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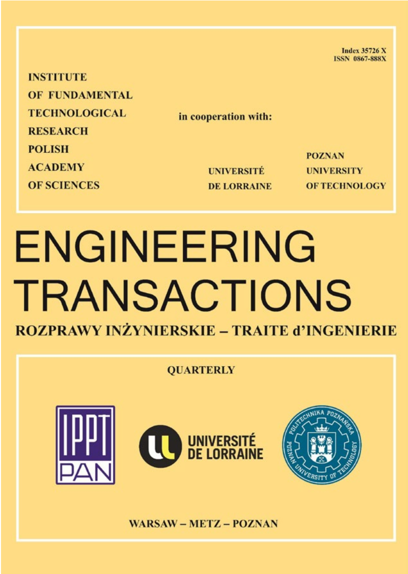


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# On the out-of-plane deviation of the bending deformation states of moderately thick bars of asymmetric cross-sections

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**Abstract**

A characteristic feature of the six–parameter theories of bars is a coupled form of the constitutive equations; in particular the equations linking transverse forces with transverse shear deformations cannot be, in general, decoupled, keeping a separated form of the remaining constitutive equations. The mentioned feature of the constitutive equations implies that within the six–parameter theories of straight elastic prismatic bars there do not exist, in general, plane states of bending/shearing deformations. Thus, any vertical load causes lateral deflections, the only exception being the pure bending problem. The present paper delivers analytical solutions: the closed formulae for shape functions, i.e. deformation states associated with kinematic loads at the ends, and solutions to selected static problems corresponding to the transverse span load. Although elementary, the presented solutions seem to be derived for the first time. In particular, the hitherto published shape functions concerned the theories of moderately thick bars in which all the constitutive equations are decoupled.

**Keywords:** six–parameter theory of bars, bars of moderate thickness, shape functions

# 1 Introduction

Among theories of linearly elastic straight prismatic bars the most important are: theory by Vlasov [1] of bars of thin and open profiles and six-parameter theories of moderately thick bars attributed to Timoshenko, see Bazoune et al [2], or to Saint-Venant, see Petrolo and Casciaro [3]. A peculiar feature of the Vlasov theory is decoupling of the constitutive equations: the axial force is linked with the axial strain, the bending moments are proportional to the corresponding bending strains, the torsional moment is linked with the measure of torsion and the bimoment is proportional to the measure of the warping due to torsion. This decoupling is possible by applying special measures:

a) assuming the principal axes $y$ and $z$ of the cross-section $\mathcal{A}$
b) appropriate choice of the starting point of the sectorial coordinate
c) appropriate choice of the position of the pole

The formalism of the Vlasov theory can be extended to the case of straight prismatic bars of arbitrary cross-sections, see Lewiński and Czarnecki [4], Sec.9, still preserving the mentioned decoupling of the constitutive equations by making appropriate measures concerning the position of the axes to which the internal forces are referred.

The theory of Vlasov neglects the transverse shear deformations thus removing the transverse forces from the set of the internal forces of the theory. However, including the transverse shear effects is possible since the Saint-Venant theory delivers the analytical expressions for the functions modelling warping due to shear, see Love [5] and Ieşan [6]. Such a bar theory has been proposed by Librescu and Song, see [4]. By neglecting in this theory the contribution of the warping due to torsion to the elastic energy one arrives at a six-parameter theory. Its form in the 3D setting is not unique. In a standard approach all the internal forces are referred to the neutral axis, i.e. the axis linking the centroids of the cross-sections, see e.g Petrolo and Casciaro [3]. In this model the axial force is proportional to the axial strain, the bending moments are linked with the corresponding measures of bending while the triple $(T_y, T_z, \mathcal{M})$ or the transverse forces and the torsional moment are linked with the triple: $(\gamma_y, \gamma_z, \rho)$, where $\gamma_y$, $\gamma_z$ stand for the measures of transverse shear and $\rho$ is the measure of torsion deformation. The 3x3 matrix linking these quantities is fully filled up, see Eq.40 in Petrolo and Casciaro [3].

The papers discussing more complicated models of bars, like Dikaros [7], El Fatmi [8], [9] teach that like in Vlasov theory it is expedient to shift the transverse forces to the axis $x_{(s)}$ linking shear centers $S$. By appropriate choice of functions modelling warping due to torsion and shear one can derive a bar model in which the torsional moment is only linked with the strain $\rho$ and the pair $(T_y, T_z)$ is linked with the pair $(\gamma_y, \gamma_z)$, the latter 2x2 matrix being, in general, fully filled up, i.e. the off diagonal components are in general non-zero. If we decouple the latter 2x2 system by a certain rotation of the axes $y$ and $z$ we introduce the coupling in the constitutive equations for the bending moments. Indeed, there is no aim to change the parametrization $y$, $z$ referred to the principal axes of the cross-section. We conclude: coupling of the constitutive equations linking $(T_y, T_z)$ with the pair $(\gamma_y, \gamma_z)$ is an immanent feature of the six–parameter theory of bars in all its versions.

The six-parameter model is the simplest theory of deformation of bars in space in which the kinematic unknowns are: $u$, $v$, $w$, $\theta$, $\varphi$, $\beta$ or, subsequently: the axial displacement, displacements of the shear center in $y$ and $z$ directions, angle of torsion, angles of rotations around the axes $(-z)$ and $y$. The internal forces of this theory are: $N$, $T_y$, $T_z$, $\mathcal{M}$, $M_y$, $M_z$ or: axial force, transverse forces, torsional moment and bending moments in $y$ and $z$ directions. The internal forces are linked with the measures of deformation: $\epsilon$, $\gamma_y$, $\gamma_z$, $\rho$, $\kappa_y$, $\kappa_z$, or: measures of axial strain, transverse shear strains, measure of torsion and measures of bending. The main feature of this theory is equality between the number of internal forces (always equal to the number of strains) and the number of kinematic fields. Thanks to this equality there exists a family of statically determinate problems, which paves the way to the force method, a helpful tool of structural mechanics.

If the cross-section of the bar is monosymmetric (or bisymmetric, in particular) then the problem of bending/transverse shearing in 3D decouples into two planar problems in $x - y$ and $x - z$ planes. If the shape of the domain $\mathcal{A}$ of the cross-section is arbitrary then there do not exist plane states of bending/transverse shearing deformation. In particular, the load in $z$ direction causes bending in two directions: $z$ and $y$. Just this problem is studied in the present paper. The aim is to deliver explicit formulae for the deformation states of a bar subjected to kinematic loads and to selected transverse span loads.

The present paper draws upon the six-parameter theory of straight prismatic bars made of an isotropic and homogeneous material developed in [4], Sec.10, called there Timoshenko-like theory. The prerequisite of the theory is construction of the solutions to the three auxiliary elliptic problems posed on the domain $\mathcal{A}$, see Secs 2-4 in [4]. Upon solving these problems one can fix the position of the shear center $S$ of coordinates $y_S$, $z_S$ referred to the principal axes and then determine all the required characteristics and stiffnesses of the bar. This algorithm will not be repeated in the present paper.

