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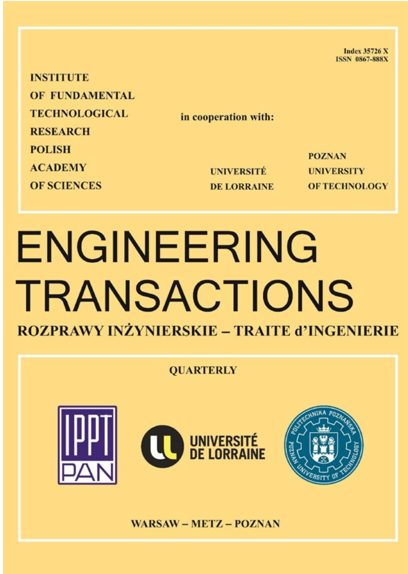

Title: Collision Integral for Non-Equilibrium Distributions of 1D Bosons with Non-Linear Dispersions

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Collision integral for non-equilibrium distributions of 1D bosons with non-linear dispersions

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Abstract

In order to understand transport phenomena in a *quasi-classical* regime the Boltzmann transport equation (BTE) is one of the most frequently used tools. Therein, the key quantity is a collision integral – the quantity that encapsulates the properties of the medium under consideration. Usually the result of this integral is approximated by one single parameter, the relaxation time. However it leaves one wondering if such situation is sufficient: for instance, if the dispersion of bosons is non-linear then what will be the influence of this non-linearity on the BTE. Here we give a fully analytic solution of the collisions' integral for 1D bosonic gases with non-linear dispersion and far out of equilibrium. Our analytic result is given in terms of the Lerch transcendent function and it has been obtained for the case of two sub-systems (one dragging another), by taking a maximum-entropy displaced Bose–Einstein *ansatz* for their distributions. Currently, there are numerous experiments performed far away from equilibrium, where distributions are massively shifted and our result will serve as a main building block to derive distributions of bosons, and later linear and non-linear transport coefficients, in such regimes.

Keywords: Boltzmann transport equation, bosons with non-linear dispersions, non-equilibrium distributions

1. Introduction

Boltzmann transport equation (BTE) has been a foundation of quasi-classical transport theory for more than a century. In a stationary state it reads:

$$\frac{\partial b}{\partial t} + \frac{p}{m} \cdot \nabla b + F \cdot \frac{\partial b}{\partial p} = \left(\frac{\partial b}{\partial t}\right)_{coll} \tag{1}$$

In this differential equation we search for an unknown distribution of particles, e.g. bosons $b(k)$, where k is the bosons' wavevector that can be associated with their momentum $p \equiv k$. On the LHS of BTE the first term is from an explicit dependence on time, the second term is the diffusion term and the last one comes from an applied external forcing field $F(r)$. These terms depend on a specific set-up where the force is applied. The RHS of BTE is the collision term, which depends on the properties of the boson gas itself and it is usually the hardest part to compute. Once the distribution $b(k)$ is known one is able to evaluate all currents j, assuming quasi-classically that the current is carried by the particles, and thus obtain the relation between $j(F)$, the key quantity of interest from an experimental viewpoint. One usually assumes *apriori* a small distortion, called $g(k)$ from the equilibrium Bose-Einstein distribution, to profit from the fact that there is no current for the equilibrium case, thus current is entirely due to the $g(k)$. Here we will not resort to this assumption, instead we shall take a non-perturbative, non-equilibrium distribution.

The aim of this work is to compute the collision term for the case of non-equilibrium distribution of bosons. In short we shall study the situation when one puts a strong current through one 1D sub-system and study how the other 1D sub-system has been affected. Several research groups are undertaking intense efforts in this direction, with papers published in the highest profile journals[1], there were also remarkable theoretical achievements in this field[2, 3, 4, 5]. It turned out that the underlying dispersion relation of bosons may play an important role. Here we wish to study one specific question: how the fact that the bosons have a non-linear dispersion affects the resulting collision term?

We intend to study this problem in a strongly non-equilibrium setting. The presence of the current always puts the system away from equilibrium and from experimental viewpoint one prefers strong current to arrive at a detection signal that is well above the noise level. From the theoretical point of view studying system away-from-equilibrium is a challenge. However over the last years significant progress has been done in this direction, many solutions for systems far away from equilibrium has been derived[6]. This drive has been fueled mostly by research on cold-atom systems[7] and plasmon-polariton systems[8] i.e. bosonic systems that can be pushed much further away from equilibrium than any traditional solid state material. We intend to build on that developments.

