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An Alternative Approach of Initial Stability Analysis of Kirchhoff Plates by the Boundary Element Method

Michał GUMINIAK

Poznan University of Technology
Piotrowo 5, 60-965 Poznań, Poland
e-mail: michal.guminiak@put.poznan.pl

An initial stability of Kirchhoff plates is analysed in the paper. Proposed approach avoids Kirchhoff forces at the plate corner and equivalent shear forces at a plate boundary. Two unknown variables are considered at the boundary element node. The governing integral equations are derived using Betti theorem. The integral equations have the form of boundary and domain integral equations. The constant type of boundary element are used. The singular and non-singular formulation of the boundary-domain integral equations with one and two collocation points associated with a single boundary element located at a plate edge are presented. To establish a plate curvature by double differentiation of basic boundary-domain integral equation, a plate domain is divided into rectangular sub-domains associated with suitable collocation points. A plate curvature can also be establish by considering three collocation points located in close proximity to each other along line pararel to one of the two axes of global coordinate system and establishment of appropriate differential operators.

Key words: the boundary element method, Kirchhoff plates, initial stability, fundamental solution.

1. Introduction

The Boundary Element Method (BEM) is one of many tools applied to the numerical analysis of structures. The main advantage of BEM is its relative simplicity of formulating and solving problems of the potential theory and the theory of elasticity. Many authors applied the boundary element method in wide aspects to static, dynamic and stability analysis of plates. Burczyński [1] described in a comprehensive manner the boundary element method and its application in a variety of fields, the theory of elasticity together with the appropriate solutions and a discussion of the basic types of boundary elements. The BEM found a wide application in the analysis of plates too. There are well known works of Altiero and Sikarskie [2], Bèzine and Gamby [3] and Stern [4] applied the boundary element method to solve the plate bending problem. The direct boundary element method in plate bending was applied by Hartmann and Zotemantel [5]. Comparisaon of the effectiveness of the boundary element method with the finite element method and application of the BEM in the analysis of thick plates was done by Debbih [6, 7]. The evaluation of boundary integrals for thin plate bending analysis was proposed by Abdel-Akher and Hartley [8]. Hartley [9] also proposed the plate bending theory for frame structures analysis. Beskos [10] and Wen, Aliabadi and Young [11] applied the BEM in the dynamic analysis of plates. Aliabadi and Wrobel [12] described an application of BEM in the thick plate analysis together with procedures for calculating singular and hypersingular integrals. A number of contributions devoted to the analysis of plates were presented by: Katsikadelis [13, 14], Katsikadelis and Yotis [15], Katsikadelis, Sapountzakis and Zorba [16], Katsikadelis and Kandilas [17], Katsikadelis and Sapountzakis [18]. Shi [19] applied BEM formulation for vibration and initial stability problem of orthotropic thin plates. In order to simplify the calculation procedures Guminiak, Okupniak and Sygulski [20] proposed a modified formulation of the boundary integral equation for a thin plate. This approach was extended for static, dynamic and stability analysis of thin plates and it is presented together with several numerical examples in many papers, e.g. [21–25]. Myślecki [26, 27] proposed BEM to static analysis of plane girders and BEM combined with approximate fundamental solutions for problem of plate bending resting on elastic foundation. Author used non-singular approach of boundary integral equations wherein the derivation of the second boundary integral equation was executed for additional collocation points located outside of a plate domain. The same approach of derivation of boundary integral equation was proposed by Myślecki and Oleńkiewicz [28, 29] to free vibration analysis of thin plates. Authors also used isoparametric, three-node boundary elemnt and applied dual reciprocity principle to dermine the inertia forces inside a plate domain. Very interesting approach was presented by Litewka and Sygulski [30, 31] who applied the Ganowicz [32] fundamental solutions for Reissner plates to static analysis of plates. Katsikadelis [33] applied BEM in a wide aspects of engineering analysis of plates.

In classic form the BEM is limited to linear problems with known fundamental solutions. To fully overcomes this drawback the conception of the Analog Equation Method (AEM) was created and introduced by Katsikadelis [34]. This version of BEM is basing on formulation of boundary-domain integral equation method and can treat efficiently not only linear problems, whose fundamental solution can not be established or it is difficult to treat numerically, but also nonlinear differential equations and systems of them as well. The method is based on the principle of the analog equation of Katsikadelis for differential equations. This conception was established to analysis of plate buckling by Nerantzaki and Katsikadelis [35] and Chinnaboon, Chucheepsakul and Katsikadelis [36]. Similarly Babouskos and Katsikadelis [37, 38] solved problem of flutter instability of dumped plate subjected by conservative and non-conservative loading. A wide review of literature devoted to application of BEM in plate analysis takes place in work of Guminiak and Litewka [39]. Authors compared thin [20, 21] and thick (Reissner) [30, 31] plate theory in therms of the modified BEM formulation and application of Ganowicz fundamental solutions [32]. Additionally, in the paper [39] the analysis of plate resting on internal flexible supports and plate with variable thickness in terms of AEM were presented.

In present paper, an analysis of plate initial stability the BEM will be presented. The analysis will focus on the modified [23] formulation of thin plate bending. The Bèzine [3] technique will be established to directly derive boundary-domain integral equation.

2. Integral formulation of plate bending and initial stability problem

The differential equation qoverning of plate initial stability has the form [40, 41]:

$$D \cdot \nabla^4 w = -\overline{p}, \tag{2.1}$$

where p is the substitute load, which has the form:

$$\overline{p} = N_x \cdot \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \cdot \frac{\partial^2 w}{\partial x \partial y} + N_y \cdot \frac{\partial^2 w}{\partial y^2}. \tag{2.2}$$

In the majority of contributions devoted to the application of BEM to the thin (Kirchhoff) plate theory, the derivation of the boundary integral equation involves the known boundary variables of the classic plate theory, i.e. the shear force and the concentrated corner forces. Thus, on the plate boundary there are considered the two physical quantities: the equivalent shear force V_n, reaction at the plate k-th corner R_k, the bending moment M_n, the corner concentrated forces and two geometric variables: the displacement w_b and the angle of rotation in the normal direction φ_n. The boundary integral equation can be derived using the Betti's theorem. Two plates are considered: an infinite plate, subjected to the unit concentrated force and a real one, subjected to the real in plane loadings N_x, N_{xy} and N_y. The plate bending problem is described in a unique way by two boundary-domain integral equations. The first equation has the form:

$$
\begin{aligned}
(2.3)\quad & c(\mathbf{x})\cdot w(\mathbf{x})+\int_{\Gamma}[V_n^*(\mathbf{y},\mathbf{x})\cdot w_b(\mathbf{y})-M_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})]\cdot \mathrm{d}\Gamma(\mathbf{y})-\sum_{k=1}^{K}R^*(k,\mathbf{x})\cdot w(k) \\
&= \int_{\Gamma}[V_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y},\mathbf{x})\cdot\varphi_n^*(\mathbf{y},\mathbf{x})]\cdot \mathrm{d}\Gamma(\mathbf{y})-\sum_{k=1}^{K}R_k\cdot w^*(k,\mathbf{x}) \\
&\quad +\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot w^*(\mathbf{y},\mathbf{x})\cdot \mathrm{d}\Omega(\mathbf{y}),
\end{aligned}
$$

where the fundamental solution of this biharmonic equation

$$
\nabla^4 w^*(\mathbf{y},\mathbf{x})=\frac{1}{D}\cdot\delta(\mathbf{y},\mathbf{x}) \tag{2.4}
$$

which is the free space Green function given as

$$
w^*(\mathbf{y},\mathbf{x})=\frac{1}{8\pi D}\cdot r^2\cdot\ln(r) \tag{2.5}
$$

for a thin isotropic plate, $r=|\mathbf{y}-\mathbf{x}|$, δ is the Dirac delta, $D=\dfrac{E\,h^3}{12(1-v^2)}$ is the plate stiffness, $\mathbf{x}$ is the source point and $\mathbf{y}$ is a field point. The coefficient $c(\mathbf{x})$ is taken as:

$c(\mathbf{x})=1$, when $\mathbf{x}$ is located inside the plate domain,
$c(\mathbf{x})=0.5$, when $\mathbf{x}$ is located on the smooth boundary,
$c(\mathbf{x})=0$, when $\mathbf{x}$ is located outside the plate domain.

