INFLUENCE OF ANISOTROPY INDUCED BY MICROCRACKS
ON EFFECTIVE ELASTIC PROPERTIES

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The influence of microcracks distribution on macroscopic elastic properties of composites with a specific structure is studied. The model predicts the properties of laminates made of materials in which fracture process leads to appearance of many microcracks distributed practically uniformly. The method of solution is based on the so-called reiterated homogenization with two different scales of inhomogeneities. The smaller scale is connected with microcracks size. After homogenization performed with the help of FEM an anisotropic homogeneous elastic material is obtained. The anisotropy is implied by directional distribution of microcracks. On the second larger scale, random mixture of two or more different anisotropic elastic materials is considered.

1. INTRODUCTION

Initiation of fracture processes in some elastic materials such like plastics is associated with appearance of microcracks. For instance, one can easily observe without any special instruments the loss of transparency in polyethylene rods subject to cyclic bending. If the distribution of small cracks is "dense" in some sense, the description of effective properties of such a fissured material involves a homogenization procedure. We mean by homogenization the method which enables to predict the macroscopic behaviour of material in which the "inhomogeneities", in this case cracks are smeared out. Homogenization methods were already applied to such types of problems, cf. the book of LEWIŃSKI and TELEGA [3] and the references therein. Theoretical considerations for finding the effective properties of elastic matrix weakened by randomly distributed microcracks with Signorini-type conditions on the crack lips were performed in [5, 6]. The effective behaviour in that case demonstrates nonlinearity, namely the elastic constitutive law is "piece-wise linear".

The present contribution is based on the results of the mathematical theory of homogenization. First, the effective anisotropy caused by the periodic distribution of thin rectangular voids is calculated by the FEM. Next, for a specific microstructure, the influence of anisotropy on effective elastic behaviour is studied. We analyze the dependence of effective moduli on such geometrical parameters like length of the crack, or angles between two directions perpendicular to differently oriented families of uniformly oriented cracks. The model is applied to finding the distribution of microcracks which minimizes (or maximizes) the stresses in a material element. The proposed model is a hybrid model which combines the periodic homogenization with random homogenization, cf. [1, 4] To perform numerical calculations and control the geometrical parameters, the necessary idealization was assumed.

2. ANISOTROPY CAUSED BY MICROCRACKS DISTRIBUTION

The first step is to calculate the effective elastic behaviour of a fissured elastic matrix. To perform it, we solve the problem of periodic homogenization. In this case, the effective elasticity tensor C^h_{ijkl} is calculated from the following formula:

$$(2.1) \qquad C^h_{ijmn} = \langle C_{ijmn} \rangle + \left\langle C_{ijpq} \frac{\partial \chi^{(mn)}_p}{\partial y_q} \right\rangle,$$

where

$$(2.2) \qquad \langle \cdot \rangle = \frac{1}{|Y|} \int_Y (\cdot) d\mathbf{y}.$$

Y – periodic functions $\chi^{(mn)}_p$ are the solution to the following local problem on Y:

$$(2.3) \qquad \frac{\partial}{\partial y_i} \left(C_{ijmn} \frac{\partial}{\partial y_n} \left(\chi^{(pq)}_m + \delta^p_m y^q \right) \right) = 0 \quad \text{in } Y,$$

The elementary unit periodic cell $Y, |Y| = 1$ is depicted in Fig. 1.

The parameter a denotes the length of a single crack. The crack is modeled by a "thin" rectangular void in which the ratio between sides is ca. 0.01. The solution to Eq. (2.3) is obtained by FEM and the anisotropic effective elasticity tensor is calculated from Eq. (2.1). This tensor exhibits anisotropy due to the directionality of cracks distribution. We observe that if a approaches 1, the technical constants calculated from Hooke's tensor behave in agreement with the fact that the material is cut along the lines parallel to the $0x$-axis, cf. [2].
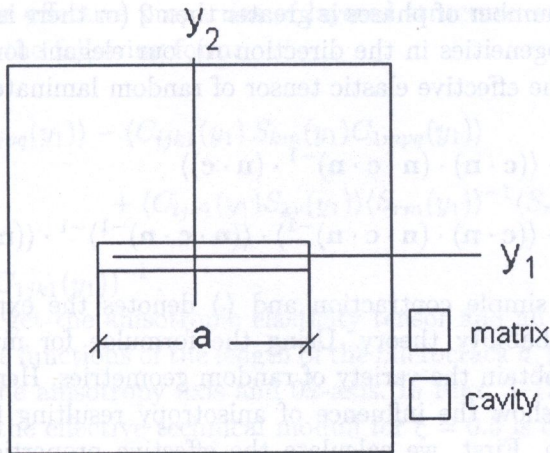
Fig. 1. Elementary unit cell.

