-hand side of formula (11.1) two values of angle χ:

$$(11.2) \qquad \chi_1 = \chi, \qquad \chi_2 = \begin{cases} \chi + \pi & \text{for } \chi \leq 0, \\ \chi - \pi & \text{for } \chi > 0. \end{cases}$$

For example, with $\chi = -60°$, one should substitute into (11.1) $\chi = \chi_1 = -60°$ and $\chi = \chi_2 = 120°$. If one chooses only one of the two positions of system $\{\xi\}^{(N)}$ in the last region N, oriented with respect to system $\{\xi\}^{(1)}$ in the first region 1, then elimination of admissible roots in solution to equation (11.1) might not be done correctly.

It becomes clear then that, with such assumptions, the solution of the field around a node does not depend on the angle $\overset{(1)}{\phi}$, and this parameter only defines field's orientation on the physical plane. Moreover, the solutions of the subsystem consisting only of conditions (11.1), (9.4) do not depend on the data on the boundaries $\nu^{0,1}$, $\nu^{N,0}$. The latter appear in the structural conditions (10.1) (possibly also in geometrical conditions, if such ones are imposed). Therefore, in technical terms, the solution of the whole system (11.1), (9.4), (10.1) may consist in determining initially the roots of system (11.1), (9.4), from among which one consequently eliminates the roots that do not satisfy the conditions (10.1).

The above principle is actually applied in the described algorithm. An additional advantage is the easiness of removing some selected limitations from the system (10.1), which gives a possibility to examine a wider set of possible solutions. One of the most interesting cases is that when only $\{\overset{(1)}{\omega}, \overset{(N)}{\omega}, \overset{(1)}{\phi}, \chi\}$ are given, which means that the lines $\mathcal{L}^{0,1}$ and $\mathcal{L}^{N,0}$ are not imposed. This case will be discussed in the examples presented in Part III.

11.2. Set of problem's unknowns

As the number of homogeneous regions N is not known, it must be counted among the problem's unknowns, similarly as the parameters $\{\boldsymbol{\omega}, \Delta\boldsymbol{\phi}\}$ in the inner homogeneous regions, and the parameters $\Delta\boldsymbol{\gamma}$ defining the directions of lines $\mathcal{L}^{\alpha,\alpha+1}$. The functions $\Delta\phi$ and $\Delta\gamma$, as well as the boundary conditions (8.3) on lines $\mathcal{L}^{0,1}$, $\mathcal{L}^{N,0}$, are expressed only by $\{\boldsymbol{\omega}, \mathbf{q}\}$, then the set of unknowns comprises:

$$\{N, \overset{(2)}{\omega}, ..., \overset{(N-1)}{\omega}, \quad q^{1,2}, q^{2,3}, .., q^{N-1,N}\}.$$

Analysing the number of the unknowns and the equations, one can immediately notice that there is only one equation (11.1), not being an identity, which can be used for solving the field around a node. The equation can be formulated for all possible paths defined by the sets of parameter values $Q^{1,2}$, $Q^{2,3}$, ... $Q^{N-1,N}$ (for example, in a field of $N = 4$ regions, one of the paths can be as follows: $\{Q^{1,2} = 2, Q^{2,3} = 1, Q^{3,4} = 2\}$). The number of unknown parameters ω is then identical with that of equations only in the case, when the field we seek for comprises $N = 3$ homogeneous regions. On the other hand, it is known that, within the structure consisting of only three regions, there are sets of data $\{\overset{(1)}{\omega}, \overset{(N)}{\omega}, \overset{(1)}{\phi}, \overset{(N)}{\phi}, \nu^{0,1}, \nu^{N,0}\}$ for which the solutions do not exist. In such cases, one typically assumes $N = 4$ or more regions, and introduces additional conditions to obtain the solution. There are also such cases in which the attempt of increasing the number of regions proves to be futile, while the decrease of N leads to finding the solution. These cases will be discussed in Sec. 15.2.

It is worth noticing that, when the condition (11.1) is formulated for all paths $Q^{1,2}$, $Q^{2,3}$, ... $Q^{N-1,N}$ in such a way that the parameters of the families Q are eliminated from it, then this condition leads to a set of equations, each of whom is expressed by the functions $\Delta\hat{\phi}\left(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}\right)$ (7.2). The system of conditions (11.1), (9.4), becomes then defined only in the space of stress parameters.

In practical applications, besides of boundary problems of the type presented here, one can also encounter problems formulated with different sets of data and unknowns. However, these are in fact only modifications of the already presented ones. A set of such problems, necessary for solving arbitrary boundary problems encountered in the fields around nodes, will be presented in Sec. 13, Part III of this work.

11.3. Transformations

The set of conditions (11.1), (9.4), (10.1) is valid in systems $\{\xi\}^{(\alpha)}$. When the results of its solutions are to be used for constructing complex fields, it is necessary to express them in the system of co-ordinates $\{a\}$ identical for all the nodes $w = 1..W$.

A general scheme of transformation of the parameters given in $\{\xi\}^{(\alpha)}$ into the field parameters defined in $\{a\}$ is denoted by the following symbolic mapping:

$$(11.3)\qquad \{\overset{(1)}{\phi}, \boldsymbol{\omega}, \boldsymbol{\Delta\phi}, \boldsymbol{\Delta\gamma}, \boldsymbol{\Delta\nu}\}_w \overset{\xi\rightarrow a}{\longrightarrow} \{\boldsymbol{\omega}, \boldsymbol{\phi}, \boldsymbol{\nu}\}_w.$$

Detailed transformation formulae used to perform the mapping (11.3) can be created on the basis of definitions of functions (7.1) and (7.3) (see also Fig. 4):

$$(11.4)\qquad \begin{aligned} &\Big\{\overset{(\alpha+1)}{\phi} = \overset{(\alpha)}{\phi} + \Delta\phi\Big(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1}\Big)\Big\}_w \qquad (\alpha = 1..(N_w-1)),\\ &\Big\{\nu^{\alpha,\alpha+1} = \overset{(\alpha)}{\phi} + \Delta\gamma\Big(\overset{(\alpha)}{\omega}, \overset{(\alpha+1)}{\omega}, q^{\alpha,\alpha+1}\Big) - \frac{\pi}{2}\Big\}_w \qquad (\alpha = 1..(N_w-1)).\end{aligned}$$

Here, the angles $\overset{(1)}{\phi}$, $\overset{(N)}{\phi}$ are either given *a priori*, or can be calculated from the boundary conditions of type (8.1).

The transformation formulae can also be created in other way, for example by using $\{\boldsymbol{\Delta\nu}\}_w$. Both methods are equivalent, in fact.

The parameters $\overset{(\alpha)}{\omega}$ are invariant for transformations of co-ordinate systems.

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