The second boundary-domain integral equation can be obtained replacing the unit concentrated force $P^*=1$ by the unit concentrated moment $M_n^*=1$. Such a replacement is equivalent to the differentiation of the first boundary integral equation (2.3) with respect to the co-ordinate n at a point $\mathbf{x}$ belonging to the plate domain and letting this point approach the boundary and taking n coincide with the normal to it. The resulting equation has the form:

$$
\begin{aligned}
(2.6)\quad & c(\mathbf{x})\cdot\varphi_n(\mathbf{x})+\int_{\Gamma}\Big[\overline{V}_n^*(\mathbf{y},\mathbf{x})\cdot w_b(\mathbf{y})-\overline{M}_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\Big]\cdot \mathrm{d}\Gamma(\mathbf{y})-\sum_{k=1}^{K}\overline{R}^*(k,\mathbf{x})\cdot w(k) \\
&= \int_{\Gamma}[V_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})]\cdot \mathrm{d}\Gamma(\mathbf{y})-\sum_{k=1}^{K}R_k\cdot\overline{w}^*(k,\mathbf{x}) \\
&\quad +\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot\overline{w}^*(\mathbf{y},\mathbf{x})\cdot \mathrm{d}\Omega(y),
\end{aligned}
$$

where

$$\left\{\overline{V}_n^*(\mathbf{y},\mathbf{x}),\overline{M}_n^*(\mathbf{y},\mathbf{x}),\overline{R}^*(\mathbf{y},\mathbf{x}),\overline{w}^*(\mathbf{y},\mathbf{x}),\overline{w}^*(\mathbf{y},\mathbf{x}),\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right\}$$
$$=\frac{\partial}{\partial n(\mathbf{x})}\left\{V_n^*(\mathbf{y},\mathbf{x}),M_n^*(\mathbf{y},\mathbf{x}),R^*(k,\mathbf{x}),w^*(k,\mathbf{x}),w^*(\mathbf{y},\mathbf{x}),\varphi_n^*(\mathbf{y},\mathbf{x})\right\}.$$

The second boundary-domain integral equation can be also derived by introducing additional collocation point, which is located in the same normal line outside the plate edge. According this approach, the second equation has the same mathematical form as the first one (2.3). This double collocation point approach was presented in publication [27–29].

The detailed procedure for the derivation of the fundamental solution, the integral representation of the solution and the two boundary-domain integral equations is presented by Katsikadelis in [33]. The issues related to the assembly of the algebraic equations in terms of the classical boundary element method are discussed in many papers, including [33].

The plate bending problem can also be formulated in a modified, simplified way using an integral representation of the plate biharmonic equation. Because the concentrated force at the corner is used only to satisfy the differential biharmonic equation of the thin plate, one can assume, that it could be distributed along a plate edge segment close to the corner. Hence, the terms in the boundary integral Eqs. (2.5) and (2.6) which correspond to the corner force R can be substituted in the following way:

$$-\sum_{k=1}^{K}R_k\cdot w^*(k,\mathbf{x})=\int_{\Gamma_k}R_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})\cdot \mathrm{d}\Gamma_k(\mathbf{y}), \tag{2.7}$$

$$-\sum_{k=1}^{K}R_k\cdot \overline{w}^*(k,\mathbf{x})=\int_{\Gamma_k}R_n(\mathbf{y})\cdot \overline{w}^*(\mathbf{y},\mathbf{x})\cdot \mathrm{d}\Gamma_k(\mathbf{y}), \tag{2.8}$$

where the subscript k denotes an unknown segment of the plate edge near the corner. In the Eqs. (2.7) and (2.8) the fundamental twisting moment $M_{ns}^*(\mathbf{y})$ must be considered, too. Hence, the boundary integral equations will take the form:

$$\begin{aligned}c(\mathbf{x})\cdot w(\mathbf{x})+\int_{\Gamma}\left[T_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-M_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})-M_{ns}^*(\mathbf{y},\mathbf{x})\cdot\varphi_s(\mathbf{y})\right]\cdot\,\mathrm{d}\Gamma(\mathbf{y})\\
=\int_{\Gamma}\left[T_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\varphi_n^*(\mathbf{y},\mathbf{x})\right]\cdot\mathrm{d}\Gamma(\mathbf{y})+\int_{\Gamma_k}R_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})\cdot\mathrm{d}\Gamma_k(\mathbf{y})\\
+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot w^*(\mathbf{y},\mathbf{x})\cdot\mathrm{d}\Omega(\mathbf{y}),\end{aligned} \tag{2.9}$$

$$(2.10)\quad c(\mathbf{x})\cdot\varphi_n(\mathbf{x})+\int_{\Gamma}\Big[\overline{T}_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-\overline{M}_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})-\overline{M}_{ns}^*(\mathbf{y},\mathbf{x})\cdot\varphi_s(\mathbf{y})\Big]\cdot \mathrm{d}\Gamma(\mathbf{y})$$
$$=\int_{\Gamma}[T_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})]\cdot \mathrm{d}\Gamma(\mathbf{y})$$
$$+\int_{\Gamma_k}R_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})\cdot \mathrm{d}\Gamma_k(\mathbf{y})+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)$$
$$\cdot\,\overline{w}^*(\mathbf{y},\mathbf{x})\cdot \mathrm{d}\Omega(\mathbf{y}).$$

Because the length k of the plate edge segment is unknown, the selected components of the Eqs. (2.9) and (2.10) can form a common integral:

$$(2.11)\quad c(\mathbf{x})\cdot w(\mathbf{x})+\int_{\Gamma}[T_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-M_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})-M_{ns}^*(\mathbf{y},\mathbf{x})\cdot\varphi_s(\mathbf{y})]\cdot \mathrm{d}\Gamma(\mathbf{y})$$
$$=\int_{\Gamma}[T_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})+R_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\varphi_n^*(\mathbf{y},\mathbf{x})]\cdot \mathrm{d}\Gamma(\mathbf{y})$$
$$+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot w^*(\mathbf{y},\mathbf{x})\cdot \mathrm{d}\Omega(\mathbf{y}),$$

$$(2.12)\quad c(\mathbf{x})\cdot\varphi_n(\mathbf{x})+\int_{\Gamma}\Big[\overline{T}_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-\overline{M}_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})-\overline{M}_{ns}^*(\mathbf{y},\mathbf{x})\cdot\varphi_s(\mathbf{y})\Big]\cdot \mathrm{d}\Gamma(\mathbf{y})$$
$$=\int_{\Gamma}[T_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})+R_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})]\cdot \mathrm{d}\Gamma(\mathbf{y})$$
$$+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot\overline{w}^*(\mathbf{y},\mathbf{x})\cdot \mathrm{d}\Omega(\mathbf{y}).$$

Then, the common factor $w*$ can be separated:

$$(2.13)\quad c(\mathbf{x})\cdot w(\mathbf{x})+\int_{\Gamma}[T_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-M_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})-M_{ns}^*(\mathbf{y},\mathbf{x})\cdot\varphi_s(\mathbf{y})]\cdot \mathrm{d}\Gamma(\mathbf{y})$$
$$=\int_{\Gamma}[(T_n(\mathbf{y})+R_n(\mathbf{y}))\cdot w^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\varphi_n^*(\mathbf{y},\mathbf{x})]\cdot \mathrm{d}\Gamma(\mathbf{y})$$
$$+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot w^*(\mathbf{y},\mathbf{x})\cdot \mathrm{d}\Omega(\mathbf{y}),$$

$$c(\mathbf{x})\cdot\varphi_n(\mathbf{x})+\int_{\Gamma}\Big[\overline{T}_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-\overline{M}_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})-\overline{M}_{ns}^*(\mathbf{y},\mathbf{x})\cdot\varphi_s(\mathbf{y})\Big]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$=\int_{\Gamma}\left[(T_n(\mathbf{y})+R_n(\mathbf{y}))\cdot\overline{w}^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot\overline{w}^*(\mathbf{y},\mathbf{x})\cdot\mathrm{d}\Omega(\mathbf{y}). \tag{2.14}$$

