Engineering Transactions, **74**(2): 231–247, 2026, https://doi.org/10.24423/engtrans.2025.3692
Institute of Fundamental Technological Research Polish Academy of Sciences (IPPT PAN)
Université de Lorraine • Poznan University of Technology

Research Paper

Collision Integral for Non-Equilibrium Distributions of 1D Bosons with Non-Linear Dispersions

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In order to understand transport phenomena in the quasi-classical regime, the Boltzmann transport equation (BTE) is one of the most frequently used tools. Therein, the key quantity is the collision integral – the quantity that encapsulates the properties of the medium under consideration. Usually the result of this integral is approximated by a single parameter, the relaxation time. However, this leaves one wondering if such a treatment is sufficient, for instance, if the dispersion of bosons is non-linear, what will be the influence of this non-linearity on the BTE. Here, we give a fully analytic solution of the collision integral for 1D bosonic gases with non-linear dispersion and far out of equilibrium. Our analytic result is given in terms of the Lerch transcendent function and it has been obtained for the case of two subsystems (one dragging the other), by taking a maximum-entropy displaced Bose–Einstein ansatz for their distributions. Currently, there are numerous experiments performed far from equilibrium, where distributions are massively shifted and our result may serve as a main building block for deriving distributions of bosons, and later linear and non-linear transport coefficients, in such regimes.

Keywords: Boltzmann transport equation, bosons with non-linear dispersions, non-equilibrium distributions.

1. Introduction

The Boltzmann transport equation (BTE) has been a foundation of quasi-classical transport theory for more than a century. In a general state, it reads:

$$\frac{\partial b}{\partial t} + \frac{p}{m} \cdot \nabla b + F \cdot \frac{\partial b}{\partial p} = \left(\frac{\partial b}{\partial t}\right)_{\text{coll}}. \tag{1.1}$$

In this differential equation we search for an unknown distribution of particles, e.g., bosons $b(k)$, where k is the boson wavevector, which can be associated with their momentum $p \equiv k$. On the left-hand side (LHS) of BTE, the first term is from an explicit dependence on time, the second term is the drift term, and the last one comes from an applied external forcing field $F(r)$. These terms depend on the specific setup where the force is applied. The right-hand side (RHS) of BTE is the collision term, which depends on the properties of the boson gas itself and is usually the hardest part to compute. Once the distribution $b(k)$ is known, one is able to evaluate all currents j, assuming quasi-classically that the current is carried by the particles, and thus obtain the relation between $j(F)$, the key quantity of interest from an experimental viewpoint. One usually assumes *a priori* a small distortion, called $g(k)$, from the equilibrium Bose–Einstein distribution to profit from the fact that there is no current in the equilibrium case; thus, current is entirely due to $g(k)$. Here, we will not resort to this assumption; instead, we shall take a non-perturbative, non-equilibrium distribution.

The aim of this work is to compute the collision term for the case of a non-equilibrium distribution of bosons. In short, we shall study the situation when one puts a strong current through one 1D subsystem and studies how another 1D subsystem is affected. Several research groups undertake intense efforts in this direction [1], there have also been remarkable theoretical achievements in this field [2–5]. It has turned out that the underlying dispersion relation of the bosons may play an important role. Here, we wish to study one specific question: how the fact that the bosons have a non-linear dispersion affects the resulting collision term?

We intend to study this problem in a strongly non-equilibrium setting. The presence of a current always puts the system away from an equilibrium and from experimental viewpoint one prefers strong current to arrive at a detection signal that is well above the noise level. From the theoretical point of view, studying system far-from-equilibrium is a challenge. However, over the last years significant progress have been made in this direction, and many solutions for systems far away from equilibrium have been derived [6]. This progress has been fueled mostly by research on cold-atom systems [7] and plasmon-polariton systems [8], i.e., bosonic systems that can be pushed much further away from equilibrium than any traditional solid-state material. We intend to build on these developments.

The outline of this work is as follows. In Sec. 2, we define the model under consideration. In Sec. 3, we outline previous theoretical work and provide an ansatz for the non-equilibrium distribution of bosons. In Sec. 4, we compute the collision integral in an analytical form and evaluate it by showing a few examples of the collision term. The paper concludes with a discussion in Sec. 5 and conclusions in Sec. 6.