One of the aims of the present paper is to deliver the explicit formulae for the so-called shape functions or the deformation forms of a bar subjected to arbitrary kinematic loads. These functions are given in compact formulae (24). Interestingly, these formulae are not available in the literature. There is only one paper, namely the paper by Schramm et al [10] in which this problem is considered at a similar level of accuracy, but the explicit form of the solution (24) has not been there published. Other papers, like e.g the articles by Kączkowski [11], Pełczyński and Gilewski [12] and the paper by Luo [13], deliver the solutions corresponding to the special case when the coupling of the constitutive equations is absent, which means that either these papers refer to the monosymmetric cross-sections or the authors assume, usually tacitly, that the constitutive equations can be accepted in their decoupled form. This assumption paves the way for the planar forms of the shape functions, as incorrect in general or referring to the special case of the cross section being mono- or bi-symmetric.

Moreover, the aim of the present paper is to derive the specific shapes of deformation caused by span loads. In particular the paper shows that the point loads generate rather unexpected lateral deflections in the form of zigzag lines.

## 2 Equations of the Theory

Consider a straight prismatic bar of the cross-section $\mathcal{A}$ for which we construct the principal axes $y$ and $z$, the centroid being its center. The axis $x$ is orthogonal to the domain $\mathcal{A}$; this axis links the centroids of all cross-sections. They are $x$ independent, which means that the bar is prismatic. The domain of the bar is filled up by a homogeneous and isotropic elastic material of Young modulus $E$ and shear modulus $G$. The center of shear $S$ has the coordinates $(y_S, z_S)$; their construction is explained in [4], Sec.2. According to the underlying theory this center coincides with the center of torsion; in more complicated discussions these centers do not coincide, see comments in [4]. By linking points $S$ we form the straight line $x_{(s)}$. Having solved the elliptic problems set up in Secs 2-4 of [4] one can compute: the area $A$ of $\mathcal{A}$, the principal moments of inertia $J_y$, $J_z$, and the shear correction factors $k_y$, $k_{yz} = k_{zy}$, $k_z$ forming the matrix $\boldsymbol{k}$; its inverse is denoted by $\boldsymbol{\alpha}$

$$\boldsymbol{k} = \begin{bmatrix} k_y & k_{yz} \\ k_{zy} & k_z \end{bmatrix}, \quad \boldsymbol{\alpha} = \boldsymbol{k}^{-1}, \quad \boldsymbol{\alpha} = \begin{bmatrix} \alpha_y & \alpha_{yz} \\ \alpha_{zy} & \alpha_z \end{bmatrix} \tag{1}$$

Both the matrices $\boldsymbol{k}$ and $\boldsymbol{\alpha}$ are positive definite. One can compute also the torsional constant $J$ and the torsional stiffness $GJ$.

The state of deformation of the bar is determined by the kinematic fields $u(x)$, $v(x)$, $w(x)$, $\theta(x)$, $\varphi(x)$, $\beta(x)$ given along the bar, see Fig.1a. The field $u$ is an average of the displacements $u_x(x, y, z)$ over $\mathcal{A}$, $\theta$ is the angle of rotation of $\mathcal{A}$ around the axis $x$, $v$ and $w$ represent displacements of the point $S$ along $y$ and $z$ axes, $\varphi$ and $\beta$ are defined by the averages

$$\beta(x) = \frac{1}{J_y} \int_{\mathcal{A}} z u_x d\mathcal{A}, \quad \varphi(x) = \frac{1}{J_z} \int_{\mathcal{A}} y u_x d\mathcal{A} \tag{2}$$

Their signs are chosen such that they cause positive displacements $u_x(x, y, z)$ if $y > 0$ and $z > 0$.

The loads acting along the bar are reduced to: the axial load of intensity $p(x)$, the transverse loads $q_y(x)$, $q_z(x)$ acting along the axes $y_{(s)}$, $z_{(s)}$ and the distributed moments $m(x)$ acting along the axis $x_{(s)}$.

The following internal forces (or stress and couple resultants) appear in the bar: the axial force $N(x)$, the transverse forces $T_y(x)$, $T_z(x)$ acting along the $y_{(s)}$ and $z_{(s)}$ directions, the bending moments $M_y(x)$, $M_z(x)$ acting along the axes $y$ and $z$, and $\mathcal{M}(x)$ or the torsional moment acting along the axis $x_{(s)}$, see Fig.1b

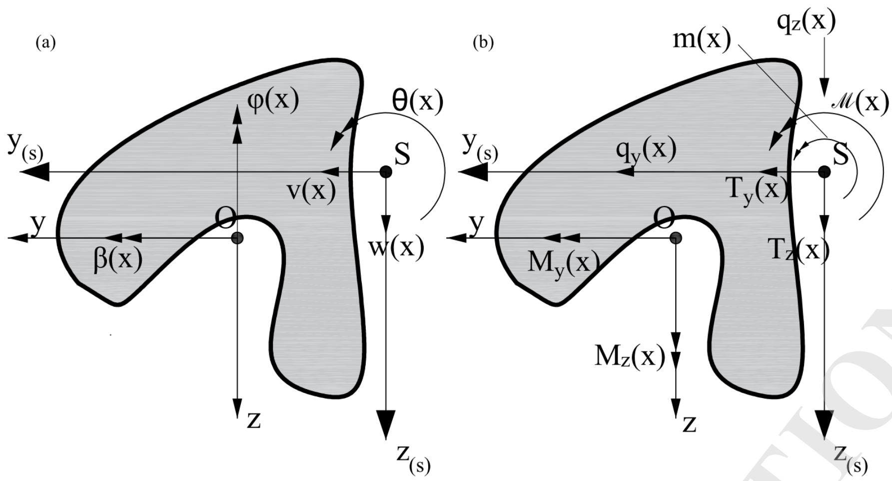



**Fig. 1**: The sign conventions concerning the kinematic unknowns (a) and the static unknowns (b) at the cross-section $x = \text{const}$.

The following strains are defined within the theory:

$$\epsilon = \frac{du}{dx}, \quad \gamma_y = \frac{dv}{dx} + \varphi, \quad \gamma_z = \frac{dw}{dx} + \beta, \quad \kappa_y = \frac{d\beta}{dx}, \quad \kappa_z = -\frac{d\varphi}{dx}, \quad \rho = \frac{d\theta}{dx} \tag{3}$$

$\epsilon$ represents the relative elongation of the bar, $\gamma_y$, $\gamma_z$ are the measures of transverse shear strains in the planes: $x-y$ and $x-z$, respectively; $\kappa_z$, $\kappa_y$ are measures of bending in the planes $x-y$ and $x-z$ and $\rho$ is the measure of torsion.