Now, the new notation is introduced:

$$\widetilde{T}_n(\mathbf{y})=T_n(\mathbf{y})+R_n(\mathbf{y}). \tag{2.15}$$

Hence, the boundary integral equations will have the form:

$$c(\mathbf{x})\cdot w(\mathbf{x})+\int_{\Gamma}\left[T_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-M_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})-M_{ns}^*(\mathbf{y},\mathbf{x})\cdot\varphi_s(\mathbf{y})\right]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$=\int_{\Gamma}\Big[\widetilde{T}_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\varphi_n^*(\mathbf{y},\mathbf{x})\Big]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot w^*(\mathbf{y},\mathbf{x})\cdot\mathrm{d}\Omega(\mathbf{y}), \tag{2.16}$$

$$c(\mathbf{x})\cdot\varphi_n(\mathbf{x})+\int_{\Gamma}\Big[\overline{T}_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-\overline{M}_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})-\overline{M}_{ns}^*(\mathbf{y},\mathbf{x})\cdot\varphi_s(\mathbf{y})\Big]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$=\int_{\Gamma}\Big[\widetilde{T}_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\Big]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot\overline{w}^*(\mathbf{y},\mathbf{x})\cdot\mathrm{d}\Omega(\mathbf{y}). \tag{2.17}$$

The expression (2.15) denotes shear force for clamped and for simply-supported edges:

$$\widetilde{T}_n(\mathbf{y})=\begin{cases}V_n(\mathbf{y}),\\ R_n(\mathbf{y}).\end{cases}$$

Because in all the cases (Eqs. (2.3), (2.6) and (2.11), (2.12)) the forces on the real plate: $V_n(\mathbf{y})$ and $T_n(\mathbf{y})$ are multiplied by the same fundamental functions $w^*(\mathbf{y},\mathbf{x})$ and $\overline{w}^*(\mathbf{y},\mathbf{x})$, the force $\widetilde{T}_n(\mathbf{y})$ can be treated as an equivalent shear force $V_n(\mathbf{y})$ on a fragment of the boundary which is located far from the corner. In the case of the free edge we must combine the angle of rotation in the tangent direction $\varphi_s(\mathbf{y})$ with the fundamental function $M^*_{ns}(\mathbf{y})$. Because the relation between $\varphi_s(\mathbf{y})$ and the deflection is known: $\varphi_s(\mathbf{y}) = \dfrac{\mathrm{d}w(\mathbf{y})}{\mathrm{d}s}$, the angle of rotation $\varphi_s(\mathbf{y})$ can be evaluated using a finite difference scheme of the deflection with two or more adjacent nodal values. In this analysis, the employed finite difference scheme includes the deflections of three adjacent nodes. As a result, the boundary integral Eqs. (2.16) and (2.17) will take the form:

$$c(\mathbf{x})\cdot w(\mathbf{x})+\int_{\Gamma}\left[T^*_n(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-M^*_{ns}(\mathbf{y},\mathbf{x})\cdot\frac{\mathrm{d}w(\mathbf{y})}{\mathrm{d}s}-M^*_n(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\right]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$=\int_{\Gamma}\left[\widetilde{T}_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\varphi^*_n(\mathbf{y},\mathbf{x})\right]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot w^*(\mathbf{y},\mathbf{x})\cdot\mathrm{d}\Omega(\mathbf{y}), \tag{2.18}$$

$$c(\mathbf{x})\cdot\varphi_n(\mathbf{x})+\int_{\Gamma}\left[\overline{T}^*_n(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-\overline{M}^*_{ns}(\mathbf{y},\mathbf{x})\cdot\frac{\mathrm{d}w(\mathbf{y})}{\mathrm{d}s}-\overline{M}^*_n(\mathbf{y},\mathbf{x})\cdot\varphi_n(y)\right]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$=\int_{\Gamma}\left[\widetilde{T}_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\overline{\varphi}^*_n(\mathbf{y},\mathbf{x})\right]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$+\int_{\Omega}\left(N_x\cdot\frac{\partial^2 w}{\partial x^2}+2N_{xy}\cdot\frac{\partial^2 w}{\partial x\partial y}+N_y\cdot\frac{\partial^2 w}{\partial y^2}\right)\cdot\overline{w}^*(\mathbf{y},\mathbf{x})\cdot\mathrm{d}\Omega(\mathbf{y}). \tag{2.19}$$

3. Construction of set of algebraic equations

The plate boundary is discretized by elements of the constant type. Three approaches of constructing the boundary integral equations are considered. According to the first one, singular approach, the collocation points are located exactly on the plate boundary (Fig. 1).

According to the second, non-singular approach, the boundary integral equations can be formulated using one collocation point (Fig. 2a) or two collocation points (Fig. 2b) located outside of the plate boundary on the line normal to the plate edge.

It is assumed that a rectangular plate is compressed only by N_x forces. Then, in the boundary integral Eqs. (2.16) and (2.17) takes a stand only the part $N_x\cdot\left(\partial^2 w/\partial x^2\right)$. The unknown variable in internal collocation points is the parameter $\kappa = \partial^2 w/\partial x^2$, the plate curvature in x direction [19, 23]. It is also assumed, that a plate has a regular shape without any holes. According to these

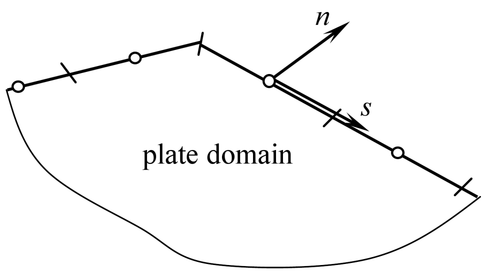

FIG. 1. Collocation point assigned to the boundary element of the constant type.

a) b)

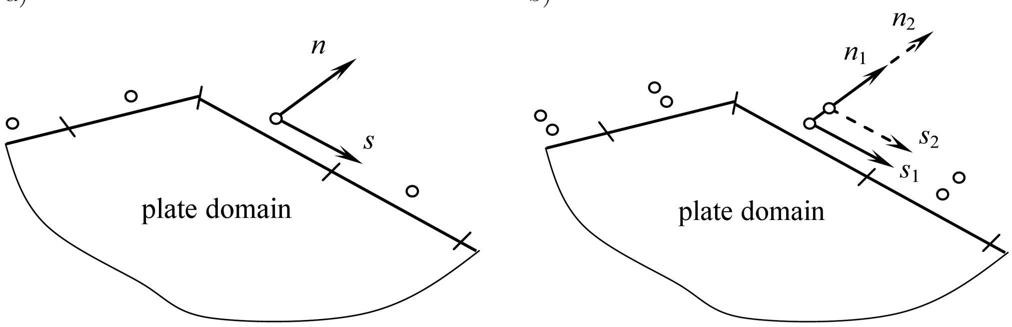

FIG. 2. One collocation point a) and two collocation points b) assigned to the boundary element of the constant type.

assumptions it is possible to accept an arbitrary linear distribution of the normal loading along plate edge perpendicular to the x direction. The plate domain Ω is divided into finite number of sub-domains just to define a plate curvature in selected internal collocation points associated with these sub-domains Ω_m. The normal loading N_x is constant on the length of the single internal sub-domain side which shows Fig. 3.

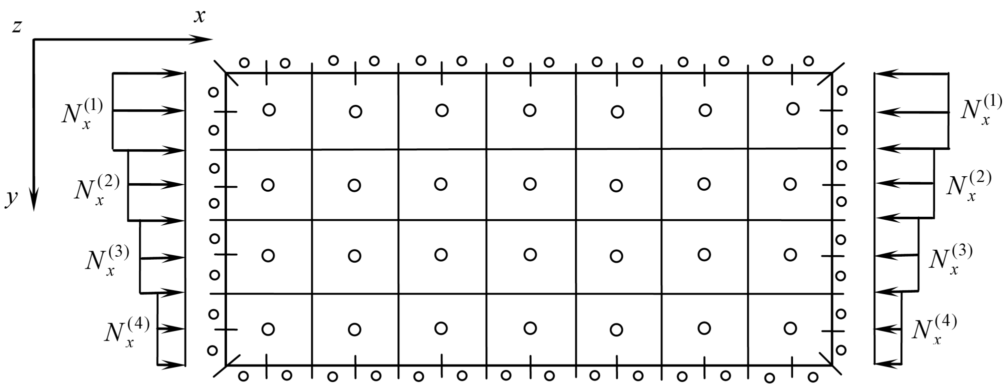

FIG. 3. Distribution of in plane loading.