2. Model

We study the problem of interacting bosons. We take 1D bosons since this simplifies the reasoning (although an isotropic higher-dimensional problem can also be tackled using spherical harmonics). The Hamiltonian of such a system generally reads:

$$H_0 = \sum_{ij} E_{ij} b_i^\dagger b_j + \sum_{ijkl} V_{ijkl} b_i^\dagger b_j b_k^\dagger b_l, \tag{2.1}$$

and the system is subjected to an external field F_{ij}:

$$H_F = \sum_{ij} F_{ij} b_i^\dagger b_j. \tag{2.2}$$

In the following, we shall assume that the system is translationally invariant; thus, the momentum is a good quantum number and diagonalizes the first term in H_0. The field F_{ij} is also defined in terms of spatial harmonics, so now $i, j \equiv k, k'$. Moreover, a part of interactions, that is, of a density-density type $\propto n_i n_l$ is assumed to be the strongest and it leads to a renormalization of the boson velocity, while keeping their dispersion linear, just as it happens in the Lieb–Liniger model [9] or in the random-phase approximation (RPA) treatment of acoustic phonons/plasmons [10]. For such a case, the quantum, $T = 0$, solution to the model given by Eq. (2.1) is known [11], and it has two phases: superfluid and insulator. However, it is also known that coupling to a strong multi-harmonic field $F(q_i)$, Eq. (2.2), leads to floating Aubry–André physics [12], and/or coupling to a dissipative environment leads to a metallic phase with a well-defined Drude peak [13, 14]. Then, using the BTE, Eq. (1.1) is justified. Our quantity of interest is then the average occupancy of bosonic modes $b(k) = \langle \widetilde{b}_k^\dagger \widetilde{b}_k \rangle$ in the high-temperature regime, when the low-energy physics may be integrated out and the eigenstates $\widetilde{b}_k$ of the interacting bosons, Eq. (2.1), emerge, see, e.g., Eq. (2.3). This implies that the effective susceptibility of the environment will contain some retardation effects, namely some non-trivial momentum dependence inherited from these averaged-out low-energy modes. In turn, this implies that the interactions $V(q)$ will have more complicated character and will not be entirely absorbed in a simple linear dispersion.

Furthermore, we assume that, upon interaction with the field F, the bosons can be divided into different ν-branches ($\equiv$ subsystems); for instance, phonons with different symmetries, such that their interaction with the external field $F(q)$ is different (e.g., due to dipole matrix selection rules). The simplest situation is that of two branches, although it can naturally be extended to more subsystems. The simplest system under consideration thus consists of two parts: 1) a 1D electron liquid (1D metal) that is subjected to a strong current (due to the applied force $F(q)$), and 2) a recipient 1D system that is subjected to the collisions with 1. Of the latter, we only need to assume that it can be described by a distribution that admits a local Taylor expansion. Thus, the locally induced voltage in the recipient system is linearly proportional to a distortion of local distribution of carriers, such that transverse transport coefficients (e.g., drag coefficients) can be potentially defined.

In a 1D system, in the presence of any density-density interaction V_0, the spectral weight is shifted from quasiparticles, to collective modes [15]. These collective modes are described by the following Hamiltonian:

$$H_{\mathrm{mod}} = \sum_{\nu} u_{\nu}(q) q \widetilde{b}_{\nu}^{\dagger}(q) \widetilde{b}_{\nu}(q), \tag{2.3}$$

which we express here in terms of bosonic operators $\widetilde{b}_{\nu}^{\dagger}$ where the index ν spans over, e.g., charge and spin degrees of freedom to capture collective fluctuations in each of these sectors. This example defines separable ν-branches ($\equiv$ subsystems). Thanks, e.g., to spin-charge separation present in 1D, each of these modes can be indeed considered separately. In general, the velocities $v_{\nu}(q) \propto q^{\alpha_{\nu}-1}$ are determined by interactions, with the linear dispersion $\alpha_{\nu} = 1$ present for purely Hartree-type interactions. In many nanostructures, that are, e.g., deposited on the surface, the 1D charges can provide only partial screening. We then obtain some finite range, Coulomb-type interaction $\widetilde{V}_{\mathrm{Coul}}(q)$, which affects these charged degrees of freedom. The velocity $v_{\rho}(q)$ is now determined by $\widetilde{V}_{\mathrm{Coul}}(q)$, which in turn depends on the dielectric constant of the nanostructured environment within which the 1D system is embedded, namely: $\widetilde{V}_{\mathrm{Coul}}(q) = V_{\mathrm{Coul}}(q)/\epsilon(q)$. Here $V_{\mathrm{Coul}}(q)$ is the bare Coulomb interaction (which can be assumed to be of the Hartree type), while $\epsilon(q)$ is the dielectric constant that has been evaluated analytically, for instance, in [16].

Our problem thus consist out of 1D charge-carrying bosons with a power-law dispersion:

$$\omega_{\rho}(q) = \widetilde{v}_{\rho} q^{\varsigma}, \tag{2.4}$$

which is equivalent to a velocity $v_{\rho}(q) \propto q^{\varsigma-1}$, where q is dimensionless and $\widetilde{v}_{\rho}$ is a proportionality constant that carries the dimension of energy.