By virtue of special measures, i.e. specific interpretations of the internal forces, strains and kinematic fields the problem of statics of a bar decomposes into three independent problems:

## $(\mathcal{P}_1)$ THE TENSION/COMPRESSION PROBLEM

find $N(x)$, $\epsilon(x)$, $u(x)$ such that

$$\frac{dN}{dx} + p = 0, \quad N = EA\epsilon, \quad \epsilon = \frac{du}{dx} \tag{4}$$

while at the ends $x = 0$, $x = l$ either $N$ or $u$ is given.

## $(\mathcal{P}_2)$ THE TORSION PROBLEM

find $\mathcal{M}(x)$, $\rho(x)$, $\theta(x)$ such that

$$\frac{d\mathcal{M}}{dx} + m = 0, \quad \mathcal{M} = GJ\rho, \quad \rho = \frac{d\theta}{dx}, \tag{5}$$

while at the ends $x = 0$, $x = l$ either $\mathcal{M}$ or $\theta$ is given

## $(\mathcal{P}_3)$ THE BENDING/TRANSVERSE SHEARING PROBLEM

find $T_y(x)$, $T_z(x)$, $M_y(x)$, $M_z(x)$, $\kappa_z(x)$, $\kappa_y(x)$, $\gamma_y(x)$, $\gamma_z(x)$, $v(x)$, $w(x)$, $\varphi(x)$, $\beta(x)$ such that

$$\frac{dT_z}{dx} + q_z = 0, \quad T_z = \frac{dM_y}{dx}, \quad M_y = EJ_y\kappa_y, \quad \kappa_y = \frac{d\beta}{dx}, \quad \gamma_y = \varphi + \frac{dv}{dx} \tag{6}$$

$$\frac{dT_y}{dx} + q_y = 0, \quad T_y = -\frac{dM_z}{dx}, \; M_z = EJ_z\kappa_z, \quad \kappa_z = -\frac{d\varphi}{dx}, \quad \gamma_z = \beta + \frac{dw}{dx} \tag{7}$$

$$T_y = GA(k_y\gamma_y + k_{yz}\gamma_z), \quad T_z = GA(k_{zy}\gamma_y + k_z\gamma_z) \tag{8}$$

At the ends $x = 0$ or $x = l$ given are:
either $w$ or $T_z$, either $\beta$ or $M_y$, either $v$ or $T_y$, and either $\varphi$ or $M_z$.

Let us stress once again that in the standard setting the bending/shearing problem and the torsion problem are coupled, see Petrolo and Casciaro [3].

# 3 Simplifications in case of mono-symmetric profiles

If the axis $y = 0$ is a symmetry axis of the domain $\mathcal{A}$ then $k_{yz} = k_{zy} = 0$, $\alpha_{yz} = \alpha_{zy} = 0$ and the bending/shearing problem splits up into two problems

a) find $T_y(x)$, $M_z(x)$, $\kappa_z(x)$, $\gamma_y(x)$, $v(x)$, $\varphi(x)$, such that

$$\begin{aligned} \frac{dT_y}{dx} + q_y = 0, &\qquad T_y = -\frac{dM_z}{dx} \\ T_y = k_y GA\gamma_y, &\qquad M_z = EJ_z\kappa_z \\ \gamma_y = \varphi + \frac{dv}{dx}, &\qquad \kappa_z = -\frac{d\varphi}{dx} \end{aligned} \tag{9}$$

At the ends $x = 0$, $x = l$ given are: either $v$ or $T_y$; either $\varphi$ or $M_z$

b) find $T_z(x)$, $M_y(x)$, $\kappa_y(x)$, $\gamma_z(x)$, $w(x)$, $\beta(x)$, such that

$$\begin{aligned} \frac{dT_z}{dx} + q_z = 0, \qquad & T_z = \frac{dM_y}{dx} \\ T_z = k_z GA\gamma_z, \qquad & M_y = EJ_y\kappa_y \\ \gamma_z = \beta + \frac{dw}{dx}, \qquad & \kappa_y = \frac{d\beta}{dx} \end{aligned} \tag{10}$$

At the ends $x = 0$, $x = l$ given are: either $w$ or $T_z$; either $\beta$ or $M_y$

The majority of analyses available in the literature concerns the problems above.

If one assumes that additionally $z = 0$ is a symmetry axis, then the domain $\mathcal{A}$ is bisymmetric and then $S$ coincides with the centroid; $(y_S = 0, z_S = 0)$; the axes $x$ and $x_{(s)}$ coincide. Then the loads $q_z$, $q_y$ applied along the $x$ axis do not cause torsion, yet the equations (9), (10) do not change. Solving the static problems of bars of a mono-symmetric cross-section is not more complicated than in case of bisymmetric cross-sections.

## 4 Deformations caused by kinematic loads

Assume that the cross-section $\mathcal{A}$ is of arbitrary shape, the bar is clamped at both the ends: $x = 0$, $x = l$. The span load is absent. The kinematical boundary conditions have the form

$$\begin{aligned} u(0) &= {}^*u, & u(l) &= u^*, & \theta(0) &= {}^*\theta, & \theta(l) &= \theta^*, \\ w(0) &= {}^*w, & w(l) &= w^*, & \beta(0) &= {}^*\beta, & \beta(l) &= \beta^*, \\ v(0) &= {}^*v, & v(l) &= v^*, & \varphi(0) &= {}^*\varphi, & \varphi(l) &= \varphi^*, \end{aligned} \tag{11}$$

The solutions to the problems ($\mathcal{P}_1$), ($\mathcal{P}_2$) are elementary

$$\begin{aligned} u(x) &= {}^*u \cdot (1 - \xi) + u^*\xi, \\ \theta(x) &= {}^*\theta \cdot (1 - \xi) + \theta^*\xi, \quad \xi = \frac{x}{l} \end{aligned} \tag{12}$$

In order to solve the problem ($\mathcal{P}_3$) let us note that the deflections satisfy the following uncoupled system of equations

$$\begin{aligned} EJ_z\frac{d^4v}{dx^4} &= q_y - \frac{EJ_z}{GA}\left(\alpha_y\frac{d^2q_y}{dx^2} + \alpha_{yz}\frac{d^2q_z}{dx^2}\right), \\ EJ_y\frac{d^4w}{dx^4} &= q_z - \frac{EJ_y}{GA}\left(\alpha_{yz}\frac{d^2q_y}{dx^2} + \alpha_z\frac{d^2q_z}{dx^2}\right) \end{aligned} \tag{13}$$

provided that the span loads $q_z(x)$, $q_y(x)$ are smooth. In our case $q_y = 0$, $q_z = 0$, hence we see that both the deflections are expressed by polynomials of degree 3, while the angles of rotation $\beta$, $\varphi$ are expressed by polynomials of degree 2. Let us introduce the non-dimensional parameters

$$\varkappa_y = \frac{12EJ_y}{l^2GA}, \quad \varkappa_z = \frac{12EJ_z}{l^2GA} \tag{14}$$