The set of $N_x^{(i)}$ forces is expressed by the comparative normal loading $N{=}N_{\rm cr}$. Hence, the set of algebraic equation can be written in the form [23]:

$$\begin{bmatrix} \mathbf{G_{BB}} & \mathbf{G_{BS}} & -\lambda \cdot \mathbf{G_{B\kappa}} \\ \mathbf{\Delta} & -\mathbf{I} & \mathbf{0} \\ \mathbf{G_{\kappa B}} & \mathbf{G_{\kappa S}} & -\lambda \cdot \mathbf{G_{\kappa\kappa}} + \mathbf{I} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{B} \\ \varphi_{\mathbf{S}} \\ \boldsymbol{\kappa} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \tag{3.1}$$

where $\lambda = N_{\rm cr}$ and $\mathbf{G_{BB}}$ and $\mathbf{G_{BS}}$ are the matrices of the dimensions of the dimension $(2N \times 2N)$ and of the dimension $(2N \times S)$ grouping boundary integrals and depend on type of boundary, where N is the number of boundary nodes (or the number of the elements of the constant type) and S is the number of boundary elements along free edge; $\mathbf{G_{B\kappa}}$ is the matrix of the dimension $(2N \times 2N)$ grouping integrals over the internal sub-domains Ω_m; $\mathbf{\Delta}$ is the matrix grouping difference operators connecting angle of rotations in tangent direction with deflections of suitable boundary nodes if a plate has a free edge.

The third matrix equation $(3.1)_3$ in the set of equation (3.1) is obtained by construction the boundary integral equations for internal collocation points associated with internal sub-domains Ω_m. According the typical approach, in this equation, the plate curvature can be derived by double differentiation of boundary integral Eq. (2.16) and by constructing one integral equation with respect to central collocation point "1" belonging to each internal sub-surface. Therefore $\mathbf{G_{\kappa B}}$ is the matrix of the dimension $(M \times 2N)$ grouping the boundary integrals of the second derivatives with respect to the co-ordinate x of the appropriate fundamental functions, where M is the number of the internal collocation points and N is the number of the boundary nodes; $\mathbf{G_{\kappa S}}$ is the matrix of the dimension $(M \times S)$ grouping the boundary integrals of the second derivatives with respect to the co-ordinate x of the appropriate fundamental functions; $\mathbf{G_{\kappa\kappa}}$ is the matrix of the dimension $(M \times M)$ grouping the integrals of the second derivatives with respect to the co-ordinate x over the internal sub-surfaces $\Omega_m \in \Omega$.

In accordance with the simplified approach, the plate curvature can be also establish by addition two internal collocation points ("2" and "3"). Due to this conception it is necessary to construct three integral equation considering three collocation points ("1", "2" and "3") and using Eq. (2.18) in unchanged form. These two approaches are illustrated in Fig. 4.

According the second approach the plate curvature at central point "1" is calculated by constructing difference quotient:

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}_x = \frac{\Delta^2 \mathbf{w}}{\Delta x^2} = \frac{\mathbf{w}_2 - 2 \cdot \mathbf{w}_1 + \mathbf{w}_3}{(\Delta x)^2}. \tag{3.2}$$

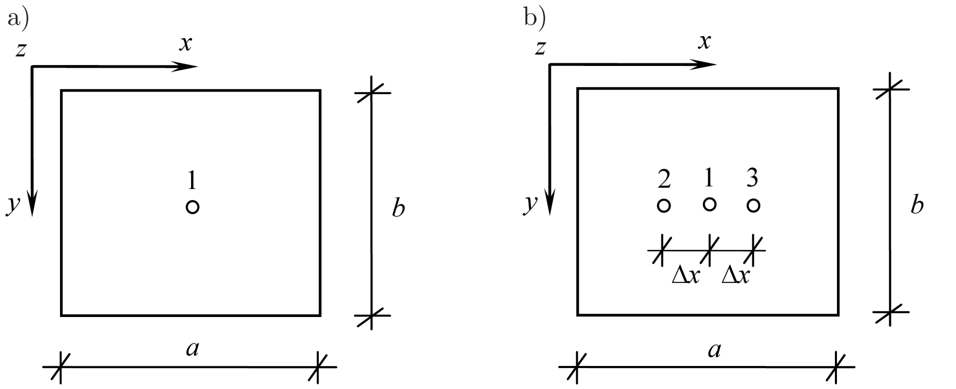

FIG. 4. Definition of the curvature in central collocation point.

Hence elements of the matrices $\mathbf{G}_{\boldsymbol{\kappa}\mathbf{B}}$, $\mathbf{G}_{\boldsymbol{\kappa}\mathbf{S}}$ and $\mathbf{G}_{\boldsymbol{\kappa}\boldsymbol{\kappa}}$ can be evaluated using three boundary integral equations based only on the boundary integral Eq. (2.18). Elimination of boundary variables $\mathbf{B}$ and $\varphi_{\mathbf{S}}$ from matrix Eq. (3.2) leads to standard eigenvalue problem:

$$\left\{\mathbf{A} - \widetilde{\lambda} \cdot \mathbf{I}\right\} \cdot \boldsymbol{\kappa} = \mathbf{0}, \tag{3.3}$$

where $\widetilde{\lambda} = 1/\lambda$ and

$$\mathbf{A} = \left\{\mathbf{G}_{\boldsymbol{\kappa}\boldsymbol{\kappa}} - (\mathbf{G}_{\boldsymbol{\kappa}\mathbf{B}} - \mathbf{G}_{\boldsymbol{\kappa}\mathbf{S}} \cdot \boldsymbol{\Delta}) \cdot [\mathbf{G}_{\mathbf{BB}} + \mathbf{G}_{\mathbf{BS}}]^{-1} \cdot \mathbf{G}_{\mathbf{B}\boldsymbol{\kappa}}\right\}. \tag{3.4}$$

The same problem of plate stability can also be formulated in terms of the Analog Equation Method. The plate bending is expressed by differential Eq. (2.1). It is assumed, that plate is compressed only by N_x forces, the governing equation will take the form

$$D \cdot \nabla^4 w + N_x \cdot \frac{\partial^2 w}{\partial x^2} = 0. \tag{3.5}$$

The real problem can be replaced by the analogous issue, which is described by the following differential equation

$$\nabla^4 w = b(x, y). \tag{3.6}$$

In this issue, the boundary conditions are the same as in the real one and $b(x, y)$ is the unknown function of a fictitious loading. The solution of Eq. (3.6) can be expressed using integral representation by two equations:

$$c(\mathbf{x})\cdot w(\mathbf{x})+\int\limits_{\Gamma}\left[T_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-M_{ns}^*(\mathbf{y},\mathbf{x})\cdot\frac{\mathrm{d}w(\mathbf{y})}{\mathrm{d}s}-M_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\right]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$=\int\limits_{\Gamma}\left[\widetilde{T}_n(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\varphi_n^*(\mathbf{y},\mathbf{x})\right]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$+\int\limits_{\Omega}b(\mathbf{y})\cdot w^*(\mathbf{y},\mathbf{x})\cdot\mathrm{d}\Omega(\mathbf{y}), \tag{3.7}$$

$$c(\mathbf{x})\cdot\varphi_n(\mathbf{x})+\int\limits_{\Gamma}\left[\overline{T}_n^*(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-\overline{M}_{ns}^*(\mathbf{y},\mathbf{x})\cdot\frac{\mathrm{d}w(\mathbf{y})}{\mathrm{d}s}-\overline{M}_n^*(\mathbf{y},\mathbf{x})\cdot\varphi_n(\mathbf{y})\right]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$=\int\limits_{\Gamma}\left[\widetilde{T}_n(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})-M_n(\mathbf{y})\cdot\overline{\varphi}_n^*(\mathbf{y},\mathbf{x})\right]\cdot\mathrm{d}\Gamma(\mathbf{y})$$
$$+\int\limits_{\Omega}b(\mathbf{y})\cdot\overline{w}^*(\mathbf{y},\mathbf{x})\cdot\mathrm{d}\Omega(\mathbf{y}), \tag{3.8}$$

where the fundamental solution is known and expressed by Eq. (2.5).