An outline of this work is as follows. In Sec.2 we define the model under consideration. In Sec.3 we outline previous theoretical works and provide an *ansatz* for the non-equilibrium distribution of bosons. In Sec.4 we compute the collision integral in an analytical form and evaluate it to show a few examples of the collision term. The paper is completed with Discussion and Conclusions sections.

2. The model

We study the problem of interacting bosons, we take one-dimensional (1D) bosons since it simplifies the reasoning (although an isotropic higher dimensional problem can be tackled using spherical harmonics). The Hamiltonian of such a system in general reads:

$$H_0 = \sum_{ij} E_{ij} b_i^\dagger b_j + \sum_{ijkl} V_{ijkl} b_i^\dagger b_j b_k^\dagger b_l \tag{2}$$

it is subjected to an external field F_{ij}:

$$H_F = \sum_{ij} F_{ij} b_i^\dagger b_j \tag{3}$$

In the following we shall assume that the system is translationally invariant thus the momentum is a good quantum number and diagonalizes the first term in H_0. The field F_{ij} is also defined in terms of spatial harmonics, so now $i,j \equiv k,k'$. Moreover a part of interactions that is of a density-density type $\propto n_i n_l$ is assumed to be the strongest and it leads to a renormalization of bosons velocity, keeping their dispersion linear, just like it happens for Lieb-Liniger model[9] or RPA treatment of acoustic phonons/plasmons[10]. For such a case the quantum, $T = 0$, solution to the model given by Eq.2 is known[11] and it has two phases superfluid and insulator. However what is also known is that coupling with a strong multi-harmonic field $F(q_i)$, Eq.3, that leads to floating Aubry-Andree physics[12], and/or a dissipative environment lead to a metallic phase with a well defined Drude peak[13, 14]. Then using the BTE, equation Eq.1, is justified. Our quantity of interest is then an average occupancy of bosonic modes $b(k) = \langle \tilde{b}_k^\dagger \tilde{b}_k \rangle$ at the high temperature regime, when the lowest energy physics may be integrated out and the eigenstates $\tilde{b}_k$ of the interacting bosons Eq.2 have emerged, see e.g. Eq.4. This implies that the effective susceptibility of an environment will contain some retardation effects, namely some non-trivial momentum dependence inherited from these averaged out lowest energy modes. In turn, this implies that the interactions $V(q)$ will have more complicated character and will not be entirely absorbed in the simple linear dispersion.

Furthermore, we assume that upon interaction with a field F the bosons can be divided into different ν-branches ($\equiv$ subsystems), for instance phonons with different symmetries, such that their interaction with the external field $F(q)$ is different (e.g. due to dipole matrix selection rules). The simplest situation is that of two branches, although it can naturally be extended to more sub-systems. The simplest system under consideration thus consists out of two parts: 1) the 1D electron liquid (1D metal) that is subjected to a strong current (due to the applied force $F(q)$) and 2) a recipient 1D system that is subjected to the collisions with 1). Of the latter part we only need to assume that it can be described by a distribution that can be locally Taylor expanded. Thus the locally induced voltage in the recipient system is linearly proportional to a distortion of local distribution of carriers therein such that transverse transport coefficients (e.g. drag) can be potentially defined.

In a 1D system, in the presence of any density-density interaction V_0 the spectral weight is shifted from quasi-particles, to collective modes[15]. These collective modes are described by the following Hamiltonian:

$$H_{mod} = \sum_{\nu} u_{\nu}(q) q \tilde{b}^{\dagger}_{\nu}(q) \tilde{b}_{\nu}(q) \tag{4}$$

which we have expressed here in terms of bosonic operators $\tilde{b}^{\dagger}_{\nu}$ where the ν index spans over e.g. charge and spin degrees of freedom to capture the collective fluctuations in each of these sectors. This example allows to define separable ν-branches ($\equiv$ subsystems). Thanks to e.g. spin-charge separation present in 1D each of these modes can be indeed considered separately. In general the velocities $v_{\nu}(q) \propto q^{\alpha_{\nu}-1}$ are determined by interactions with the linear dispersion $\alpha_{\nu} = 1$ present for purely Hartree-type interactions. In many nanostructures, that are e.g. deposited on the surface, the 1D charges can provide only partial screening. We then have some finite range, Coulomb-type interaction $\tilde{V}_{Coul}(q)$ which affects these charge degrees of freedom. The velocity $v_{\rho}(q)$ is now determined by $\tilde{V}_{Coul}(q)$ which in turn depends of dielectric constant of the nanostructured environment within which the 1D system is embedded, namely: $\tilde{V}_{Coul}(q) = V_{Coul}(q)/\epsilon(q)$. Here $V_{Coul}(q)$ is the bare Coulomb interaction (that can be assumed to be Hartree type) while $\epsilon(q)$ is the dielectric constant that has been evaluated analytically for instance in Ref.[16].