a	0.8	0.9	0.95	0.99	0.9999
E_1	0.724	0.724	0.723	0.723	0.723
E_2	0.706	0.684	0.662	0.585	0.032
E_3	0.724	0.724	0.723	0.723	0.723
ν_{12}	0.449	0.435	0.412	0.373	0.020
ν_{21}	0.460	0.460	0.460	0.460	0.460
ν_{13}	0.460	0.460	0.460	0.460	0.460
ν_{31}	0.460	0.460	0.460	0.460	0.460
ν_{23}	0.460	0.460	0.460	0.460	0.460
ν_{32}	0.449	0.435	0.421	0.373	0.020
G_{12}	0.239	0.228	0.209	0.124	0.002
G_{13}	0.248	0.248	0.248	0.248	0.247
G_{23}	0.230	0.220	0.205	0.125	0.002

3. A RANDOM STRUCTURE WITH FIXED DIRECTION. LAMINATION FORMULAE

The fact that random structure of a two-phase composite can be approached by multiple rank coated lamination process is examined in [1, 4]. In general, the number of parameters which should be determined is infinite. The simplest case where randomness appears is a laminate of rank one. In this case, the direction of lamination is fixed and the probabilities of finding the phases in this direction

are given. If the number of phases is greater then 2 (or there is continuous distribution of inhomogeneities in the direction **n**), our elegant formula (in absolute notation) gives the effective elastic tensor of random laminate:

$$(3.1) \quad \mathbf{c}^h = \langle \mathbf{c} \rangle - \langle (\mathbf{c} \cdot \mathbf{n}) \cdot (\mathbf{n} \cdot \mathbf{c} \cdot \mathbf{n})^{-1} \cdot (\mathbf{n} \cdot \mathbf{c}) \rangle$$

$$+ \langle (\mathbf{c} \cdot \mathbf{n}) \cdot (\mathbf{n} \cdot \mathbf{c} \cdot \mathbf{n})^{-1} \rangle \cdot \langle (\mathbf{n} \cdot \mathbf{c} \cdot \mathbf{n})^{-1} \rangle^{-1} \cdot \langle (\mathbf{n} \cdot \mathbf{c} \cdot \mathbf{n})^{-1} \cdot (\mathbf{n} \cdot \mathbf{c}) \rangle$$

where · denotes simple contraction and $\langle \rangle$ denotes the expectation value in the sense of probability theory. Using the formulae for multiple lamination (cf. [3]) one can obtain the variety of random geometries. Here we study simple examples which show the influence of anisotropy resulting from the microfissures distribution. First, we calculate the effective properties of directionally uniformly distributed cracks. The effective material properties are transversally isotropic. Below, the technical constants of a cracked matrix are compared with technical constants of an uncracked one (the length of every crack is equal to 0.9):

	E_1	E_2	E_3	ν_{12}	ν_{21}	ν_{13}	ν_{31}	ν_{23}	ν_{32}	G_{12}	G_{13}	G_{23}
cracked	0.56	0.56	0.72	0.56	0.56	0.46	0.36	0.46	0.36	0.18	0.12	0.12
uncracked	0.73	0.73	0.73	0.46	0.46	0.46	0.46	0.46	0.46	0.50	0.50	0.50