The plate domain may be discretized using internal sub-surfaces acting as constant domain elements $\Omega_m \in \Omega$ or linear elements [33, 35]. In each internal collocation point the fictitious loading vector $\mathbf{b}$ is introduced. The boundary-domain integral Eqs. (3.7) and (3.8) allow to specify the boundary conditions on each plate edges and the second derivetives of the plate displacement in each of the internal collocation points. Substitution of Eq. (3.6), Eqs. (3.7) and (3.8) which express the boundary conditions and double-differentiated Eq. (3.7) which describes second derivatives with respect to the x global coordinate into governing Eq. (3.5) leads to the standard eigenvalue problem where the eigen multiplier is equal $\widetilde{\lambda} = 1/N_{\mathrm{cr}}$. If the plate domain is divided into rectangular sub-surfaces of the constant type (each sub-surface is associated with one central collocation point in which the plate curvature is established) the AEM approach becomes special case equivalent to the direct Bèzine technique.

4. Modes of buckling

The elements of the eigenvector $\boldsymbol{\kappa}$ obtained after solution of the standard eigenvalue problem (3.3) present the plate curvatures. The set of the algebraic equation indispensable to calculate the eigenvector $\mathbf{w}$ elements has a form:

$$\begin{bmatrix} \mathbf{G_{BB}} & \mathbf{G_{BS}} & \mathbf{0} \\ \boldsymbol{\Delta} & -\mathbf{I} & \mathbf{0} \\ \mathbf{G_{\kappa B}} & \mathbf{G_{\kappa S}} & \mathbf{I} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{B} \\ \varphi_{\mathbf{s}} \\ \mathbf{w} \end{Bmatrix} = \begin{Bmatrix} \lambda\cdot\mathbf{G_{B\kappa}}\cdot\boldsymbol{\kappa} \\ \mathbf{0} \\ \lambda\cdot\mathbf{G_{\kappa\kappa}}\cdot\boldsymbol{\kappa} \end{Bmatrix}. \tag{4.1}$$

In the set of the Eq. (4.1) the first and second Eqs. $(4.1)_1$ and $(4.1)_2$ are obtained from the first and second equations of (3.1) and the third Eq. $(4.1)_3$ is gotten by construction the boundary integral equations for calculating the plate deflection in internal collocation points. Elimination of the boundary variables $\mathbf{B}$ and $\boldsymbol{\varphi}_\mathbf{S}$ from Eq. (4.1) gives the elements of the wanted displacement vector:

$$\mathbf{w} = \lambda \cdot \left[\mathbf{G}_{\boldsymbol{\kappa}\boldsymbol{\kappa}} - (\mathbf{G}_{\boldsymbol{\kappa}\mathbf{B}} - \mathbf{G}_{\boldsymbol{\kappa}\mathbf{S}} \cdot \boldsymbol{\Delta}) \cdot [\mathbf{G}_{\mathbf{BB}} + \mathbf{G}_{\mathbf{BS}}]^{-1} \cdot \mathbf{G}_{\mathbf{B}\boldsymbol{\kappa}}\right] \cdot \boldsymbol{\kappa}. \tag{4.2}$$

5. Numerical examples

The initial stability problem of a square and rectangular plates, simply-supported on each edge and a square plate simply- supported on two opposite edges with two remaining free edges is considered. For each of them the critical value of the normal loading is investigated. Each of plate edge is divided by the boundary elements of the constant type with the same length. The set of the internal collocation points is regular. The plate properties are: Young modulus $E = 205$ GPa, Poisson ratio $v = 0.3$. The following notations are assumed:

BEM I – singular formulation of governing boundary-domain integral Eqs. (2.18) and (2.19) with second equation obtained by single differentiation of Eq. (2.18), the vector of curvatures is established by double differentiation of the first governing boundary-domain integral Eq. (2.18);

BEM II – non-singular formulation of governing boundary-domain integral Eqs. (2.18) and (2.19), with second Eq. (2.19) obtained by differentiation of Eq. (2.18), the vector of curvatures is established by double differentiation of the first governing boundary-domain integral Eq. (2.18). The collocation point of single boundary element is located outside, near the plate edge. For one collocation point: $\varepsilon_1 = \widetilde{\delta}_1/d = 0.001$ [23] where $\widetilde{\delta}_1$ is distance of collocation point from the plate edge and d is the boundary element length;

BEM III – non-singular formulation of governing boundary-domain integral Eqs. (2.18) and (2.19), with second Eq. (2.19) obtained for the set of additional collocation points with the same fundamental solution w^*, the vector of curvatures is established by constructing difference quotient (3.2) and fundamental solution w^*. Localization of two collocation points for single boundary element is determined by: $\varepsilon_1 = 0.001$ and $\varepsilon_2 = \widetilde{\delta}_2/d = 0.01$. For three collocation point belonging for each internal sub-domain element: $\varepsilon_\Delta = \Delta x/a = 0.001$.

FEM – regular finite element mesh 0.5 m × 0.5 m and element type of S4R (four node with three degree of freedom per node) of ABAQUS program with reduced integration were assumed into comparative analysis.

The critical force $N_{\rm cr}$ is expressed using non-dimensional term:

$$\widetilde{N}_{\rm cr} = \frac{N_{\rm cr}}{D} \cdot l_x \cdot l_y. \tag{5.1}$$

5.1. A simply-supported rectangular plate under uniformly constant normal loading

Static and loading scheme is shown in the Fig. 5.

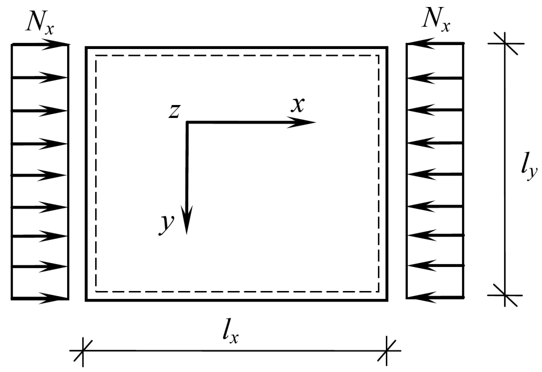

FIG. 5. A simply-supported rectangular plate under uniformly constant normal loading.

Two plates are considered: a) square and b) rectangular. In case a) the plate boundary was discretized using 256 number of boundary elements. The number of internal sub-surfaces used to describe the plate curvature is equal: 256. The plate geometry is defined as: $l_x = l_y = l = 2.0$ m. In case b) the plate boundary was discretized using 128 number of boundary elements. The number of internal sub-surfaces used to describe the plate curvature is equal: 512. The plate geometry is defined as: $l_x = 0.5 \cdot l_y = 2.0$ m. In both cases the following the plate thickness is equal $h = 0.05$ m. Each plate edge is divided into number of 64 in case a) and 32 in case b) boundary elements of the same length. The set of internal square sub-domains is reagular. The results of calculation are presented in Tables 1–3. The influence of localization of internal collocation points on critical force values for square plate a) using BEM III approach is presented in Table 2.

Table 1. Critical force values: $l_x/l_y = 1.0$.

$\widetilde{N}_{\rm cr}$	Analytical solution [40, 41]	BEM I solution	BEM II solution	BEM III solution
1	39.4784	39.6198	39.6228	39.6350
2	61.6850	62.1887	62.1916	62.1996
3	109.6623	111.3933	111.3962	111.4057

Table 2. Critical force values: $l_x/l_y = 0.5$.