Our problem thus consist out of 1D charge-carrying bosons with a power-law dispersion:

$$\omega_\rho(q) = \tilde{v}_\rho q^\varsigma \tag{5}$$

which is equivalent to velocity $v_\rho(q) \propto q^{\varsigma-1}$, in the expression above q is dimensionless and $\tilde{v}_\rho$ is a proportionality constant that carries the dimension of energy.

In the following we shall consider two copies of such 1D systems that are coupled through a momentum carrying interaction:

$$H_{tot} = H_{mod}^{(1)} + H_{mod}^{(2)} + H_d^{(1-2)} \tag{6}$$

where $H_d^{(1-2)}$ is the coupling term which we assume to be weak enough to not destroy the modes' separation into the sub-systems. Moreover we assume that one of the modes is in the strongly non-equilibrium situation, in the sense that strong current is flowing through it.

The momentum relaxation term $H_d^{(1-2)}$ between the sub-systems, with an amplitude $\equiv \Gamma_0 = cste$, can potentially allow to transfer a part of the current to the other sub-system (in fact Γ_0 has to have some UV cut-off, like in Yukawa potential). The remanent interaction induces the non-linearity of the spectrum (that can be interpreted as a self-energy effect), but they will also induce a vertex correction. From Ward identity we know that the self-energy $\Sigma(k \pm q)$ and vertex correction $\Gamma(q)$ have to be related if the system is conserving. To be precise[17], in the low energy limit

$\Gamma(q) \propto \partial\Sigma(k)/\partial k|_{k=q}$. The non-linearity of the dispersion will thus induce an anharmonic behaviour of bosons, i.e. boson-boson interactions. We then expect that in our particular case, when there is a deviation $Re[\Sigma(k)] \approx \Delta\omega(k)$ from the linear dispersion, the vertex correction reads:

$$\Gamma(q) = (\partial_q \Delta\omega_\rho(q)/\partial q) = (\varsigma - 1)\tilde{v}_\rho q^{\varsigma-1} \tag{7}$$

thus the vertex correction will be given by another power law. In deriving this we have made a Taylor expansion of $\omega(k \pm q)$ for a small values of exchanged momenta q and kept only the first term. This is justified by the fact that the postulated interactions $\tilde{V}_{Coul}(q)$ is of a screened Coulomb type, thus has a preference for a small momentum exchange processes. The most important implication of Eq.7 is that in such generalized dispersion model the vertex corrections are unavoidable. We emphasize that the formalism that we shall derive below can accommodate the vertex correction that is given by any power-law, the specific form given by Eq.7 is however useful to show that such correction will necessarily appear if one want to keep the approximation conserving.

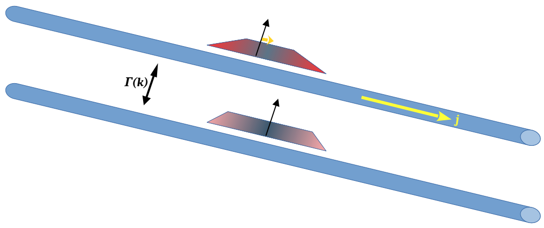

Figure 1. Schematic illustration of our model. Two 1D sub-systems are coupled by a momentum dependent amplitude $\Gamma(k)$. There is a strong current j (yellow arrow) flowing through one of them. Each sub-system (mode) has a different distributions of bosons, shown (trimmed) on the top of each sub-system. We see different temperatures and a shift of upper distribution due to the current flow. The *ansatz* for these distributions is given in Sec.3.

