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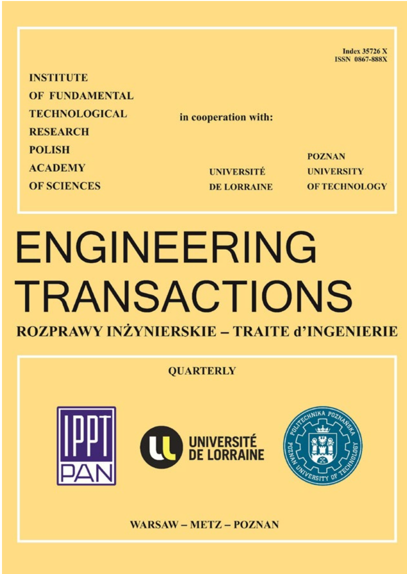

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Torsion of Functionally Graded Material Structures: An Overview

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Abstract

Examining torsion in functionally graded materials (FGMs) is crucial because their properties vary spatially. FGMs with continuously graded architectures provide a robust basis for investigating mechanical behavior. Current understanding of torsional response draws on analytical, numerical, and experimental approaches. This review synthesizes how material gradation influences stress distribution, stiffness, and failure modes, and compares advances in FGM torsion across diverse models and geometries. The theoretical background is framed by classical torsion theories, including Saint-Venant, Prandtl's membrane analogy, and Vlasov formulations. We further discuss modeling with isoparametric finite elements and summarize established homogenization schemes for FGMs. A tabulated overview of torsion-related results is also provided. The novelty of this review lies in its exclusive focus on torsion in FGMs, the systematic tabulation of prior contributions, and a coherent exposition of homogenization models and torsion theories tailored to FGM structures. To our knowledge, this is among the first reviews to focus specifically on torsion of FGM structures, distinguishing it from prior overviews that address torsion only briefly. Methodologically, we conduct a structured scoping review that screens peer-reviewed sources, classifies studies by geometry, torsion theory, homogenization scheme, and numerical strategy, and synthesizes observed trends. Finally, we present concise conclusions and future research directions. This review covers analytical, numerical, and experimental studies of torsion in FGMs, identified via a structured Google Scholar search and prioritized by citation impact and relevance.

Keywords: torsion; FGM structures; torsional stiffness; overview.

1 Introduction

The concept of functionally graded materials (FGMs) was developed in Japan in the mid-1980s [1]. The established idea of materials providing a high through-thickness thermal barrier was crucial for space shuttle construction; Japanese engineers and scientists proposed functionally graded variations in thermal coefficients [2]. Since then, the FGM concept has advanced steadily. The literature on bending, tension, and compression is relatively extensive, whereas torsion remains comparatively underexplored. The spatial variation of graded structure in such

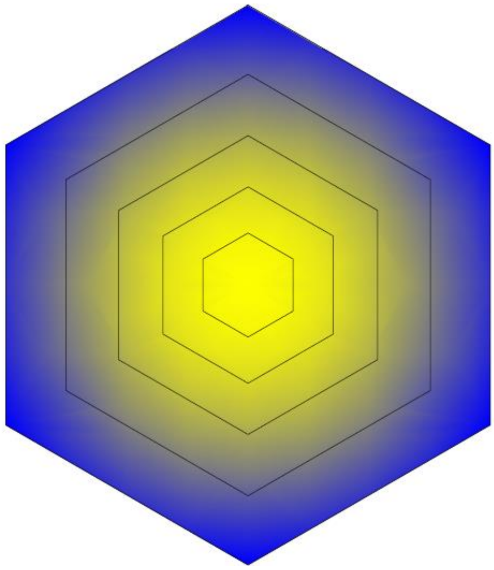

Fig. 1. Example of FGM structure.

materials underpins their usefulness in many automotive, aerospace, and biomedical applications. See Fig. 1, which presents an example of property gradation in the x- and y- directions.

One of the most important aspects is the behavior of such materials under torsional loads. A thorough exploration of the FGM torsion problem may yield substantial improvements and enable greater use of FGM layers in the design of parts that transmit significant twisting moments. These may include structural elements such as support beams with various cross-sectional areas, machine shafts, or even aircraft wings. Furthermore, the development of precise numerical torsion models could reduce design costs and time, thereby encouraging broader adoption of FGM structures in place of conventional materials. Therefore, a substantial task is to consider existing torsion theories and reflect on their potential improvements.