Let $(\alpha_1, \alpha_2)$ and $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ are solutions to the systems:

$$\begin{bmatrix} k_y + \varkappa_z & k_{yz} \\ k_{yz} & k_z + \varkappa_y \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \varkappa_y \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} k_z + \varkappa_y & k_{yz} \\ k_{yz} & k_y + \varkappa_z \end{bmatrix} \begin{bmatrix} \widetilde{\alpha}_1 \\ \widetilde{\alpha}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \varkappa_z \end{bmatrix} \tag{16}$$

Let us introduce the polynomials

$$\begin{aligned} &a(\xi) = 1 - 3\xi^2 + 2\xi^3, \quad b(\xi) = \xi - 3\xi^2 + 2\xi^3, \quad c(\xi) = \xi - 2\xi^2 + \xi^3, \\ &d(\xi) = 6\xi - 6\xi^2, \quad e(\xi) = 1 - 4\xi + 3\xi^2, \end{aligned} \tag{17}$$

their plots for $0 \leq \xi \leq 1$ being presented by Fig. 2.

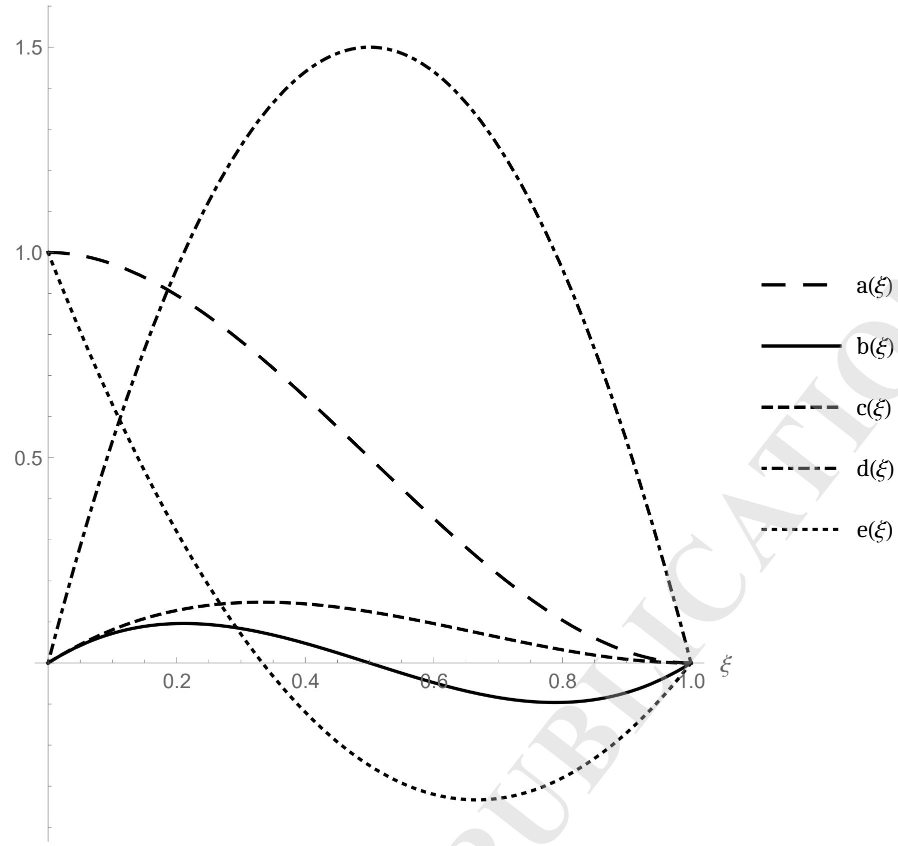



**Fig. 2**: Plots of the polynomials $a$, $b$, $c$, $d$, $e$

The factorized forms of these polynomials read

$$\begin{aligned} &a(\xi) = (1-\xi)^2(1+2\xi), \quad b(\xi) = \xi(1-\xi)(1-2\xi), \quad c(\xi) = \xi(1-\xi)^2, \\ &d(\xi) = 6\xi(1-\xi), \quad e(\xi) = (1-\xi)(1-3\xi) \end{aligned} \tag{18}$$

Let us introduce the angles of slopes in the $x-z$ and $x-y$ planes

$$\psi = \frac{1}{l}\left(w^* - {}^*w\right), \quad \chi = \frac{1}{l}\left(v^* - {}^*v\right) \tag{19}$$

as well as the mean values of the angles of rotation of both the ends of the bar

$$\beta_o = \frac{1}{2}\left({}^*\beta + \beta^*\right), \quad \varphi_o = \frac{1}{2}\left({}^*\varphi + \varphi^*\right) \tag{20}$$

According to the Bernoulli-Euler theory (i.e. the theory of thin bars) the deflection functions and the functions representing variation of the angles of rotations of the bar's cross-sections are expressed by the formulae:

$$
\begin{aligned}
\hat{w}(x) &= l\left[\frac{{}^*w}{l}a(\xi) + \frac{w^*}{l}a(1-\xi) - {}^*\beta c(\xi) + \beta^* c(1-\xi)\right] \\
\hat{v}(x) &= l\left[\frac{{}^*v}{l}a(\xi) + \frac{v^*}{l}a(1-\xi) - {}^*\varphi c(\xi) + \varphi^* c(1-\xi)\right] \\
\hat{\beta}(x) &= {}^*\beta e(\xi) + \beta^* e(1-\xi) - \psi d(\xi) \\
\hat{\varphi}(x) &= {}^*\varphi e(\xi) + \varphi^* e(1-\xi) - \chi d(\xi)
\end{aligned}
\tag{21}
$$

where

$$
a(1-\xi) = 3\xi^2 - 2\xi^3, \quad c(1-\xi) = \xi^2 - \xi^3, \quad e(1-\xi) = -2\xi + 3\xi^2 \tag{22}
$$

Thus, the angles of rotation are linked with the deflection functions by

$$
\hat{\beta} = -\frac{d\hat{w}}{dx}, \quad \hat{\varphi} = -\frac{d\hat{v}}{dx} \tag{23}
$$

since the theory of thin bars imposes constraints on the angles of rotation of cross-sections assuring the zero values of the transverse shear deformations in both the planes: $x-z$, $x-y$.