3. Postulated non-equilibrium bosons distributions

The non-equilibrium distribution of bosons is postulated in the form of a displaced Bose-Einstein distribution function (taking $k_B = 1$):

$$F_d = \frac{1}{2\pi}\left\{exp[\omega(k)/T_e - kv_{drag}/T_d]\right\}^{-1} \tag{8}$$

the same as assumed for a strongly dragged (i.e. non-equilibrium) gas of bosons with arbitrary dispersion relation $\omega(|k|)$. This form is known in the literature and has been used many times before, e.g. Ref.[18, 19, 20] Here T_e is the temperature of a gas in equilibrium of the same energy density, and v_{drag} and T_d are so called drift velocity and drift temperature respectively. For the state of a gas of bosons represented by a distribution function $b[\chi(k);T]$ the energy density and the momentum density reads:

$$\epsilon := \frac{1}{\Omega_{BZ}}\int_{BZ}\omega(|k|)b[\chi(k);T_{e,d}]dk \ , \ \ p := \frac{1}{\Omega_{BZ}}\int_{BZ}kb[\chi(k);T_{e,d}]dk \tag{9}$$

In Ref.[20, 21], the distribution of the form Eq.8 was obtained from optimization procedure of entropy written as a functional of the distribution $b(k)$:

$$s[b(k)] = \frac{1}{2\pi}\int\{(1+2\pi b(k))\ln(1+2\pi b(k)) - 2\pi b(k)\ln(2\pi b(k))\}dk \tag{10}$$

under a constrains corresponding to fixed values of energy ϵ and momentum p, namely:

$$\delta\left(s(b(k)) + \alpha\left(\epsilon - \frac{1}{\Omega_{BZ}}\int_{BZ}\omega(k)b(k)dk\right) + a\left(p - \frac{1}{\Omega_{BZ}}\int_{BZ}kb(k)dk\right)\right) = 0 \tag{11}$$

where α and a are the Lagrange multipliers of this variational problem with constrains. Solving Eq.11 for $b[]$ and comparing the result with Eq.8 we obtained[20, 21] that the variational problem is indeed optimized by:

$$b[\chi(k)] = F_d[\chi] = \frac{1}{2\pi\left(e^{\chi(k)} - 1\right)} \quad , \quad \chi(k) = \alpha\,\omega(k) + ak \tag{12}$$

where:

$$\alpha = T_e{}^{-1}, \;\; a = -\frac{v_{drag}}{T_d} \tag{13}$$

Based on this result, in the following, to make correspondence with standard condensed matter notation, we take that in the i−th subsystem:

$$\alpha_i \equiv \beta_i = 1/T_i \tag{14}$$

Moreover, we shall re-scale the β_i with the "bare" velocity $\tilde{v}_{i\rho}$ that is $\tilde{v}_{i\rho}\beta_i \to \beta_i$ a shorthand dimensionless notation which we are going to use in the latter sections.

Naturally, the solution of the constrained variational problem reproduces the constrains, i.e.

$$\epsilon = \tfrac{1}{\Omega_{BZ}} \int_{BZ} \omega(|k|) F_d[\chi(k)] dk \text{ and } p = \tfrac{1}{\Omega_{BZ}} \int_{BZ} k F_d[\chi(k)] dk.$$

Moreover, when the gas of bosons tends to equilibrium then a constraint $p \to 0$ and then the corresponding Lagrange multiplier in the distribution $a \to 0$, thus the distribution itself becomes a Bose-Einstein distribution. As written explicitly in Sec.5.1, for the case of linear dispersion the quantities T_e, T_d and v_{drag} can be expressed[21] as analytic functions of a given ϵ and p.

To close this sub-section we would like to emphasize that naturally the proposed ansatz for the distribution is obviously an approximation in which we assume that the emergent modes are weakly interacting. Such assumption can be justified for instance when interactions $V(q)$ are chosen such that the system is in the vicinity of Luther-Emery points. We will come back to it later in Discussion section.

4. Collision term and resulting integral

4.1. *Derivation of the collision integral*

We can substitute the postulated non-equilibrium distribution function F_d into Boltzmann transport equation Eq.1. Specifically, we are interested in evaluating RHS of this equation, the collision integral, that depends on the inherent properties of the system, less than on the shape of the externally applied field. The collision integral that we want to evaluate reads:

$$\bar{\Gamma}_{coll}(k) = \int_0^{k_0} dq \Gamma_0 \Gamma(q) b_{d1}(k+q) b_{d2}(k-q) \tag{15}$$

where k_0 is the upper limit of the theory (e.g. Brillouin zone boundary) and b_{di} is a short-hand notation for the distribution with drift, Eq.12, in the i−th subsystem. This is Bose-Einstein distribution with a drag, a distribution parameterized by a pair (β_i, a_i). In order to make progress we take a Taylor expansion of $\omega_\rho(k \pm q)$ which leads to a term $\beta((1+\varsigma)k^\varsigma \pm +\varsigma k^{\varsigma-1} q)$. The underlying assumption of a small q exchanges' dominance has already been discussed below Eq.7 and clearly manifests in the last term in Eq.22. Upon substituting it to the distributions we find that:

$$b_{d1}(k+q)b_{d2}(k-q) = \tag{16}$$

$$exp\Big\{-\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^{\varsigma}\Big\} \tag{17}$$

$$\Bigg(exp\Big\{\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^{\varsigma}\Big\}\exp\Big\{[(\beta_1+\beta_2)\varsigma k^{\varsigma-1}+(a_1+a_2)]q\Big\}+ \tag{18}$$

$$exp(\frac{1}{2}(\beta_1-\beta_2)(1+\varsigma)k^{\varsigma})\exp\Big\{[(\beta_1-\beta_2)\varsigma k^{\varsigma-1}+(a_1-a_2)]q\Big\}+ \tag{19}$$

$$exp(\frac{1}{2}(-\beta_1+\beta_2)(1+\varsigma)k^{\varsigma})\exp\Big\{[(-\beta_1+\beta_2)\varsigma k^{\varsigma-1}+(-a_1+a_2)]q\Big\}+ \tag{20}$$

$$exp\Big\{-\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^{\varsigma}\Big\}\Bigg)^{-1} \tag{21}$$

Thus the collision integral can now be casted in a following form::

$$\bar{\Gamma}_{coll}(k) = exp\Big\{-\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^{\varsigma}\Big\}\int dq \frac{(\varsigma-1)\Gamma_0 q^{\varsigma-1}\exp\Big\{\frac{1}{2}[(\beta_1-\beta_2)\varsigma k^{\varsigma-1}+(a_1-a_2)]q\Big\}exp[-q/k_0]}{z(k)\exp\Big\{[(\beta_1+\beta_2)\varsigma k^{\varsigma-1}+(a_1+a_2)]q\Big\}-1} \tag{22}$$

where the numerator contains the inverse boson scattering amplitude $\Gamma(k)$ times Yukawa potential (that is solid-state setting is equivalent to Thomas-Fermi screening), which we introduced both to accommodate the upper limit of the integral and physically the finite range of the interaction. In the denominator there is a following function:

$$z(k) = \frac{exp\Big\{\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^{\varsigma}\Big\}}{exp(-\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^{\varsigma})+exp(\frac{1}{2}(\beta_1-\beta_2)(1+\varsigma)k^{\varsigma})+exp(\frac{1}{2}(-\beta_1+\beta_2)(1+\varsigma)k^{\varsigma})} \tag{23}$$

It is well known that many integrals of canonical distribution functions can be given in terms of poly-logarithms[22]. This is not sufficient to solve the above Eq.22, we then shall move to a generalization of a poly-logarithm function which is a Lerch transcendent function. Its integral form reads[23]:

$$\Phi(z,s,\bar{\bar{a}}) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}e^{-\bar{\bar{a}}x}}{1-ze^{-x}}dx \tag{24}$$

where the denominator has a form similar to a denominator of the canonical distribution and in the numerator there is a trimmed power-law.

By comparing this with Eq.22 we observe that with substitution $x \to q$ our integral is indeed solvable through the Lerch transcendent function. The result of this integral reads:

$$\bar{\Gamma}_{coll}(k) = \frac{exp\Big\{-\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^{\varsigma}\Big\}}{[(\beta_1+\beta_2)\varsigma k^{\varsigma-1}+(a_1+a_2)]}\frac{(\varsigma-1)\Gamma_0}{\Gamma(\varsigma-1)}\Phi\Big(z(k),(\varsigma-1),-y(k)\Big) \tag{25}$$

where

$$y(k) = \frac{\frac{1}{2}[(\beta_1-\beta_2)\varsigma k^{\varsigma-1}+(a_1-a_2)]-1/k_0}{[(\beta_1+\beta_2)\varsigma k^{\varsigma-1}+(a_1+a_2)]} \tag{26}$$

and $\Phi(,,)$ is a Lerch transcendent function, and $\overline{\Gamma()}$ is the gamma function. Actually, the Lerch transcendent special function, a generalization of poly-logarithm functions (known to be integrals of quantum distributions), has already appeared in the literature as an integral of distributions with drag, see Ref.[24], however therein the full dependence on ς has not been explored. The Eq.25, together with expressions for $z(k)$ (Eq.19) and $y(k)$ (Eq.21) provide a closed analytic form for the momentum resolved collision integral. This is the central result of our work.

This result also allows to explore the energy weighted and the momentum weighted integrals which both must exist and be positive. This is because Lagrange multipliers (as defined in Sec.3) must be reproduced by these moments of distribution. From properties of Lerch function we can now deduce that the positivity condition is equivalent to positivity of distribution's argument.