In the following, we shall consider two copies of such 1D systems coupled through a momentum-carrying interaction:

$$H_{\mathrm{tot}} = H_{\mathrm{mod}}^{(1)} + H_{\mathrm{mod}}^{(2)} + H_{d}^{(1-2)}, \tag{2.5}$$

where $H_{d}^{(1-2)}$ is the coupling term, which we assume to be weak enough to not destroy the modes' separation into the subsystems. Moreover, we assume that one of the modes is in a strongly non-equilibrium state, in the sense that a strong current flows through it.

The momentum relaxation term $H_d^{(1-2)}$ between the subsystems, with amplitude $\equiv \Gamma_0 =$ constant, can potentially allow a transfer of part of the current to the other subsystem (in fact, Γ_0 has to have some ultra-violet (UV) cut-off, as in Yukawa potential). The remanent interaction induces the non-linearity of the spectrum (which can be interpreted as a self-energy effect), but it will also induce a vertex correction. From the Ward identity, we know that the self-energy $\Sigma(k \pm q)$ and the vertex correction $\Gamma(q)$ have to be related if a system is conserving. To be precise [17], in the low-energy limit $\Gamma(q) \propto \partial\Sigma(k)/\partial k|_{k=q}$. The non-linearity of the dispersion will thus induce anharmonic behavior of the bosons, i.e., boson-boson interactions. We then expect that in our particular case, when there is a deviation $\mathrm{Re}[\Sigma(k)] \approx \Delta\omega(k)$ from the linear dispersion, the vertex correction reads:

$$\Gamma(q) = (\partial_q \Delta\omega_\rho(q)/\partial q) = (\varsigma - 1)\widetilde{v}_\rho q^{\varsigma-1}, \tag{2.6}$$

and is therefore given by another power law. In deriving this we performed a Taylor expansion of $\omega(k \pm q)$ for a small values of exchanged momenta q and kept only the first term. This is justified by the fact that the postulated interactions $\widetilde{V}_{\mathrm{Coul}}(q)$ is of screened Coulomb type and thus favors small exchanged momentum processes. The most important implication of Eq. (2.6) is that, with such a generalized dispersion model, the vertex corrections are unavoidable. We emphasize that the formalism that we shall derive further can accommodate a vertex correction that is given by any power law. However, the specific form given by Eq. (2.6) is useful for demonstrating that such a correction will necessarily appear if one wants to keep a conserving approximation. Figure 1 presents a schematic illustration of our model. Two 1D subsystems are coupled by a momentum-dependent amplitude $\Gamma(k)$. There is a strong current j (yellow arrow) flowing through one of them. Each subsystem (mode) has a different distributions of bosons, shown (trimmed) on the top of each subsystem. We see different temperatures and a shift of the upper distribution due to the current flow. The ansatz for these distributions is given in Sec. 3.

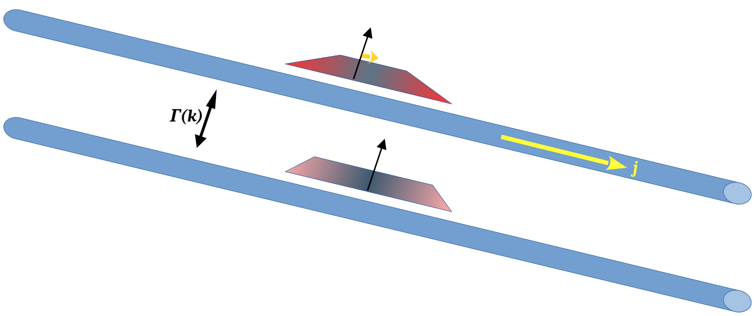

FIG. 1. Schematic illustration of our model.

3. Postulated non-equilibrium bosonic distributions

The non-equilibrium distribution of bosons is postulated in the form of a displaced Bose–Einstein distribution function (taking $k_B = 1$):

$$F_d = \frac{1}{2\pi}\left\{\exp[\omega(k)/T_e - kv_{\mathrm{drag}}/T_d]\right\}^{-1}, \tag{3.1}$$

the same as the one assumed for a strongly dragged (i.e., non-equilibrium) gas of bosons with an arbitrary dispersion relation $\omega(|k|)$. This form is known in the literature and has been used many times before, e.g., in [18–20]. Here, T_e is the temperature of a gas in equilibrium with the same energy density, and v_{drag} and T_d are the so-called drift velocity and drift temperature, respectively. For a bosonic gas represented by a distribution function $b[\chi(k);T]$ the energy density and momentum density read:

$$\epsilon := \frac{1}{\Omega_{\mathrm{BZ}}}\int\limits_{\mathrm{BZ}} \omega(|k|)b[\chi(k);T_{e,d}]\,\mathrm{d}k, \qquad p := \frac{1}{\Omega_{\mathrm{BZ}}}\int\limits_{\mathrm{BZ}} kb[\chi(k);T_{e,d}]\,\mathrm{d}k. \tag{3.2}$$