$\widetilde{N}_{\mathrm{cr}}$	Analytical solution [40, 41]	BEM I solution	BEM II solution	BEM III solution
1	30.8425	30.9227	30.9230	30.9253

Table 3. Critical force values: $l_x/l_y = 1.0$. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$.

$\widetilde{N}_{\mathrm{cr}}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	39.6350	39.6350	39.6351	39.6362	39.6398
2	62.1993	62.1996	62.1996	62.2073	62.2311
3	111.4057	111.4057	111.4057	111.4385	111.5362

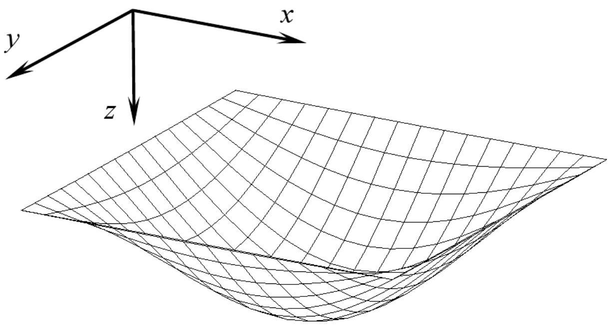

FIG. 6. The first buckling mode, $l_x/l_y = 1.0$.

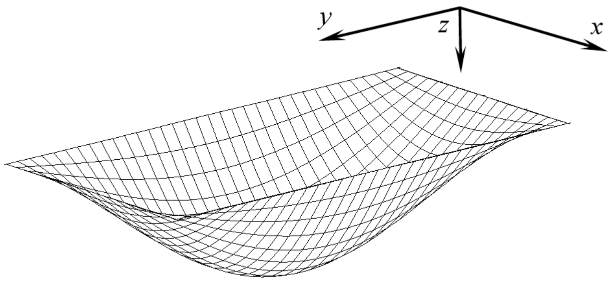

FIG. 7. The first buckling mode, $l_x/l_y = 0.5$.

5.2. A simply-supported rectangular plate under uniformly linear normal loading

Static and loading scheme is shown in the Fig. 8.

Two plates are considered: a) square and b) rectangular. In case a) the plate boundary was discretized using 256 number of boundary elements (64 elements

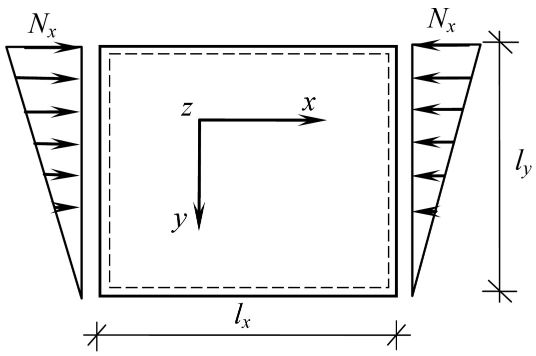

FIG. 8. A simply-supported rectangular plate under uniformly linear normal loading.

on each edge). The number of internal sub-surfaces used to describe the plate curvature is equal: 256. The plate geometry is defined as: $l_x = l_y = l = 2.0$ m. In case b) the plate boundary was discretized using 128 number of boundary elements (32 elements on each edge). The number of internal sub-surfaces used to describe the plate curvature is equal: 384. The plate geometry is defined as: $l_x = 1.5 \cdot l_y = 3.0$ m. In both cases the plate properties were assumed identically as in Example 5.1. The results of calculation are presented in Tables 4–6.

Table 4. Critical force values $l_x/l_y = 1.0$.

$\widetilde{N}_{\text{cr}}$	Analytical solution [41]	BEM I solution	BEM II solution	BEM III solution
1	76.9829	77.3858	77.3918	77.4153
2	–	115.7346	115.7401	115.7566
3	–	194.3706	194.3797	194.4041

Table 5. Critical force values: $l_x/l_y = 1.0$. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$.

$\widetilde{N}_{\text{cr}}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	77.4153	77.4153	77.4153	77.4175	77.4245
2	115.7566	115.7566	115.7566	115.7707	115.8148
3	194.4038	194.4041	194.4043	194.4599	194.6296

Table 6. Critical force values $l_x/l_y = 1.5$.

$\widetilde{N}_{\text{cr}}$	Analytical solution [41]	BEM I solution	BEM II solution	BEM III solution
1	124.3570	124.5210	124.5241	124.5283

The influence of localization of internal collocation points on critical force values for square plate a) using BEM III approach is presented in Table 5.

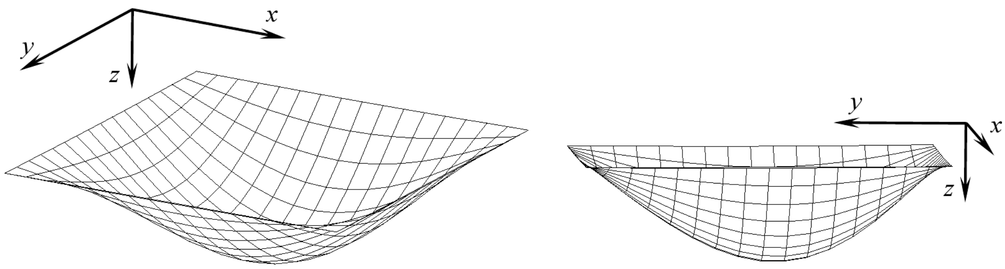

FIG. 9. The first buckling mode, $l_x/l_y = 1.0$.

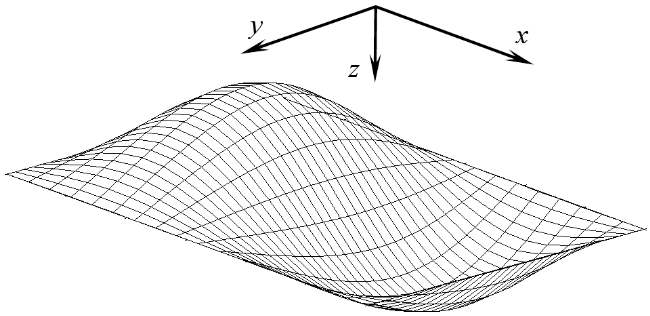

FIG. 10. The first buckling mode, $l_x/l_y = 1.5$.

5.3. A simply-supported rectangular plate under uniformly linear normal loadings

Static and loading scheme is shown in the Fig. 11.

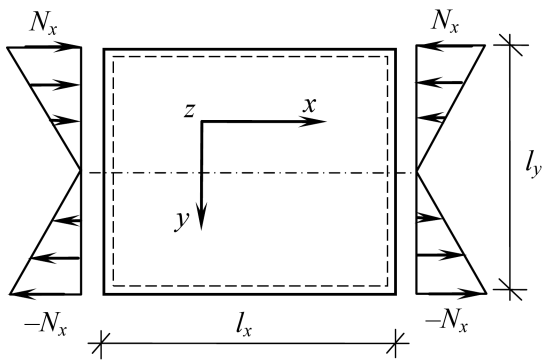

FIG. 11. A simply-supported rectangular plate under uniformly linear normal loadings.

Two plates are considered: a) square and b) rectangular. In both cases the plate geometry, properties and discretization were assumed identically as in Example 5.2. The results of calculation are presented in Tables 7–9. Number of real critical force value is given in the first column. Number of computational value is indicated beside by roman numerals. The influence of localization of internal collocation points on critical force values for square plate a) using BEM III approach is presented in Table 8.

Table 7. Critical force values, $l_x/l_y = 1.0$.

$\widetilde{N}_{\text{cr}}$	Analytical solution [41]	BEM I solution	BEM II solution	BEM III solution
1 (I)	252.6619	254.8014	254.8211	254.8753
2 (III)	–	269.1932	269.2128	269.2808
3 (V)	–	340.3840	340.4153	340.4976

Table 8. Critical force values: $l_x/l_y = 1.0$. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$.

$\widetilde{N}_{\text{cr}}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1 (I)	254.8753	254.8753	254.8753	254.9058	254.9984
2 (III)	269.2813	269.2808	269.2808	269.2872	269.3072
3 (V)	340.4969	340.4976	340.4983	340.5935	340.8843

Table 9. Critical force values, $l_x/l_y = 1.5$.