4.2. *Results*

We shall now illustrate the results of Eq.25. Firstly, in Fig.2 we plot a collision integral as a function of $a_1 \equiv v_{drag}$. For simplicity we take the case when one of the distributions is not shifted, i.e. $a_2 = 0$. We observe that the strongest collision amplitude is for the small electron momenta k. For larger value of ς (right panel), for small values of k, we see that the function is clearly non-monotonous: there exist an optimal value of Lagrange multiplier a_1 for which the drag effect is the strongest. Naturally, for the smallest values of a_1 the effect is increasing with the shift of the distribution. However, when distribution is shifted too strongly then the carriers within receiving subsystem cannot catch up with the rapid motion of non-equilibrium carriers. This is the reason why we call a_1 the drag velocity v_{drag}. For larger k the effect monotonously increasing with v_{drag} although this tendency is much less pronounced. Same behavior for large k is observed on the left panel (smaller ς). Interestingly, here for small k we cannot notice such a clear resonance: there is a noisy variation at small v_{drag} and then the collision amplitude is monotonously decreasing.

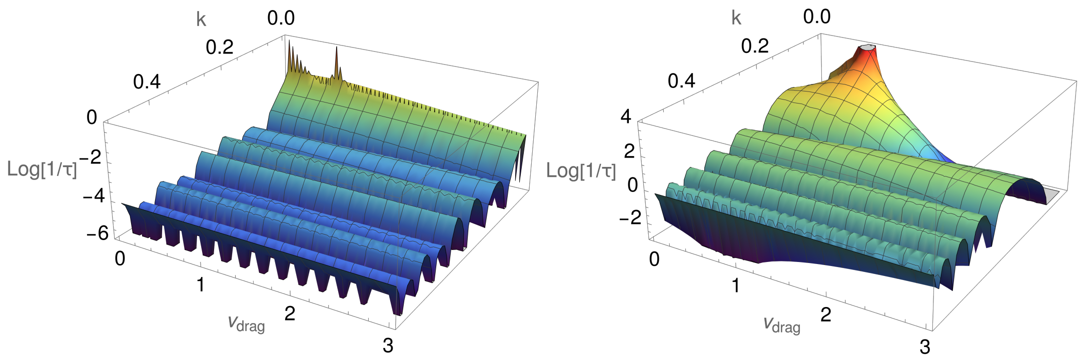

Figure 2. (Logarithm) of a collision amplitude as a function of electrons' momentum k and drag velocity v_{drag} ($\equiv a_1$). On the left we show result for $\varsigma = 1.5$, on the right for $\varsigma = 2.5$. Both calculations were done for $\beta_1 = 30$, $\beta_2 = 20$ and $a_2 = 0$ (no initial shift of distribution). Other parameters: $\Gamma_0 = 2/3$, $q_0 = 3$. All the quantities are given in a units of energy where $\bar{v}_\rho = 1$, which is set by the characteristic kinetic energy of the system and units of momentum set by 1D density of particles n, just like in Lieb's seminal paper [9]

.

In Fig.3 we plot the result of a collision amplitude as a function of temperature.

We set the temperature of recipient mode at $b_2 = 20$ and vary the temperature of the dragging sub-system. We observe that the largest effect, a strong maximum, exist when the temperature $b_1 = 20$, thus $b_1 = b_2$. The effect indicate that the collision integral is strongest when the two distributions are most similar to each other.

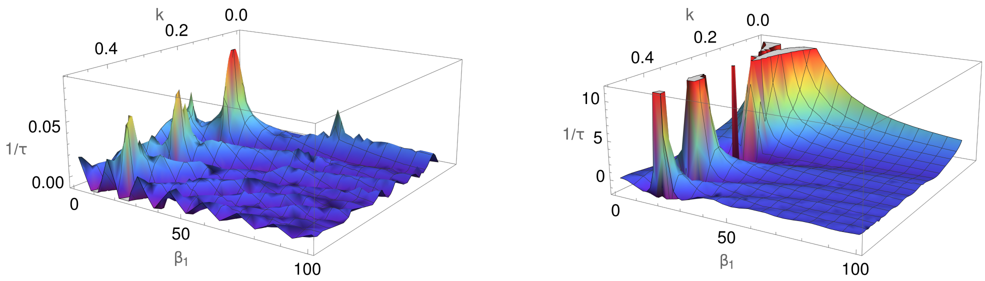

Figure 3. Collision integral amplitude as a function of electrons' momentum k and temperature $\beta_1 = T_{e1}^{-1}$. On the left we show result for $\varsigma = 1.5$, on the right for $\varsigma = 2.5$. Both calculations were done for $a_1 = 0.5$, $\beta_2 = 20$ and $a_2 = 0$ (no initial shift of the distribution). Other parameters: $\Gamma_0 = 2/3$, $q_0 = 3$.