In [20, 21], the distribution of the form given by Eq. (3.1) was obtained from an optimization procedure of entropy written as a functional of the distribution $b(k)$:

$$s\,[b(k)] = \frac{1}{2\pi}\int \{(1+2\,\pi\,b(k))\ln(1+2\,\pi\,b(k)) - 2\,\pi\,b(k)\ln(2\,\pi\,b(k))\}\,\mathrm{d}k, \tag{3.3}$$

under a constrains corresponding to fixed values of the energy ϵ and momentum p, namely:

$$\delta\left(s(b(k)) + \alpha\left(\epsilon - \frac{1}{\Omega_{\mathrm{BZ}}}\int\limits_{\mathrm{BZ}} \omega(k)\,b(k)\,\mathrm{d}k\right) + a\left(p - \frac{1}{\Omega_{\mathrm{BZ}}}\int\limits_{\mathrm{BZ}} kb(k)\,\mathrm{d}k\right)\right) = 0, \tag{3.4}$$

where α and a are the Lagrange multipliers of this variational problem with constrains. Solving Eq. (3.4) for $b[\,]$ and comparing the result with Eq. (3.1) we find [20, 21] that the variational problem is indeed optimized by

$$b[\chi(k)] = F_d\,[\chi] = \frac{1}{2\pi\,\left(e^{\chi(k)}-1\right)}, \qquad \chi(k) = \alpha\,\omega(k) + ak, \tag{3.5}$$

where

$$\alpha = {T_e}^{-1}, \qquad a = -\frac{v_{\mathrm{drag}}}{T_d}. \tag{3.6}$$

Based on this result, in the following, to make correspondence with standard condensed-matter notation, we define for the i-th subsystem:

$$\alpha_i \equiv \beta_i = 1/T_i. \tag{3.7}$$

Moreover, we shall re-scale β_i with the 'bare' velocity $\widetilde{v}_{i\rho}$, that is, $\widetilde{v}_{i\rho}\beta_i \to \beta_i$ which is a shorthand dimensionless notation used in subsequent sections.

Naturally, the solution of the constrained variational problem reproduces the constrains, i.e., $\epsilon = \frac{1}{\Omega_{\mathrm{BZ}}} \int\limits_{\mathrm{BZ}} \omega(|k|) F_d[\chi(k)]\,\mathrm{d}k$ and $p = \frac{1}{\Omega_{\mathrm{BZ}}} \int\limits_{\mathrm{BZ}} k F_d[\chi(k)]\,\mathrm{d}k$. Moreover, when the gas of bosons tends to equilibrium, the constraint $p \to 0$ and consequently the corresponding Lagrange multiplier in the distribution $a \to 0$; thus, the distribution itself becomes the Bose–Einstein distribution. As shown explicitly in Subsec. 5.1, for the case of linear dispersion the quantities T_e, T_d, and v_{drag} can be expressed [21] as analytic functions of the given ϵ and p.

To close this subsection, we would like to emphasize that naturally the proposed ansatz for the distribution is obviously an approximation in which we assume that the emergent modes are weakly interacting. Such an assumption can be justified, for instance, when the interactions $V(q)$ are chosen such that the system is in the vicinity of Luther–Emery points. This issue is discussed further in Sec. 5.

4. Collision term and resulting integral

4.1. Derivation of the collision integral

We can substitute the postulated non-equilibrium distribution function F_d into the Boltzmann transport Eq. (1.1). Specifically, we are interested in evaluating the RHS of this equation, the collision integral, which depends on the inherent properties of the system, rather than on the specific form of the externally applied field. The collision integral that we want to evaluate reads:

$$\overline{\Gamma}_{\mathrm{coll}}(k) = \int\limits_0^{k_0} \mathrm{d}q \Gamma_0 \Gamma(q) b_{d1}(k+q) b_{d2}(k-q), \tag{4.1}$$

where k_0 is the upper cut-off momentum of the theory (e.g., the Brillouin zone boundary) and b_{di} is a short-hand notation displaced distribution, Eq. (3.5), in the i-th subsystem. This is a Bose–Einstein distribution with drag, a distribution parameterized by the pair (β_i, a_i). In order to make progress, we take a Taylor expansion of $\omega_\rho(k \pm q)$, which leads to a term $\beta((1+\varsigma)k^\varsigma \pm +\varsigma k^{\varsigma-1} q)$. The underlying assumption of a small q exchanges' dominance has already been discussed (Eq. (2.6)) and it clearly manifests in the last term of Eq. (22). Upon substituting it into the distributions, we find that:

(4.2)
$$b_{d1}(k+q)b_{d2}(k-q) = \exp\left\{-\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^\varsigma\right\}$$
$$\cdot\left(\exp\left\{\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^\varsigma\right\}\exp\left\{[(\beta_1+\beta_2)\varsigma k^{\varsigma-1}+(a_1+a_2)]q\right\}\right.$$
$$+\exp\left(\frac{1}{2}(\beta_1-\beta_2)(1+\varsigma)k^\varsigma\right)\exp\left\{[(\beta_1-\beta_2)\varsigma k^{\varsigma-1}+(a_1-a_2)]q\right\}$$
$$+\exp\left(\frac{1}{2}(-\beta_1+\beta_2)(1+\varsigma)k^\varsigma\right)\exp\left\{[(-\beta_1+\beta_2)\varsigma k^{\varsigma-1}+(-a_1+a_2)]q\right\}$$
$$\left.+\exp\left\{-\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^\varsigma\right\}\right)^{-1}.$$

Thus, the collision integral can now be cast in the following form:

(4.3)
$$\overline{\Gamma}_{\mathrm{coll}}(k) = \exp\left\{-\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^\varsigma\right\}$$
$$\int \mathrm{d}q\frac{(\varsigma-1)\Gamma_0 q^{\varsigma-1}\exp\left\{\frac{1}{2}[(\beta_1-\beta_2)\varsigma k^{\varsigma-1}+(a_1-a_2)]q\right\}\exp[-q/k_0]}{z(k)\exp\left\{[(\beta_1+\beta_2)\varsigma k^{\varsigma-1}+(a_1+a_2)]q\right\}-1},$$

where the numerator contains the inverse bosonic scattering amplitude $\Gamma(k)$ multiplied a Yukawa potential (which in the solid-state setting is equivalent to Thomas–Fermi screening), which we introduced both to accommodate the upper limit of the integral and to physically model the finite range of the interaction. In the denominator, there is a following function:

(4.4)
$$z(k) = \frac{\exp\left\{\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^\varsigma\right\}}{a^*},$$

where

$$a^* = \exp\left(-\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^\varsigma\right)+\exp\left(\frac{1}{2}(\beta_1-\beta_2)(1+\varsigma)k^\varsigma\right)$$
$$+\exp\left(\frac{1}{2}(-\beta_1+\beta_2)(1+\varsigma)k^\varsigma\right).$$

It is well known that many integrals of canonical distribution functions can be given in terms of polylogarithms [22]. This is not sufficient to solve Eq. (4.3), we therefore shall move to a generalization of polylogarithm function, which is the Lerch transcendent function. Its integral form reads [23]:

(4.5)
$$\Phi(z,s,\overline{\overline{a}}) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}e^{-\overline{\overline{a}}x}}{1-ze^{-x}}\,\mathrm{d}x,$$

where the denominator has a form similar to a denominator of the canonical distribution and in the numerator there is a modified power law factor.

By comparing this with Eq. (4.3), we observe that after that substitution $x \to q$ our integral is indeed solvable through the Lerch transcendent function. The result of this integral reads:

$$\overline{\Gamma}_{\text{coll}}(k) = \frac{\exp\left\{ -\frac{1}{2}(\beta_1+\beta_2)(1+\varsigma)k^{\varsigma}\right\}}{[(\beta_1+\beta_2)\varsigma k^{\varsigma-1} + (a_1+a_2)]} \frac{(\varsigma-1)\Gamma_0}{\overline{\Gamma(\varsigma-1)}} \Phi\left(z(k),(\varsigma-1),-y(k)\right), \tag{4.6}$$

where

$$y(k) = \frac{\frac{1}{2}[(\beta_1-\beta_2)\varsigma k^{\varsigma-1} + (a_1-a_2)] - 1/k_0}{[(\beta_1+\beta_2)\varsigma k^{\varsigma-1} + (a_1+a_2)]}, \tag{4.7}$$

and $\Phi(\,)$ is the Lerch transcendent function, and $\overline{\Gamma(\,)}$ is the gamma function. Actually, the Lerch transcendent special function, a generalization of polylogarithm functions (known to be integrals of quantum distributions), has already appeared in the literature as an integral of distributions with drag, see [24]. However, the full dependence on ς has not been explored. Equation (4.6), together with expressions for $z(k)$ (Eq. (4.4)) and $y(k)$ (Eq. (4.7)) provides a closed analytic form for the momentum-resolved collision integral. This is the central result of our work.

This result also allows the investigation of the energy-weighted and momentum-weighted integrals, which both must exist and be positive. This is because the Lagrange multipliers (as defined in Sec. 3) must be reproduced by these moments of the distribution. From properties of the Lerch function, we can now deduce that the positivity condition is equivalent to the positivity of distribution argument.