Valuable issues related to torsion in FGM structures have been presented in many articles. An analysis of the available literature shows that most torsion cases are approximated as linear elastic composites, often treated as isotropic models. Many of these works are based on anisotropic elasticity models developed by Lekhnitskii [3]. Horgan and Chan [4] investigated the influence of material inhomogeneity on the torsional behavior of linear elastic isotropic rods. They extended the works of Rooney et al. [5] and Lekhnitskii [3] by formulating the shear modulus as a function of the cross-sectional position. Batra [6] solved the torsion problem for an FGM cylinder in compressible and incompressible linear elastic materials with spatially varying moduli only in the axial direction. Arghavan and Hematiyan [7] formulated numerical models of FGM hollow tubes with arbitrary non-circular shapes. In another study, Horgan [8] extended the notation introduced by Chen and Wai [9], deriving unified formulas for the absence of warping effects in rods with elliptical cross sections. Barretta and Luciano [10] demonstrated a novel analogy between the Kirchhoff plate problem and the Saint-Venant torsion problem.

In recent years, numerous papers have examined the influence of torsion on FGM nanotubes and nanobars. Li and Hu [11] analyzed the behavior of 2D FGM microtubes under torsion using the modified couple-stress theory. Barretta et al. [12] investigated the torsion of FGM nanobeams based on Eringen's nonlocal elasticity. Moreover, many recent studies employ the Saint-Venant torsion theory, as evidenced by works of Omid and Lashkarbolok [13] and Kutlu et al. [14]. The number of studies on beams with circular and square cross sections is substantial, whereas cases involving shafts with triangular, regular polygonal, and other non-circular cross sections are less frequent. The results of Akinlabi et al. [15] on torsion in triangular cross sections indicate clear room for expansion of this topic.

Furthermore, many investigations rely on finite element methods. Ganczarski, Szubartowski, and Kumor [16] offered a different perspective by solving torsion for Al–Ti FGM noncircular shafts using the finite difference method. This work highlights significant potential for future research employing methods other than the finite element method. In addition to aerospace and automotive applications, the torsion and shear-stress behavior of FGM structures is also relevant to the medical and energy sectors. Consequently, research in this area can support the design of innovative components across these industries.

Considering the current advances in the torsional behavior of FG materials, we would like to highlight several of the most important findings from studies conducted in recent years. The work of Hao at al.[100] analyzes bursting oscillations arising from bending–torsion coupling in cantilevered FGM conical sandwich panels driven by a static preload and slow in-plane harmonic forcing, and demonstrates with a nonautonomous, temperature-graded model that the onset is governed by symmetry-breaking pitchfork bifurcations. In turn, subsequent studies address the torsional behavior of nanotubes, nanorods, and microtubes. Using modified couple stress theory with radial, axial gradation, the [101] derives, numerically solves, and validates torsion equations for bi-directional FG microtubes, quantifying how phase profile and geometry control twist and shear under distributed torque.

In turn, Civalek at al.[102] present an exact nonlocal-elasticity solution for the torsional free vibration of restrained FGM nanotubes—modeling end restraints with torsional springs, deriving a characteristic matrix for natural frequencies, validating against prior results, and quantifying the effects of the FG index and length scale. Shakhlaviet et al. [103] study von Kármán nonlinear torsional vibrations of FGM carbon nanotubes via nonlocal elasticity, deriving Hamiltonian equations, computing clamped–clamped free natural frequencies with multiple scales, and quantifying FG index, size, amplitude, and mode effects for design. Another work by Beni [104] examines the size-dependent, coupled electromechanical torsional behavior of porous, functionally graded flexoelectric micro, nanotubes. Barati and Norouzi [105] presents a nonlocal model for the static torsion of bi-directional FG microtubes under a longitudinal magnetic field—deriving the governing equation via minimum potential energy, validating GDQ against a Galerkin solution, and showing that torsional angle depends on the nonlocal parameter. Finally, Zarezadeh et al. [106] develop a nonlocal elasticity model for an FG nanorod on a torsional foundation under an axial magnetic field deriving the Navier equations and Hamilton's principle, solving them with GDQM, and showing that size effects through the nonlocal length scale soften the response and reduce the natural frequencies.

Taking the above into consideration, issues related to the torsion problem, existing torsion theories, and methods of torsion modeling will be discussed and compared, and a critical perspective on the topic will be presented.