$\widetilde{N}_{\text{cr}}$	Analytical solution [41]	BEM I solution	BEM II solution	BEM III solution
1 (I)	356.7861	359.7330	359.7432	359.7538

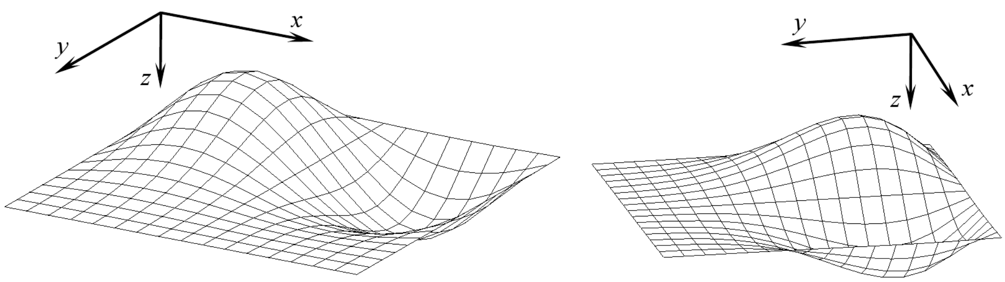

FIG. 12. The first buckling mode, $l_x/l_y = 1.0$.

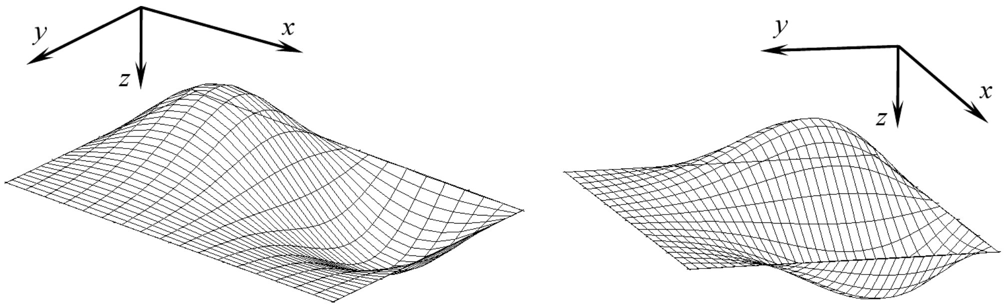

FIG. 13. The first buckling mode, $l_x/l_y = 1.5$.

5.4. *A rectangular plate simply-supported on two opposite edges with two remaining edges free under uniformly constant normal loading*

Static and loading scheme is shown in the Fig. 14.

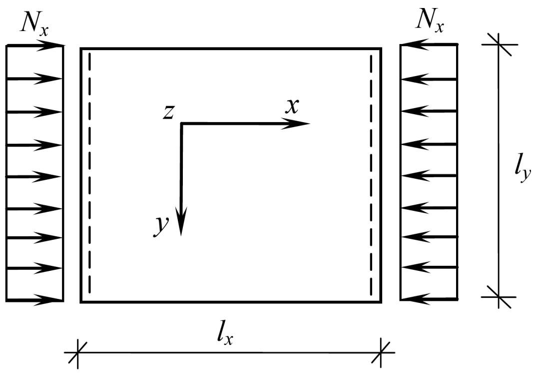

FIG. 14. A rectangular plate simply-supported on two opposite edges with two remaining free edges under uniformly constant normal loading.

Two plates are considered: a) square and b) rectangular. In both cases the plate geometry, properties and discretization were assumed identically as in Example 5.2. The results of calculation are presented in Tables 10–12. The influence

Table 10. Critical force values, $l_x/l_y = 1.0$.

$\widetilde{N}_{\mathrm{cr}}$	FEM solution	BEM II solution	BEM III solution
1	9.4603	9.8082	9.5546
2	26.5097	25.7486	25.4393
3	39.4389	38.2109	38.1065

Table 11. Critical force values: $l_x/l_y = 1.0$. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$.

$\widetilde{N}_{\mathrm{cr}}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	9.5546	9.5546	9.5527	9.5527	9.5477
2	25.4394	25.4393	25.4266	25.4266	25.3927
3	38.1064	38.1065	38.1014	38.1015	38.0892

Table 12. Critical force values, $l_x/l_y = 1.5$.

$\widetilde{N}_{\mathrm{cr}}$	FEM solution	BEM II solution	BEM III solution
1	6.1887	6.4995	6.4041

of localization of internal collocation points on critical force values for square plate a) using BEM III approach is presented in Table 11.

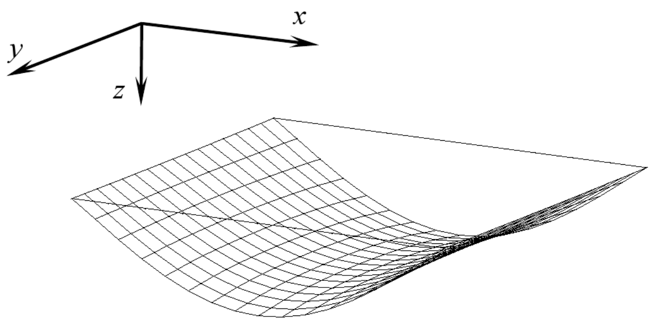

FIG. 15. The first buckling mode, $l_x/l_y = 1.0$.

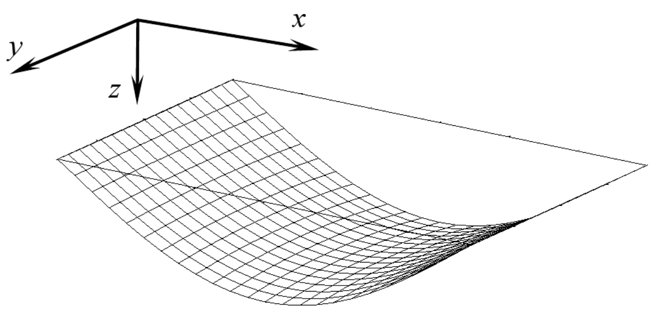

FIG. 16. The first buckling mode, $l_x/l_y = 1.5$.

5.5. A rectangular plate simply-supported on two opposite edges with two remaining free edges under uniformly linear normal loading

Static and loading scheme is shown in the Fig. 17.

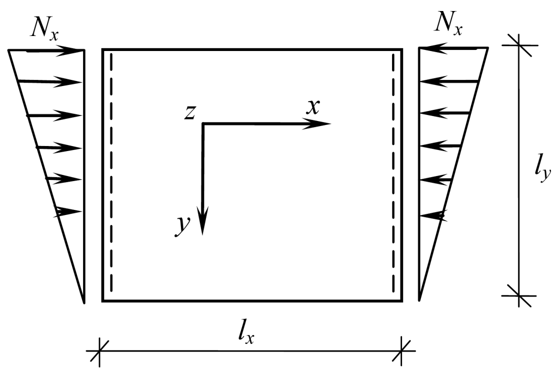

FIG. 17. A rectangular plate simply-supported on two opposite edges with two remaining free edges under uniformly linear normal loading.

Two plates are considered: a) square and b) rectangular. In both cases the plate geometry, properties and discretization were assumed identically as in Example 5.2. The results of calculation are presented in Tables 13–15. The influence of localization of internal collocation points on critical force values for square plate a) using BEM III approach is presented in Table 14.

Table 13. Critical force values, $l_x/l_y = 1.0$.

$\widetilde{N}_{\rm cr}$	FEM solution	BEM II solution	BEM III solution
1	16.7221	16.3165	16.3162
2	56.6365	54.5680	54.2951
3	83.2673	81.7806	81.0573

Table 14. Critical force values: $l_x/l_y = 1.0$. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$.

$\widetilde{N}_{\rm cr}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1	16.3162	16.3162	16.3162	16.3112	16.2983
2	54.2953	54.2951	54.2951	54.2766	54.2288
3	81.0572	81.0573	81.0569	81.0345	80.9748

Table 15. Critical force values, $l_x/l_y = 1.5$.