4.2. Results

We shall now illustrate the results of Eq. (4.6). First, in Fig. 2, we plot the collision integral as a function of $a_1 \equiv v_{\text{drag}}$. For simplicity, we take the case when one of the distributions is not shifted, i.e., $a_2 = 0$. We observe that the collision amplitude is strongest for the small electron momenta k. For larger value of ς (right panel), for small values of k, we see that the function is clearly non-monotonic: there exists an optimal value of the Lagrange multiplier a_1 for which the drag effect is strongest. Naturally, for the smallest values of a_1 the effect increases with the shift of the distribution. However, when the distribution is shifted too strongly, the carriers within the receiving subsystem cannot catch up with the rapid motion of non-equilibrium carriers. This is the reason why we call a_1 the drag velocity v_{drag}. For larger k, the effect monotonically increases with v_{drag}, although this tendency is much less pronounced. The same behavior

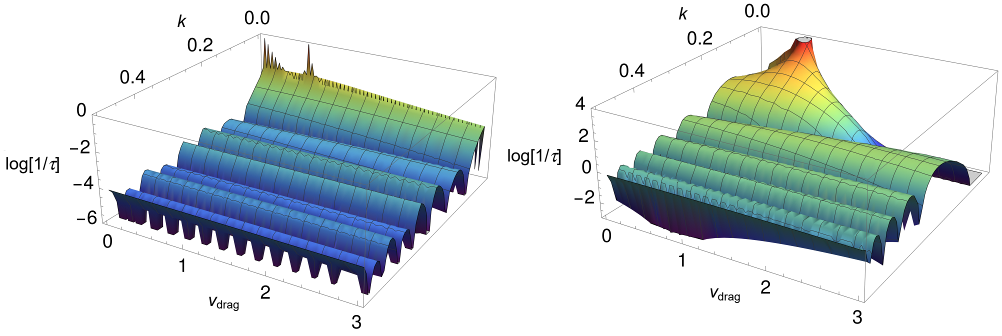

FIG. 2. Logarithm of the collision amplitude as a function of the electron momentum k and the drag velocity v_{drag} ($\equiv a_1$). On the left we show result for $\varsigma = 1.5$, on the right for $\varsigma = 2.5$. Both calculations were done for $\beta_1 = 30$, $\beta_2 = 20$, and $a_2 = 0$ (no initial shift of distribution). Other parameters: $\Gamma_0 = 2/3$, $q_0 = 3$. All the quantities are given in units of energy, where $\overline{v}_\rho = 1$, which is set by the characteristic kinetic energy of the system and in units of momentum set by the 1D density of particles n, just as in [9].

for large k is observed in the left panel (smaller ς). Interestingly, here for small k we cannot notice such a clear resonance: there is a noisy variation at small v_{drag} and then a collision amplitude monotonically decreases.

In Fig. 3, we plot the result of the collision amplitude as a function of temperature. We set the temperature of the recipient mode to $b_2 = 20$ and vary the temperature of the dragging subsystem. We observe that the largest effect, a strong maximum, exists when the temperature $b_1 = 20$, i.e., when $b_1 = b_2$. The effect indicates that the collision integral is strongest when the two distributions are most similar to each other.

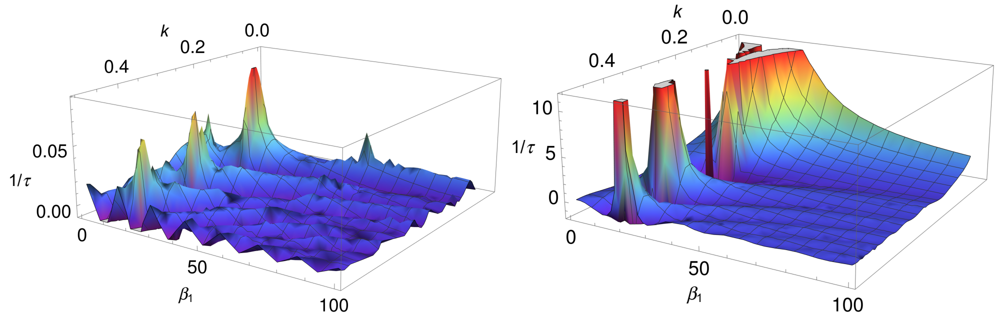

FIG. 3. Collision integral amplitude as a function of the electron momentum k and temperature $\beta_1 = T_{e1}^{-1}$. On the left we show result for $\varsigma = 1.5$, on the right for $\varsigma = 2.5$. Both calculations were done for $a_1 = 0.5$, $\beta_2 = 20$, and $a_2 = 0$ (no initial shift of the distribution). Other parameters: $\Gamma_0 = 2/3$, $q_0 = 3$.

However, the most important conclusions from all these calculations is the very strong dependence on ς: moving away from $\varsigma = 1$ can cause the result to grow by orders of magnitude. Thus, we see that the amplitude of the collision integral is a very susceptible measure of the non-linearity encoded in the value of ς.