The deflection functions and the functions representing the variation of the angles of rotation within the six-parameter theory of bars differ from the above solutions by terms involving the quantities $\psi + \beta_o$, $\chi + \varphi_o$, namely

$$
\begin{aligned}
w(x) &= \hat{w}(\xi) + l\left[\alpha_2(\psi + \beta_o) + \widetilde{\alpha}_1(\chi + \varphi_o)\right] b(\xi), \\
\beta(x) &= \hat{\beta}(\xi) + \left[\alpha_2(\psi + \beta_o) + \widetilde{\alpha}_1(\chi + \varphi_o)\right] d(\xi) \\
v(x) &= \hat{v}(\xi) + l\left[\alpha_1(\psi + \beta_o) + \widetilde{\alpha}_2(\chi + \varphi_o)\right] b(\xi), \\
\varphi(x) &= \hat{\varphi}(\xi) + \left[\alpha_1(\psi + \beta_o) + \widetilde{\alpha}_2(\chi + \varphi_o)\right] d(\xi)
\end{aligned}
\tag{24}
$$

Note that the quantities $\psi+\beta_o$, $\chi+\varphi_o$ are discrete deformation measures of the *natural approach* by Argyris, see comments in Petrolo and Casciaro [3].

Since $T_y = \text{const}$ and $T_z = \text{const}$ the deformation measures $\gamma_y = \gamma_y^o$, $\gamma_z = \gamma_z^o$ are constant and equal

$$
\begin{aligned}
\gamma_y^o &= \alpha_1(\psi + \beta_0) + \widetilde{\alpha}_2(\chi + \varphi_0) \\
\gamma_z^o &= \alpha_2(\psi + \beta_0) + \widetilde{\alpha}_1(\chi + \varphi_0)
\end{aligned}
\tag{25}
$$

The shear deformation is expressed by the functions

$$\begin{aligned} w_T(x) &= l\gamma_z^o b\left(\frac{x}{l}\right), \\ \beta_T(x) &= \gamma_z^o d\left(\frac{x}{l}\right), \\ v_T(x) &= l\gamma_y^o b\left(\frac{x}{l}\right), \\ \varphi_T(x) &= \gamma_y^o d\left(\frac{x}{l}\right) \end{aligned} \tag{26}$$

Thus, the final deformation (24) is also expressed by superposition as below

$$\begin{aligned} w(x) &= \hat{w}(x) + w_T(x), \\ \beta(x) &= \hat{\beta}(x) + \beta_T(x), \\ v(x) &= \hat{v}(x) + v_T(x), \\ \varphi(x) &= \hat{\varphi}(x) + \varphi_T(x) \end{aligned} \tag{27}$$

The shear load is skew–symmetric with respect to the plane $x = l/2$, hence the functions $w_T(x)$, $v_T(x)$ are skew–symmetric, while the functions $\beta_T(x)$, $\varphi_T(x)$ are symmetric with respect to $x = l/2$. The extremal values of the latter functions are attained in the middle of the bar and read

$$\beta_T\left(\frac{l}{2}\right) = \frac{3}{2}\gamma_z^o, \quad \varphi_T\left(\frac{l}{2}\right) = \frac{3}{2}\gamma_y^o \tag{28}$$

Let us conclude that the final deformation is a sum of the bending deformation predicted by the thin bar theory and the shear deformation added by the six-parameter theory.

The bar theoretical results (27) determine the shapes of deformation of the bar viewed as a 3D body; in particular one can predict the transverse deformation of the cross-sections $x = \text{const}$; it is given by the function $u_x(x, y, z)$ which represents the displacement along the $x$ axis of the point $(x, y, z)$. According to El-Fatmi's kinematical hypothesis this function has the form, see [4]:

$$\begin{aligned} u_x =& u(x) + y\varphi(x) + z\beta(x) + \omega(y,z)\rho(x) + \\ &+ [\eta(y,z) - y]\gamma_y(x) + [\zeta(y,z) - z]\gamma_z(x) \end{aligned} \tag{29}$$

where $\rho$ is the measure of torsion, $\omega(y, z)$ is the warping function due to torsion, $\eta(y, z)$, $\zeta(y, z)$ characterize the warping due to transverse shear in the $x - y$ and $x - z$ planes. In the problem discussed here the axial deformation and torsion are not present (which is a justified assumption due to the decoupling phenomena of the discussed theory).

Since

$$\begin{aligned} &\int_{\mathcal{A}} y(\eta - y)d\mathcal{A} = 0, \quad \int_{\mathcal{A}} z(\zeta - z)d\mathcal{A} = 0 \\ &\int_{\mathcal{A}} z(\eta - y)d\mathcal{A} = 0, \quad \int_{\mathcal{A}} y(\zeta - z)d\mathcal{A} = 0 \end{aligned} \tag{30}$$

the formulae (2) hold; consequently the last two terms in (29) do not affect the angles of rotation $\beta$, $\varphi$ understood as the averaged quantities, see Eqs (2). In the figures below, for simplicity, we shall not show the deformations generated by these quantities; the deformation of the orthogonal net of lines along $x$ and $z$ axes will be approximated by: $u_x = y\varphi(x) + z\beta(x)$, $u_z = w(x)$, or within the assumption of planar cross-sections.

For the illustration of the deformation state of the bar assume that

$$w(0) > 0, \quad w(l) > w(0), \quad \frac{dw}{dx}(0) > 0, \quad \frac{dw}{dx}(l) > 0, \quad \gamma_z^o > 0$$

Before the deformation the net $x = \text{const}$, $z = \text{const}$ is orthogonal, see Fig.3a. The fields $\hat{w}$, $\hat{\beta}$ determine the bending deformation preserving the orthogonality property, see Fig.3b. The transverse end forces act in a skew–symmetric manner and generate the deflection $w_T(x)$ characterizing the shear deformation: the angles of rotation of tangents to the neutral line are equal $\gamma_z^o$ at both the ends and equal $(-1/2)\gamma_z^o$ in the middle of the bar, see Fig.3c. Here the angles of rotation of the transverse cross-sections vanish. The angles of rotation generated by the shear forces are expressed by the function $\beta_T(x)$; this function vanishes at both the ends and is extremal in the middle of the bar, see Fig.3d; the corresponding deflection is zero.

The kinematic loads in the plane $x - z$ generate, in general, a deformation in the plane $x - y$. This phenomenon will be discussed below by considering two kinds of kinematic loads.