However, the most important conclusions from all these calculations is very strong dependence on ς: moving away from $\varsigma = 1$ can cause the result to grow by orders of magnitude. Thus we see that the amplitude of the collision integral is a very susceptible measure of the non-linearity encoded in the value of ς.

5. Discussion

5.1. *The case of linear dispersion*

For the case of linear dispersion $\omega = c|k|$ we can derive[21] analytic expressions for the Lagrange multipliers of the $\chi(k)$ variable, which now can be called an equilibrium temperature $T_e = \beta^{-1}$ and a drag temperature $T_d = a^{-1}\bar{v}_{drag}$ where we defined the bare (normalized by T_d) drag velocity $\bar{v}_{drag}$.

$$T_d = \frac{T_e(1-u)}{\sqrt{1+u}} \tag{27}$$

$$u = \frac{\epsilon - \sqrt{\epsilon^2 - c^2 p^2}}{\epsilon + \sqrt{\epsilon^2 - c^2 p^2}} \tag{28}$$

where, like before in Sec.3, the distribution $F_d[...]$ determines the average energy and momentum (normalized by a volume of the 1st Brillouin zone, Ω_{BZ}) that enter into the above formulas:

$$\epsilon = \frac{1}{\Omega_{BZ}} \int_{BZ} c|k| F[\chi(k)); T_e] dk \tag{29}$$

$$p = \frac{1}{\Omega_{BZ}} \int_{BZ} k F[\chi(k)); T_e] dk \tag{30}$$

where, because of the fixed relation Eq.27, we can use only one temperature to parameterize the distribution. The last quantity, the average momentum in a sub-system p, can be related to the bare drag velocity:

$$\bar{v}_{drag} = c \frac{cp}{\epsilon + \sqrt{\epsilon^2 - c^2 p^2}} \tag{31}$$

Thus we see that for the linear distribution there is a unique solution of a shifted density problem. Importantly, for the linear dispersion in 1D, the two integrals Eq.29 and Eq.30 up to a unitary factor $sign(k)$ have the same form.

This observation can be generalized. It can be shown that in the case of linear dispersion, i.e. $\varsigma = 1$, the drag effect is expected to be zero, which is in agreement with our prediction. This is because the moment expansion $M_n = \int k^n F(\chi(k)))dk$ of a distribution forms a closed set, as we fixed the value of ϵ (total k) and p (difference of positive and negative k). As it was shown in Ref.[25] the drag is proportional to third cumulant and from statistics we know that third cumulant is equal to a third moment (skewness). This general argument justifies our result. Similar conclusion, for more specific case of thermoelectric effect, has been reached in Ref.[26] where it was traced back to Mott relation for transport coefficients.

5.2. *Limitations of our reasoning*

There are two key limitations of our approach. The first one is that we have neglected known quantum phases of the 1D boson model: superfluid and insulator. This may be justified in the regime of high temperature and dissipative environment, actually the same regime where the quasi-classical BTE approach itself is known to be applicable. Nevertheless, one has to keep in mind that our calculations are valid in the quasi-classical regime and be cautious when applying them to low-temperature systems subjected to highly coherent field $F(q)$.

Secondly, in Sec.3 we have taken an *ansatz* for non-interacting bosonic distributions which is obviously an approximation when boson-boson interactions are present. Simply taking $V_{ijkl} = 0$ in Eq.2 will not ameliorate the situation as then we shall arrive at the bosonic liquid with a very high propensity towards Bose-Einstein condensation, a quantum phenomenon not accounted for in a simple BTE. There are two arguments that can justify our model. The first one is based on the existence of Luther-Emery points: for specific values of V_{ijkl} the 1D system behaves as a system of an emergent non-interacting particles. This is why we emphasized that the modes on which the Hamiltonian Eq.4 is defined are emergent, collective eigen modes. This is also the reason why for these modes both self-energies effects and vertex corrections may be substantial. The second argument is more pragmatic: the quantity on which we focus in this work is only the collision integral. We limited ourselves and decided to not move towards full solution of the BTE. In this way our result is more generic and can be applied for various configurations of the external $F(q)$. But in this way we also compute only a basic building block of potentially more advanced approximation. One can consider including interaction corrections to the distribution $b(k)$ in a perturbative series (with all possible particle-hole cancellations) and then at each term the expressions provided by us here can be used. Nevertheless, we admit that future research, that would give collision integral for fully interacting distributions of bosons, known[15] as hypergeometric Beta functions, will be an exciting and important direction of research.