5. Discussion

5.1. Case of linear dispersion

For the case of linear dispersion $\omega = c|k|$ we can derive analytic expressions for the Lagrange multipliers of the variable $\chi(k)$ [21], which now can be called the equilibrium temperature $T_e = \beta^{-1}$ and the drag temperature $T_d = a^{-1}\overline{v}_{\text{drag}}$, where we define the bare (normalized by T_d) drag velocity $\overline{v}_{\text{drag}}$:

$$T_d = \frac{T_e\,(1-u)}{\sqrt{1+u}}, \tag{5.1}$$

$$u = \frac{\epsilon - \sqrt{\epsilon^2 - c^2p^2}}{\epsilon + \sqrt{\epsilon^2 - c^2p^2}}, \tag{5.2}$$

where, as before in Sec. 3, the distribution $F_d[...]$ determines the average energy and momentum (normalized by the volume of the first Brillouin zone, Ω_{BZ}) that enter into the before mentioned formulas:

$$\epsilon = \frac{1}{\Omega_{\text{BZ}}} \int\limits_{\text{BZ}} c|k|\, F[\chi(k)); T_e]\,\mathrm{d}k, \tag{5.3}$$

$$p = \frac{1}{\Omega_{\text{BZ}}} \int\limits_{\text{BZ}} kF[\chi(k)); T_e]\,\mathrm{d}k, \tag{5.4}$$

where, because of the fixed relation Eq. (5.1), we can use only one temperature to parameterize the distribution. The latter quantity, the average momentum in a subsystem p, can be related to the bare drag velocity:

$$\overline{v}_{\text{drag}} = c\frac{cp}{\epsilon + \sqrt{\epsilon^2 - c^2p^2}}. \tag{5.5}$$

Thus, we see that for the linear distribution there is a unique solution to the shifted-density problem. Importantly, for the linear dispersion in 1D, the two integrals in Eq. (5.3) and Eq. (5.4) have the same form up to a unitary factor $\operatorname{sign}(k)$.

This observation can be generalized. It can be shown that in the case of linear dispersion, i.e., $\varsigma = 1$, the drag effect is expected to be zero, which is in agreement with our prediction. This is because the moment expansion $M_n = \int k^n F(\chi(k))\,\mathrm{d}k$ of a distribution forms a closed set, as we fixed the value of ϵ (total k) and p (difference between positive and negative k). As it was shown in [25], the drag is proportional to third cumulant, and from statistics we know that third cumulant is equal to the third moment (skewness). This general argument justifies our result. A similar conclusion, for more specific case of thermoelectric effect, was reached in [26], where it was traced back to the Mott relation for transport coefficients.

5.2. Limitations of our reasoning

There are two key limitations of our approach. The first one is that we have neglected the known quantum phases of the 1D boson model: superfluid and insulator. This may be justified in the regime of high temperature and in a dissipative environment, which is actually the same regime where the quasi-classical BTE approach itself is known to be applicable. Nevertheless, one has to keep in mind that our calculations are valid only in the quasi-classical regime and caution should be excercised when applying them to low-temperature systems subjected to highly coherent field $F(q)$.

Second, in Sec. 3, we adopted an ansatz for non-interacting bosonic distributions which is obviously an approximation when boson-boson interactions are present. Simply taking $V_{ijkl} = 0$ in Eq. (2.1) will not ameliorate the situation, as then we shall arrive at a bosonic liquid with a very high tendency toward Bose–Einstein condensation, a quantum phenomenon not accounted for in a simple BTE. There are two arguments that may justify our model. The first one is based on the existence of Luther–Emery points: for specific values of V_{ijkl}, the 1D system behaves as a system of an emergent non-interacting particles. This is why we emphasized that the modes on which the Hamiltonian in Eq. (2.3) is defined are emergent collective eigen modes. This is also the reason why, for these modes, both self-energies effects and vertex corrections may be substantial. The second argument is more pragmatic: the quantity on which we focus in this work is solely the collision integral. We limited ourselves and decided to not move toward full solution of the BTE. In this way, our result is more generic and can be applied to various configurations of the external $F(q)$. But, in this way, we also compute only a basic building block of a potentially more advanced approximation. One can consider including interaction corrections to the distribution $b(k)$ in a perturbative series (with all possible particle-hole cancellations) and then, at each term, the expressions provided by us here can be used. Nevertheless, we admit that future research, aiming at deriving the collision integral for fully interacting distributions of bosons, known as hypergeometric Beta functions [15], would be an exciting and important direction of research.