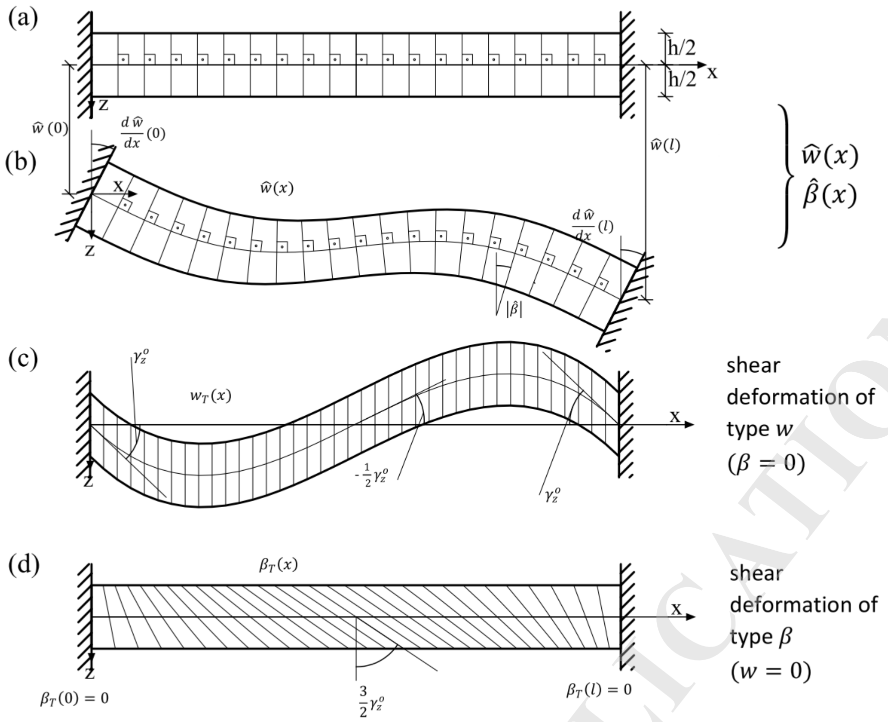



**Fig. 3**: The orthogonal net in the plane $x - z$ (a); the bending deformation according to the thin bar theory (b); the deformation due to shear for which $\beta = 0$ (c); the deformation due to shear for which $w = 0$ (d)

***Case 1***

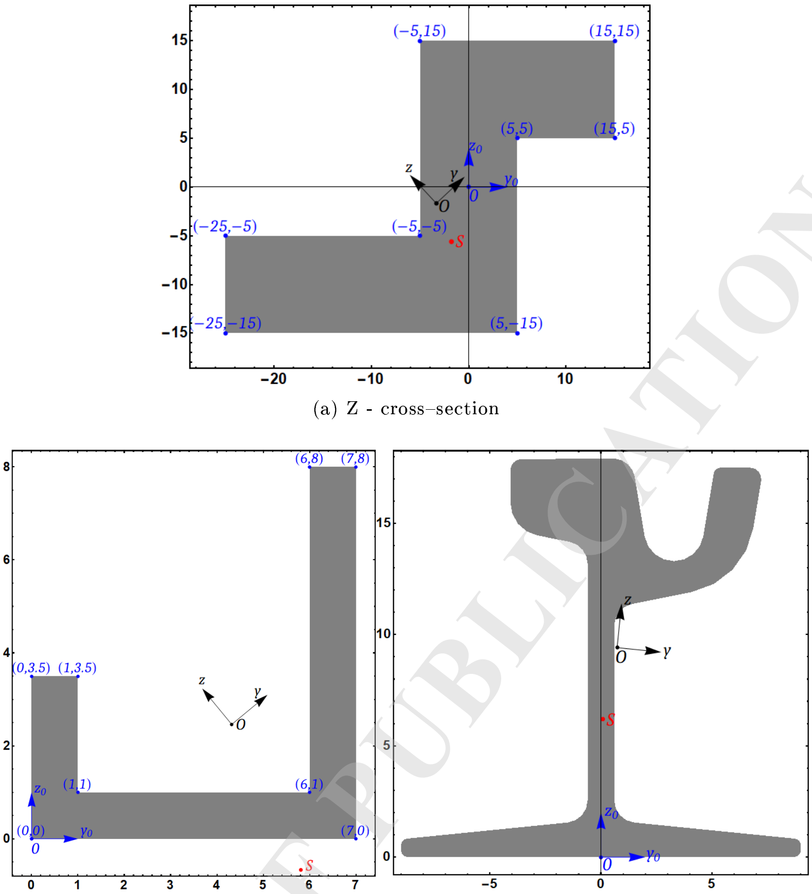
Consider the kinematic load:

$$^*w = 1, \quad w^* = {}^*v = v^* = 0, \quad {}^*\beta = \beta^* = {}^*\varphi = \varphi^* = 0 \tag{31}$$

The deformation of the bar is given by

$$\begin{aligned} w(x) &= (1 - 3\xi^2 + 2\xi^3)\underline{-\alpha_2(\xi - 3\xi^2 + 2\xi^3)} \\ \beta(x) &= \frac{1}{l}(1 - \alpha_2)(6\xi - 6\xi^2) \\ v(x) &= -\alpha_1(\xi - 3\xi^2 + 2\xi^3) \\ \varphi(x) &= -\frac{1}{l}\alpha_1(6\xi - 6\xi^2), \quad \xi = x/l \end{aligned} \tag{32}$$

To be specific let us fix the domains $\mathcal{A}$ as Z -shape, U -shape and RI60 rail denoted as A1, A2, A3, see Figs.4a, 4b, 4c. Their characteristics are set up in Table 1. Assume that: the length of the bar equals $l = 1$m, case (a), $l = 2$m case (b); $E = 210$GPa, $G = 81$GPa.



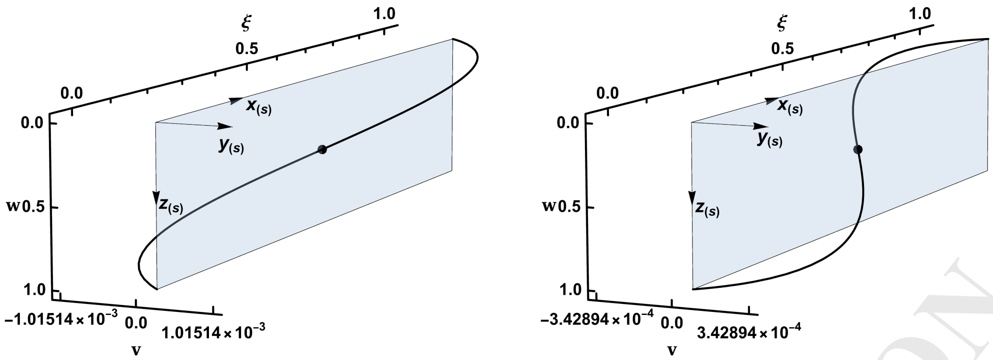

(a) Z - cross–section

(b) U - cross–section

(c) RI60 rail cross–section

**Fig. 4**: The chosen cross-sections of the bar, called A1 (a), A2 (b), A3(c)

The shape of the line $x_{(s)}$ upon deformation, determined by the formulae (32), is spatial, see Fig.5; the points $(x, y_S, z_S)$ displace in two directions: $z$ (let us call it transverse) and $y$ (let us call it lateral).