The second structure of a random composite which is considered here is a layered medium composed of two anisotropic elastic materials. One describes the distribution of cracks parallel to the $0x$-axis, whereas the second microstructure is generated by cracks of differents orientations, cf. Fig. 2.
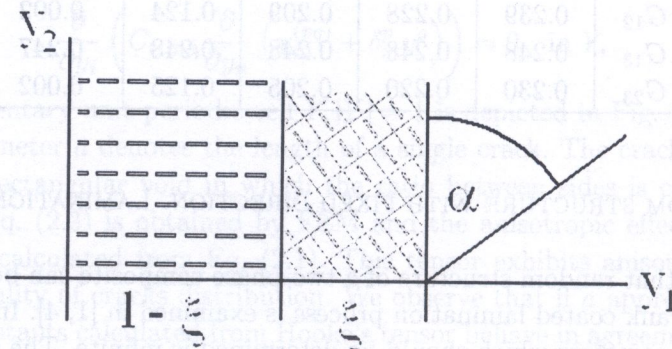

FIG. 2. Layered structure of cracked material.

To calculate the effective properties of layered structure we use the inditial notions of (3.1) in the following form:

$$(3.2) \quad C_{ijpq}^h = \langle C_{ijpq}(y_1) \rangle - \langle C_{ijk1}(y_1) S_{km}(y_1) C_{1mpq}(y_1) \rangle$$

$$+ \langle C_{ijk1}(y_1) S_{kr}(y_1) \rangle \langle S_{rm}(y_1) \rangle^{-1} \langle S_{mn}(y_1) C_{1npq}(y_1) \rangle.$$

where $S_{jk}(y_1) = (C_{1jk1}(y_1))^{-1}$.

As a result we get the anisotropic elasticity tensor and all technical moduli are found. They are functions of the length of the microcrack a, fraction ξ and the angle α between the anisotropy axis and $0x$-axis. In Figs. 3–12 the dependence on the angle α of the effective technical moduli for $\xi = 0.5$ is depicted.
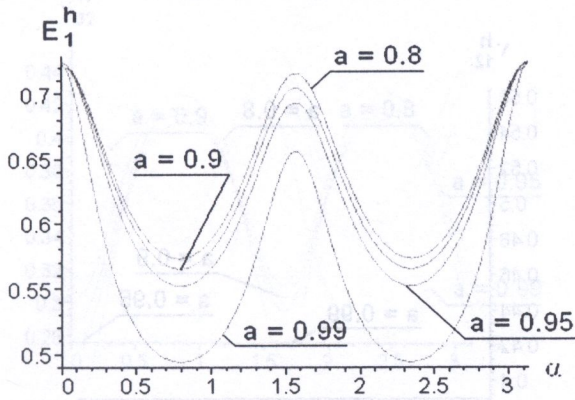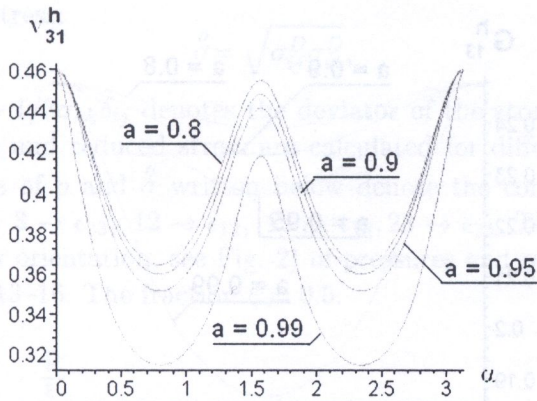

FIG. 3. E_1^h versus angle α.

FIG. 4. E_2^h versus angle α.

The calculated effective properties of layered medium and the distribution of inhomogeneities in the domain (involving collocation method of [13] in the notation) gives the effective elastic tensor of random laminate:

$$(3.2) \quad C_{ijpq} = \langle C_{ijpq}(y) \rangle - \langle C_{ijkl}(y) S_{kn}(y) C_{mnpq}(y) \rangle$$

$$+ \langle C_{ijkl}(y) \rangle \langle S_{kn}(y) \rangle \langle C_{mnpq}(y) \rangle,$$

where $S_{kn}(y) = \langle C_{ijkl}(y) \rangle^{-1}$...