$\widetilde{N}_{\mathrm{cr}}$	FEM solution	BEM II solution	BEM III solution
1	11.4891	11.9982	11.8209

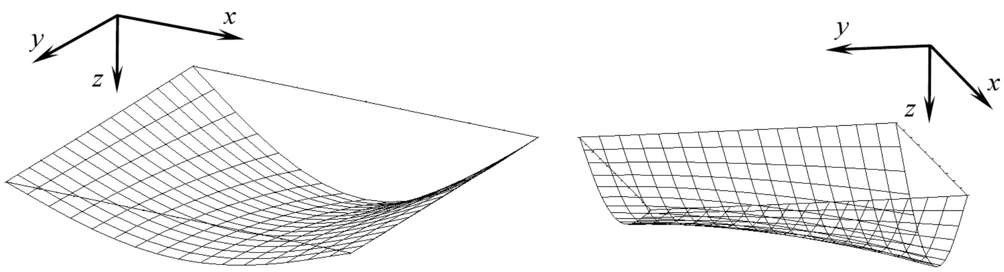

FIG. 18. The first buckling mode, $l_x/l_y = 1.0$.

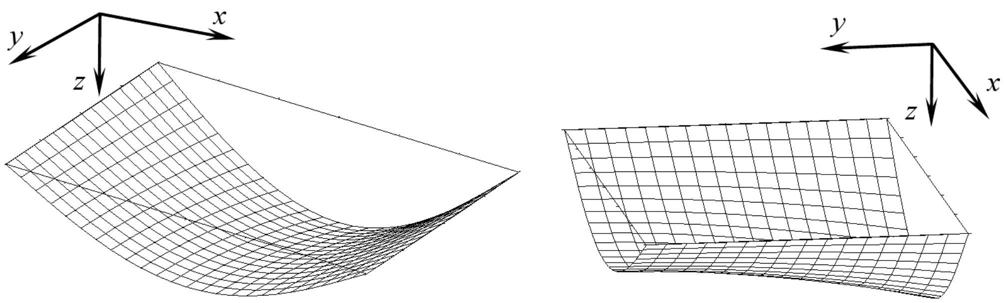

FIG. 19. The first buckling mode, $l_x/l_y = 1.5$.

5.6. A rectangular plate simply-supported on two opposite edges with two remaining free edges under uniformly linear normal loadings

Static and loading scheme is shown in the Fig. 20.

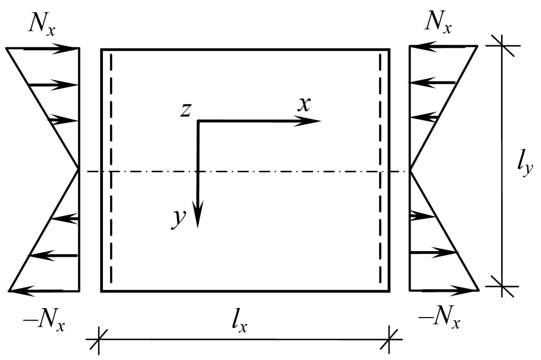

FIG. 20. A rectangular plate simply-supported on two opposite edges with two remaining free edges under uniformly linear normal loadings.

Two plates are considered: a) square and b) rectangular. In both cases the plate geometry, properties and discretization were assumed identically as in Example 5.2. The results of calculation are presented in Tables 16–18. Number of real critical force value is given in the first column. Number of computational value is indicated beside by roman numerals. The influence of localization of internal collocation points on critical force values for square plate a) using BEM III approach is presented in Table 17.

Table 16. Critical force values, $l_x/l_y = 1.0$.

$\widetilde{N}_{\mathrm{cr}}$	FEM solution	BEM II solution	BEM III solution
1 (I)	25.9506	25.9804	25.4095
2 (III)	71.5806	68.6190	68.1714
3 (V)	150.0035	134.4278	133.1770

Table 17. Critical force values: $l_x/l_y = 1.0$. Solution BEM III for different value of $\varepsilon_\Delta = \Delta x/a$.

$\widetilde{N}_{\mathrm{cr}}$	$\varepsilon_\Delta = \Delta x/a$				
	0.0001	0.001	0.01	0.1	0.2
1 (I)	25.4094	25.4095	25.4093	25.3956	25.3586
2 (III)	68.1714	68.1714	68.1709	68.1346	68.0386
3 (V)	133.1770	133.1770	133.1759	133.1156	132.9640

Table 18. Critical force values, $l_x/l_y = 1.5$.

$\widetilde{N}_{\mathrm{cr}}$	FEM solution	BEM II solution	BEM III solution
1 (I)	23.5430	23.9012	23.5336

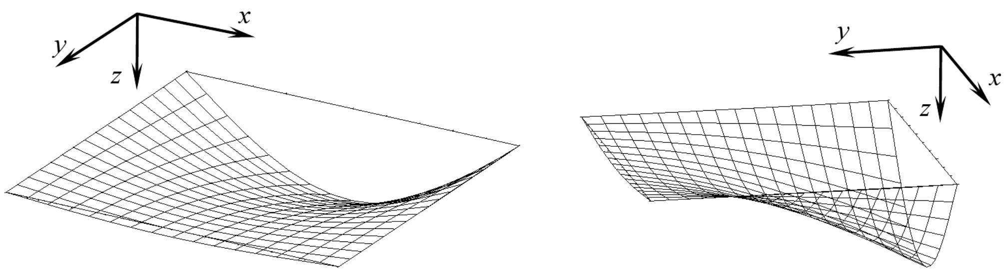

FIG. 21. The first buckling mode, $l_x/l_y = 1.0$.

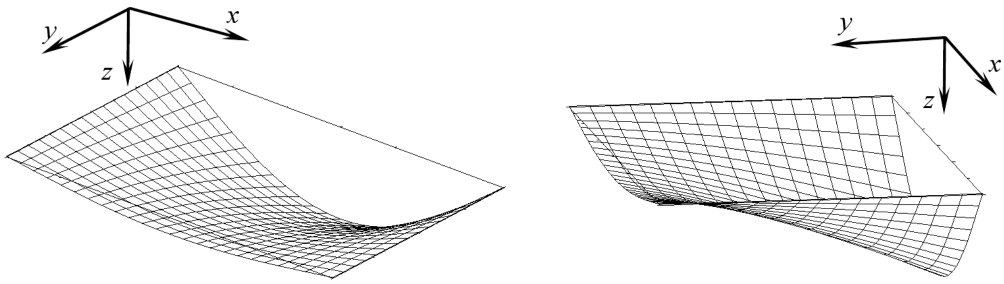

FIG. 22. The first buckling mode, $l_x/l_y = 1.5$.

6. Conclusions

An initial stability of thin plates using the boundary element method is presented. This problem was solved with the modified approach, in which the boundary conditions are defined so that there is no need to introduce equivalent boundary quantities dictated by the boundary value problem for the biharmonic differential equation. The collocation version of boundary element method with singular and non-singular calculations of integrals were employed and the constant type of the boundary element is introduced. The Bèzine technique was used to establish the vector of curvatures inside a plate domain which was divided into rectangular sub-surfaces. A plate can be subjected in plane by loading which distribution can be arbitrary, constant along selected edge of the single sub-domain element. The high number of boundary elements and internal sub-surfaces is not required to obtain sufficient accuracy. The loaded plate edge must be supported. This condition is required in proposed formulation of buckling analysis.

In case of normal conservative loading along the plate free edge, the boundary integral equation must be expanded by additional part:

$$\int_{\Gamma} -N_x \cdot \frac{\partial w_b}{\partial x} \cdot w^*(\mathbf{y}, \mathbf{x}) \cdot \mathrm{d}\Gamma(\mathbf{y}).$$

Then, construction of set of algebraic equation in matrix notation and formulation of the standard eigenvalue problem are much more complicated. To solve this problem, the first and second derivatives of deflection inside the plate area $(\partial^2 w/\partial x^2)$ and at the boundary $(\partial w_b/\partial x)$ can be establish and calculated for example approximately by constructing a differential expression using deflections of suitable neighbouring internal collocation points belonging to different internal sub-domains and collocation points located at the plate free edge.

The boundary element results obtained for presented conception of thin plate bending issue demonstrate the sufficient effectiveness and efficiency of the proposed approach which can be useful in engineering analysis of the buckling problem.

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