(a) for Z - cross–section (b) for U - cross–section

**Fig. 5**: The deflections caused by the kinematic load in Case 1 for the length of the bar 1m and the cross sections A1 (a), A2 (b)

**Table 1**: Characteristics of the cross-sections shown in Fig.4.

| | $A$ $[cm^2]$ | $J_y$ $[cm^4]$ | $J_z$ $[cm^4]$ | $J$ $[cm^4]$ | $y_{0(O)}$ $[cm]$ | $z_{0(O)}$ $[cm]$ | $y_{0(S)}$ $[cm]$ | $z_{0(S)}$ $[cm]$ | $\measuredangle(y_0, y)$ $[°]$ | $k_y$ $[-]$ | $k_{yz}{=}k_{zy}$ $[-]$ | $k_z$ $[-]$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **A1** | 600 | 19082 | 92585 | 19338 | -3.333 | -1.667 | -1.812 | -5.598 | 43.0 | 0.555 | -0.075 | 0.587 |
| **A2** | 16.5 | 50 | 140.43 | 5.438 | 4.318 | 2.462 | 5.815 | -0.675 | 39.5 | 0.351 | 0.0669 | 0.471 |
| **A3** | 77.1 | 3367.3 | 902.05 | 129.67 | 0.727 | 9.415 | 0.0792 | 6.189 | -5.83 | 0.491 | 0.0027 | 0.288 |

**Table 2**: Characteristics of the bars of the cross-sections shown in Fig.4 of length equal 1m (Ana) and length equal 2m (Anb), $n = 1, 2, 3$

| | $\varkappa_y$ | $\varkappa_z$ | $\alpha_1$ | $\alpha_2$ | $\widetilde{\alpha}_1$ | $\widetilde{\alpha}_2$ |
|---|---|---|---|---|---|---|
| **A1a** | 0.09894 | 0.4801 | 0.01055 | 0.1455 | 0.05119 | 0.4674 |
| **A1b** | 0.02474 | 0.1200 | 0.004561 | 0.04102 | 0.02213 | 0.1802 |
| **A2a** | 0.009428 | 0.02648 | -0.003563 | 0.02013 | -0.01001 | 0.07186 |
| **A2b** | 0.002357 | 0.006620 | -0.0009561 | 0.005116 | -0.002685 | 0.01900 |
| **A3a** | 0.1359 | 0.03640 | -0.001674 | 0.3209 | -0.0004484 | 0.06907 |
| **A3b** | 0.03397 | 0.009100 | -0.0005812 | 0.1056 | -0.0001557 | 0.01821 |

The term underlined in $(32)_1$ denoted here by $w_\alpha(x)$ represents the deflection due to transverse shear. The extremal values of this term are attained for $x = ((3-\sqrt{3})/6)l$ and $x = ((3+\sqrt{3})/6)l$ and equal to $-(\sqrt{3}\alpha_2)/18$, $(\sqrt{3}\alpha_2)/18$ respectively. The lateral deflection $v(x)$ is skew–symmetric with respect to $x = l/2$.

**Table 3**: Maximal absolute value of horizontal displacement $v$ and maximal absolute value of $w_\alpha$ for the analyzed bars.

| | **A1a** | **A1b** | **A2a** | **A2b** | **A3a** | **A3b** |
|---|---|---|---|---|---|---|
| $10^3 \max\|v\|$ | 1.015 | 0.4389 | 0.343 | 0.0920 | 0.1611 | 0.0559 |
| $10^3 \max\|w_\alpha\|$ | 14.00 | 3.948 | 1.937 | 0.4923 | 30.88 | 10.17 |

***Case 2***

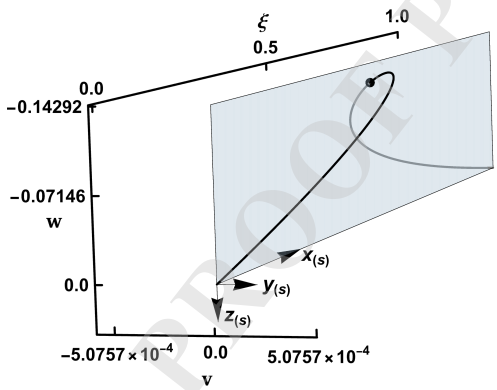
We shall consider the state of deformation caused by the kinematic load:

$$^*w = 0, \quad w^* = 0, \quad ^*v = v^* = 0, \quad ^*\beta = 1, \quad \beta^* = 0, \quad ^*\varphi = \varphi^* = 0 \tag{33}$$

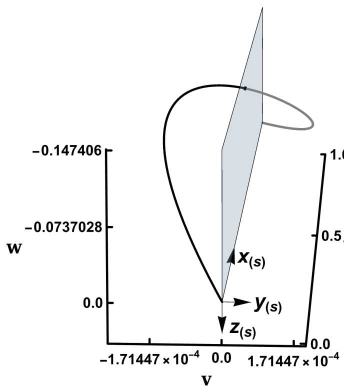
The deformation of the bar is determined by

$$\begin{aligned} w(x) &= -l(\xi - 2\xi^2 + \xi^3) + \frac{l}{2}\alpha_2(\xi - 3\xi^2 + 2\xi^3) \\ \beta(x) &= (1 - 4\xi + 3\xi^2) + \frac{1}{2}\alpha_2(6\xi - 6\xi^2) \\ v(x) &= \frac{l}{2}\alpha_1(\xi - 3\xi^2 + 2\xi^3) \\ \varphi(x) &= \frac{1}{2}\alpha_1(6\xi - 6\xi^2), \quad \xi = x/l \end{aligned} \tag{34}$$

The deformation is spatial, a lateral deflection $v(x)$ appears, it is skew–symmetric with respect to $x = l/2$, see Fig.6.




(a) for Z - cross–section




(b) for U - cross–section

**Fig. 6**: The deflections caused by the kinematic load in Case 2 for the length of the bar 1m and the cross sections A1 (a), A2 (b)

## 5 Deformations caused by span loads

### 5.1 The bar clamped at both the ends, subjected to the transverse load $q_z = \text{const}$

The load $q_z = q = \text{const}$ acts along the axis $x_{(s)}$ linking the shear centers. The bar is clamped at both the ends, i.e.

$$\begin{aligned} &w(0) = 0, \quad w(l) = 0, \quad v(0) = 0, \quad v(l) = 0, \\ &\beta(0) = 0, \quad \beta(l) = 0, \quad \varphi(0) = 0, \quad \varphi(l) = 0. \end{aligned} \tag{35}$$

The solution to the problem ($\mathcal{P}_3$) has the form

$$\begin{aligned} w &= \frac{ql^4}{24EJ_y}\xi^2(1-\xi)^2 + \alpha_z \frac{ql^2}{2GA}\xi(1-\xi), \\ \beta &= -\frac{ql^3}{12EJ_y}(\xi - 3\xi^2 + 2\xi^3), \\ v &= \alpha_{yz}\frac{ql^2}{2GA}\xi(1-\xi), \\ \varphi &= 0. \end{aligned} \tag{36}$$