As a result we get the dependence of the values of the technical moduli are found. They are functions of the microstructure a, fraction ζ and the angle α between the...

on the angle α of the effective results for $\gamma = 0.5$ is depicted.

$$E_3^h$$

FIG. 5. E_3^h versus angle α.

$$\nu_{12}^h$$

FIG. 6. ν_{12}^h versus angle α.

$$\nu_{21}^h$$

FIG. 7. ν_{21}^h versus angle α.

had the reduced str...

where σ_{ij}^D... deviator of the stress tensor. The hydrostatic pressure... corresponds to different strain states. The upper indices... $1 \leftarrow x_1$, $2 \leftarrow x_2$... the corresponding strains... on the angle α (crack orientation)... the induced stresses are depicted in Figs...

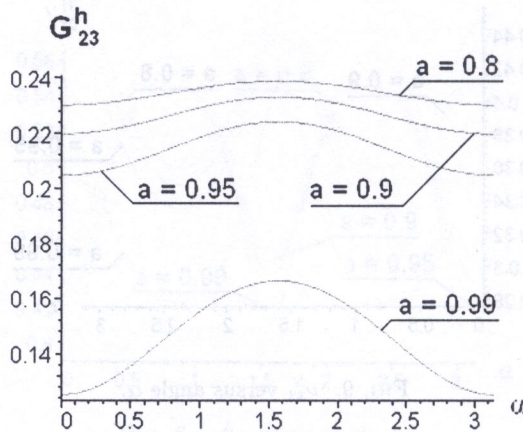FIG. 8. ν_{31}^h versus angle α.

FIG. 9. ν_{32}^h versus angle α.

FIG. 10. G_{12}^h versus angle α.

FIG. 11. G_{13}^h versus angle α.

FIG. 12. G_{23}^h versus angle α.

The Poisson coefficients ν_{13}^h and ν_{23}^h do not depend on the orientation α of cracks distribution.

4. DISTRIBUTION OF MICROCRACKS WHICH MINIMIZE (OR MAXIMIZE) THE STRESSES

To consider the influence of crack distribution on different states of stresses we calculate the hydrostatic pressure:

$$p = \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$

and the reduced stress:

$$\overset{o}{\sigma} = \sqrt{\sigma_{ij}^D \sigma_{ij}^D},$$

where $\sigma_{ij}^D = \sigma_{ij} - 1/3\sigma_{kk}\delta_{ij}$ denotes the deviator of the stress tensor. The hydrostatic pressure and reduced stress are calculated for different strain states. The upper indices of p and $\overset{o}{\sigma}$ written below denote the corresponding strain: $1 \to \epsilon_{11}$, $2 \to \epsilon_{22}$, $3 \to \epsilon_{33}$, $12 \to \epsilon_{12}$, $13 \to \epsilon_{13}$, $23 \to \epsilon_{23}$. The dependences on the angle α (crack orientation, see Fig. 2) of pressures and reduced stresses are depicted in Figs. 13–16. The fraction $\xi = 0.5$.
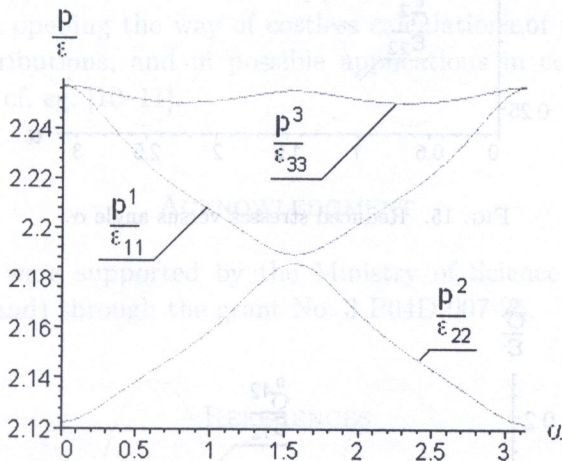
FIG. 13. Hydrostatic pressures versus angle α.
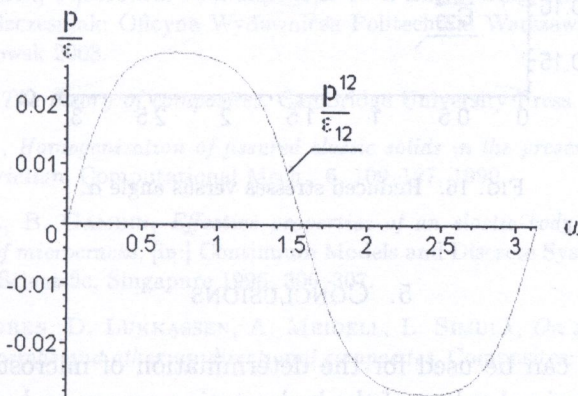
FIG. 14. Hydrostatic pressure versus angle α.
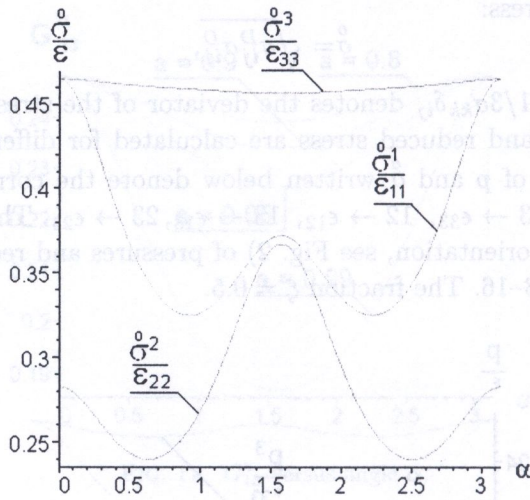
FIG. 15. Reduced stresses versus angle α.
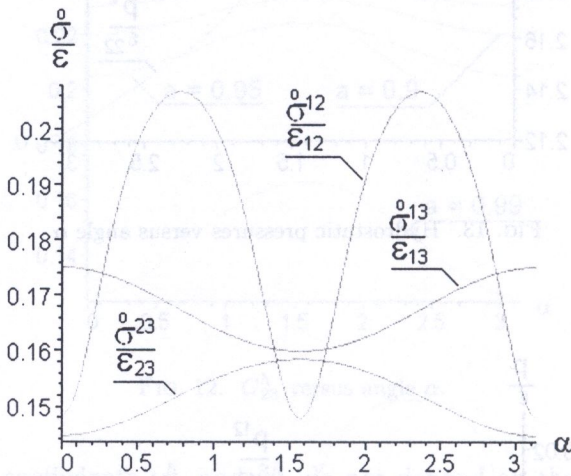
FIG. 16. Reduced stresses versus angle α.

5. CONCLUSIONS

Figures 13–16 can be used for the determination of microstructure rendering minimal or maximal values of the hydrostatic pressure and reduced stresses. If the cracks are mutually orthogonal, the hydrostatic pressure p^2 and reduced stresses $\overset{o}{\sigma}{}^2$ and $\overset{o}{\sigma}{}^{23}$ achieve maxima whereas p^1, $\overset{o}{\sigma}{}^{12}$, $\overset{o}{\sigma}{}^{13}$ achieve minima.

The stress $\overset{o}{\sigma}{}^{12}$ achieves maxima value for $\alpha = \pi/4$. For the same microstructure the hydrostatic pressure p^{12} is maximal. The reduced stress $\overset{o}{\sigma}{}^{1}$ reaches maximum for parallel distribution of cracks whilst minimal values are reached for cracks distributed at angle greater than $\pi/4$. Finally, the reduced stress $\overset{o}{\sigma}{}^{2}$ achieves minimum for cracks distributed at angle smaller than $\pi/4$. The results of using our costless semi-analytical numerical procedures (in Maple) were compared with calculations made in [7–9] by standard FEM (with more than 1000 elements in periodic cell) and the agreement of the results is quite satisfactory; the differences appeared on the 4-th decimal place. The main value of the presented paper consists in opening the way of costless calculations of more complicated microcracks distributions, and in possible applications in continuous damage micromechanics, cf. eg. [10–11].

ACKNOWLEDGMENT

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