One physical realization of the proposed model would be either a multi-wire or a multi-orbital 1D system. Here it is pretty intuitive that the field F can interact strongly only with one selected orbital or - in case of coaxial tubes (like MWCNTs) or parallel wires - only with an outermost sub-system. Then we naturally obtain sub-systems with different distributions, with different shifts. Moreover, since the sub-system are nearly orthogonal (in Hilbert space) then tunneling does require to re-constitute the entire boson. This has been evidenced by experiments[27]: the perpendicular transport measures the single particle density of states. In such a specific case it makes sense to use the shifted Bose-Einstein distributions to compute collision integrals between these re-constituted particles.

5.3. *Examples of experimental relevance*

The connection between our postulated theoretical model and its real-life experimental realizations is quite complicated but also fascinating problem. The two most promising platforms are nanostructured wires and cold atoms. In each of these systems the ongoing research is facing slightly different problems:

In cold atoms, where emergent bosons are created by trapping atoms in optical lattice, the interactions can be relatively easily modified by changing the external magnetic field (in this way one tunes scattering cross-section through Feshbach resonance). Moreover one can add fast, anti-adiabatic, fermionic component to a mixture of two bosons and vary their interaction through RKKY type mechanism. This has been experimentally employed in a seminal paper published in *Nature* [28]. Therein one finds that characteristic energy of kinetic energy can be of order of $6.5Hz$ while at the same time the inter-species scattering can be tuned to be as small as $\Gamma_0 \sim 0.5Hz$. From the point of view of energy scales of these bosons temperatures of order nK are indeed in the high temperature regime that we discuss in our paper. The difficulty here lies in extracting the dispersion relation: the interaction is that of an extended Lieb-Liniger model and from the Bethe-ansatz solution of Lieb-Liniger model we know that there are higher energy bosons with super-linear dispersion and low energy bosons with sub-linear dispersions. A separate issue is how these dispersions will be modified by the optical trapping potential, see e.g. [29] where an influence of a parabolic trap was analyzed. From experimental viewpoint, the exponent ς can be extracted only indirectly from time-of-flight experiments that are destructive for the trapped bosonic system itself.

In a system of nano-wires the plasmonic dispersion is governed by the geometry of the system. While in 3D the plasmons have a constant frequency, in 2D their dispersion obeys $k^{1/2}$ while in 1D q^1 (this is all assuming a finite range interaction, full Coulomb interaction would give a $log(q)$ correction). These results are derived for a plain geometry. For a corrugated surface, the textbook result, tells us that interaction is in general given by a power law with a characteristic exponent dependent on a corrugation angle. However in such a solid-state system inter-boson interactions are much harder to control. One can modify Thomas-Fermi screening length or submerge the system into a dielectric, thus modifying the interaction $\tilde{V}(r)$ which implies also changing inter-boson interaction (scattering with finite momentum exchange). However all these efforts are only indirect. An interesting situation, where inter-boson scattering indeed has played a crucial role, was a very recent experiment reported in *Science Advances* [30] where plasmons' kinetic energy was reported as $1.2eV$ and the scattering amplitude was $\Gamma_0 \approx 0.2eV$. Other solid state systems will have similar order of magnitude and here we need to take working temperature to be room temperature or higher (as in the case of an aforementioned experiment) to work in the high-temperature regime considered in this work.

6. Conclusions

In the paper we have obtained a closed analytic formula for the collision integral between two 1D systems. This is a basic building block to provide solutions of BTE, Eq.1, and give transport coefficients in many experimentally/technologically motivated applications. The collision term of BTE is the one that determines resistivity and the one that is notoriously the hardest to compute especially in a non-linear non-equilibrium case. This enables the researches to not only study the strength of the non-linearity ς in the charge sector, but also to directly access the non-equilibrium distributions with Lagrange multipliers that are characterizing them. With this knowledge the researchers will be able to quantify the non-equilibrium state of the system for various experimental set-ups as defined by LHS of Eq.1.

Upon completing this paper we learned that experiments capable of probing this regime were indeed very recently performed, see Ref.[31].

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