Let us note that $\beta_T = 0$, $\hat{v} = 0$, $\hat{\varphi} = 0$, $\varphi_T = 0$. The deformation is spatial, there appears a lateral deflection $v(x)$ in the shape of a parabola. Its extremal value equals $\alpha_{yz}ql^2/(8GA)$ and appears at $x = l/2$. The distribution of the internal forces is the same as in the thin bar theory and is given by

$$\begin{aligned} &M_y = -\frac{ql^2}{12}(1 - 6\xi + 6\xi^2), \\ &T_z = ql(\frac{1}{2} - \xi), \\ &M_z = 0, T_y = 0. \end{aligned} \tag{37}$$

### 5.2 A bar simply supported in two directions at both the ends, subjected to the transverse load $q_z = \text{const}$

The load $q_z = q = \text{const}$ acts along the axis $x_{(s)}$ linking the shear centers. The bar is simply supported in two directions at both the ends, i.e.

$$\begin{aligned} &w(0) = 0, \quad w(l) = 0, \quad M_y(0) = 0, \quad M_y(l) = 0, \\ &v(0) = 0, \quad v(l) = 0, \quad M_z(0) = 0, \quad M_z(l) = 0. \end{aligned} \tag{38}$$

The solution to the problem ($\mathcal{P}_3$) has the form

$$\begin{aligned}
w &= \frac{ql^4}{24EJ_y}\xi(1-\xi)(1+\xi-\xi^2) + \alpha_z\frac{ql^2}{2GA}\xi(1-\xi),\\
\beta &= -\frac{ql^3}{24EJ_y}(1-6\xi^2+4\xi^3),\\
v &= \alpha_{zy}\frac{ql^2}{2GA}\xi(1-\xi),\\
\varphi &= 0.
\end{aligned} \tag{39}$$

We see that the lateral deflection is the same as in the previous example. The distribution of the internal forces is the same as in the thin bar theory

$$\begin{aligned}
M_y &= \frac{1}{2}ql^2\xi(1-\xi),\\
T_z &= ql(\frac{1}{2}-\xi),\\
M_z &= 0, T_y = 0.
\end{aligned} \tag{40}$$

### 5.3 A cantilever under a point load $P$ at its end

The boundary conditions read

$$\begin{aligned}
&w(0) = 0, \quad v(0) = 0, \quad \varphi(0) = 0, \quad \beta(0) = 0,\\
&T_y(l) = 0, \quad T_z(l) = P, \quad M_y(l) = 0, \quad M_z(l) = 0.
\end{aligned} \tag{41}$$

The solution to the problem ($\mathcal{P}_3$) has the form

$$\begin{aligned}
w &= \frac{Pl^3}{6EJ_y}\xi^2(3-\xi) + \alpha_z\frac{Pl}{GA}\xi,\\
\beta &= -\frac{Pl^2}{2EJ_y}\xi(2-\xi),\\
v &= \alpha_{yz}\frac{Pl}{GA}\xi,\\
\varphi &= 0.
\end{aligned} \tag{42}$$

There appears a lateral deflection in the shape of a straight line; the extremal value of this deflection is $\alpha_{yz}Pl/(GA)$.

The distribution of the internal forces are the same as in the thin bar theory, i.e.

$$M_y = -Pl(1-\xi), \quad M_z = 0, \quad T_z = P, \quad T_y = 0. \tag{43}$$

### 5.4 Pure bending

Consider a bar simply supported in two directions at both the ends, loaded by two end moments in opposite directions at both the ends. The boundary conditions read

$$\begin{aligned} &w(0)=0, \quad w(l)=0, \quad v(0)=0, \quad v(l)=0, \\ &M_y(0)=M, \quad M_y(l)=M, \quad M_z(0)=0, \quad M_z(l)=0. \end{aligned} \tag{44}$$

The solution to the problem ($\mathcal{P}_3$) has the form

$$\begin{aligned} &w=\frac{Ml^2}{2EJ_y}\xi(1-\xi), \\ &\beta=-\frac{Ml}{2EJ_y}(1-2\xi), \\ &v=0, \\ &\varphi=0 \\ &M_y=M, \quad T_z=0, \quad M_z=0, \quad T_y=0. \end{aligned} \tag{45}$$

and coincides with that predicted by the thin bar theory.

## 6 Final remarks

The present paper discusses the predictions of the six–parameter theory of bars of arbitrary cross-sections proposed in Sec. 10 of [4]). The presence of the coupling terms in the constitutive equations linking the transverse forces with the transverse shear measures induces asymmetry of the solutions and effects of deplanation of the deformation states. This is clearly seen in the form of the shape functions corresponding to the kinematic loads. The response to the static load is not planar in general. There is only one problem in which the deplanation is absent: the problem of pure bending.

If the plane $x = l/2$ is the section of symmetry of the static problem, the constant load acting in the $z$ direction applied along the $x_{(s)}$ axis generates a parabolic deflection in the $y$ direction, but does not cause the angles of rotation $\varphi$ along the $(-z)$ axis, while the state of stress resultants is kept planar, i.e. $M_z = 0, T_y = 0$. Moreover, the angles of rotation $\beta$ in $y$ direction coincide with those predicted by the thin bar theory, i.e. are not affected by the transverse shear.

If the static problem is asymmetric with respect to the plane $x = l/2$, then the constant load along the $z$ direction induces all stress resultants and the deflections along $y$ direction are given by polynomials of degree 3. For instance, in the bar subjected to the load $q_z$ which is clamped at the left end and simply supported in both $y$ and $z$ directions at the right end there appear there two non-zero reactions in $y$ and $z$ directions at the right end thus inducing the stress resultants in both the planes. Moreover, note that the point loads applied to cantilevers in $z$ direction cause lateral deflections of straight shape. Thus, the lateral deflection $v(x)$ of the cantilever loaded in $z$ direction by several point loads wil assume the shape of a zigzag line. The lateral deflection is caused by the transverse forces $T_z$ which bring about the shear deformations $\gamma_y$.

Let us stress that within the standard six–parameter theory of bars, like in Petrolo and Casciaro [3], the load along the $z$ direction will, in general, generate torsion. Moreover, the kinematic loads will cause a torsional deformation. This shows that the 6-parameter theory discussed in the presented paper makes the static problems as simple as possible.

We conclude that, in general, neither of the six-parameter theories admit planar bending/shearing states of deformation. This phenomenon is of vital importance while analyzing stability of bending states, since the standard description of the lateral buckling phenomenon assumes an ideally planar initial state of bending.

Let us stress here that this problem of occurring spatial deformation states is not observed in the theory by Vlasov (and in the Vlasov-like theory developed in [4]) since then the transverse shear deformation is neglected and all the constitutive equations may be decoupled by appropriate choice of the free parameters of this theory, like the position of the pole. Thus, the standard theory of lateral buckling works there correctly, as the initial planar bending states are in general admissible. This shows how difficult is to extend the old results concerning Vlasov thin bars to moderately thick bars. As usually, a direct extension of the known predictions is impossible.

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