One posibble physical realization of the proposed model would be either a multi-wire or a multi-orbital 1D system. Here, it is pretty intuitive that the field F can interact strongly only with one selected orbital or – in the case of coaxial tubes (such multi-walled carbon nanotube (MWCNT)) or parallel wires – only to the outermost subsystem. Then, we naturally obtain subsystems with different distributions, and with different shifts. Moreover, since the subsystems are nearly orthogonal (in Hilbert space), tunneling requires reconstitution of the entire boson. This has been evidenced by experiments in [27]: the perpendicular transport measures the single-particle density of states. In such a specific case, it makes sense to use shifted Bose–Einstein distributions to compute collision integrals between these reconstituted particles.

5.3. Examples of experimental relevance

The connection between our postulated theoretical model and its real-life experimental realizations is a quite complicated but also fascinating problem. The two most promising platforms are nanostructured wires and cold atoms. In each of these systems, the ongoing research faces slightly different problems.

In cold atoms, where emergent bosons are created by trapping atoms in an optical lattice, the interactions can be relatively easily modified by changing the external magnetic field (in this way one tunes the scattering cross section through a Feshbach resonance). Moreover, one can add a fast, anti-adiabatic, fermionic component to a mixture of two bosons and vary their interaction through a Ruderman–Kittel–Kaqsuya–Yosida (RKKY) type mechanism. This has been experimentally employed in [28]. Therein, one finds that the characteristic kinetic energy can be of the order of 6.5 Hz while at the same time the interspecies scattering can be tuned to be as small as $\Gamma_0 \sim 0.5\,\mathrm{Hz}$. From the point of view of energy scales of these bosons, temperatures of the order of nK are indeed in the high-temperature regime that we discuss in our paper. The difficulty here lies in extracting the dispersion relation: the interaction is that of an extended Lieb–Liniger model, and from the Bethe-ansatz solution of the Lieb–Liniger model we know that there are higher-energy bosons with super-linear dispersion and low-energy bosons with sub-linear dispersions. A separate issue is how these dispersions will be modified by the optical trapping potential, see, e.g., [29], where the influence of a parabolic trap was analyzed. From the experimental viewpoint, the exponent ς can be extracted only indirectly from time-of-flight experiments, which are destructive for the trapped bosonic system itself.

In a system of nanowires, the plasmonic dispersion is governed by the geometry of the system. While in 3D plasmons have a constant frequency, in 2D their dispersion obeys $k^{1/2}$ whereas in 1D q^1 (this is all assuming a finite range interaction, the full Coulomb interaction would give a $\log(q)$ correction). These results are derived for simple geometry. For a corrugated surface, textbook result tells us that the interaction is, in general, given by a power law with a characteristic exponent dependent on a corrugation angle. However, in such a solid-state system, inter-boson interactions are much harder to control. One can modify the Thomas–Fermi screening length or submerge the system into a dielectric medium, thus modifying the interaction $\widetilde{V}(r)$, which implies also changing inter-boson interaction (scattering with finite momentum exchange). However, all these efforts are only indirect. An interesting situation, where inter-boson scattering indeed appear to play crucial role is reported in [30], where the plasmons' kinetic energy was reported as $1.2eV$ and the scattering amplitude was $\Gamma_0 \approx 0.2eV$. Other solid-state systems will have similar orders of magnitude and here we will need to take the operating temperature to be a room temperature or higher (as in the case of an aforementioned experiment) to remain in the high-temperature regime considered in this work.

6. Conclusions

In this paper, we have obtained a closed analytic formula for the collision integral between two 1D systems. This is a basic building block for obtaining solutions of the BTE, Eq. (1.1), and for determining transport coefficients in many experimentally/technologically motivated applications. The collision term in the BTE is the one that determines resistivity and the one that is notoriously the hardest to compute, especially in a non-linear non-equilibrium case. Our results enable researches to not only study the strength of the non-linearity ς in the charge sector, but also to directly access the non-equilibrium distributions through the Lagrange multipliers that are characterizing them. With this knowledge, researchers will be able to quantify the non-equilibrium state of the system for various experimental setups, as defined by the LHS of Eq. (1.1).

Upon completing this paper, we learned that experiments capable of probing this regime had indeed been performed very recently, see [31].

Fundings

Piotr Chudziński acknowledges financial support from Polish National Science Centre, grant no. 2021/43/B/ST8/03207.

Conflict of interest

The authors declare that there are no known competing financial interests or personal relationships that could have influenced the work described in this paper.

Authors' contributions

Both authors were involved in the conception of the paper. Wieslaw Larecki designed and drafted Sec. 3 and Subsec. 5.1. Piotr Chudziński designed and drafted all other sections including analysis and interpretation of the data in Sec. 4. All authors reviewed and approved the final manuscript.

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Received November 13, 2025; revised December 2, 2025; accepted December 3, 2025; available online December 12, 2025; version of record June 16, 2026; published issue June 24, 2026.