2. FGM Torsion Problem

Torsional behavior of FGM structures is crucial for understanding and for designing proper and effective elements. The use of FG materials can provide comprehensiveness and a more favorable stress gradient, which may result in better-designed components. Torsion of highly anisotropic or orthotropic materials differs markedly from that of isotropic structures. Nonhomogeneity, and thus variation of properties in all directions, makes the modeling process significantly more difficult. Another important problem is the definition of the graded composition. The most popular homogenization methods are based on linear approximations of the modulus of elasticity and the Poisson ratio. For torsion of FG materials, it is possible to obtain the Kirchhoff modulus through homogenization, as shown by Reuss, Voigt, Hashin–Strikman, and Mori–Tanaka in their works. Isotropic torsion is much easier to analyze. The distribution of shear stresses is relatively straightforward for shafts with isotropic and homogeneous microstructure. For a circular shaft, shear stresses are distributed linearly across the cross section and are an increasing function of the radius. This is not obvious for twisting shafts with a graded structure, since differences in elastic moduli, and consequently various Kirchhoff moduli, lead to a significantly different distribution of shear stresses across the section, see Fig. 2.

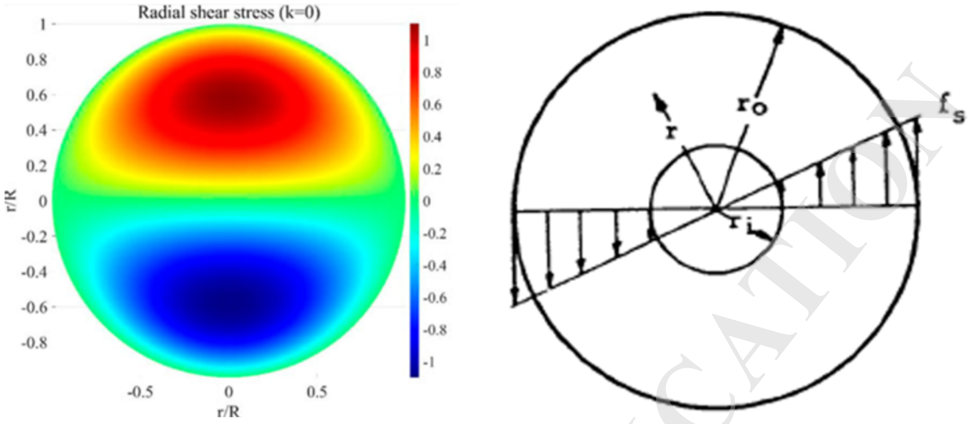

Fig. 2. Shear stress distribution in isotropic round shaft and FGM shaft with metal core and ceramic inner surface, after Duan et al. [17].

The next issue of concern is the angle of twist. For an isotropic shaft with uniform torsional stiffness, the angle of twist is identical at every point on the surface of a circular bar. By contrast, for a functionally graded structure—even under linear homogenization the torsional stiffness of each layer differs, which in turn affects the total angle of twist of a shaft subjected to a torque M, see Fig. 3.

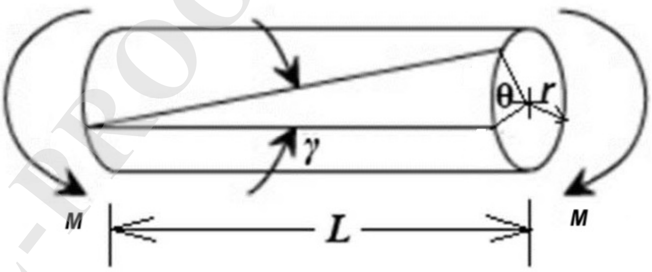

Fig. 3. Angle of twist in isotropic shaft.

Therefore, when a round shaft is considered, the distribution of shear stresses and the angle of twist present a substantial challenge. An even greater problem arises for noncircular cross sections such as rectangular, elliptical, polygonal, or asymmetrical shapes, where, even with isotropic and homogeneous materials, obtaining accurate calculations and distributions of shear stresses and angles of twist requires multiple approximations and experimental methods. The first assumptions regarding torsion are based on Prandtl's membrane analogy. The membrane analogy is used to visualize the Prandtl stress function for any contour of a twisted shaft's cross section. The values of the Prandtl function at specific points within a cross section that follows a defined contour are related to the distance from this cross section to a membrane surface. This membrane is stretched across the contour and subjected to a uniform pressure acting perpendicular to the cross section. The Prandtl membrane analogy plays a significant role in describing the torsion of FGM structures. The torsion equation is derived from the stress in a thin membrane due to the applied pressure p, which is always perpendicular to the membrane surface, see Fig. 4.

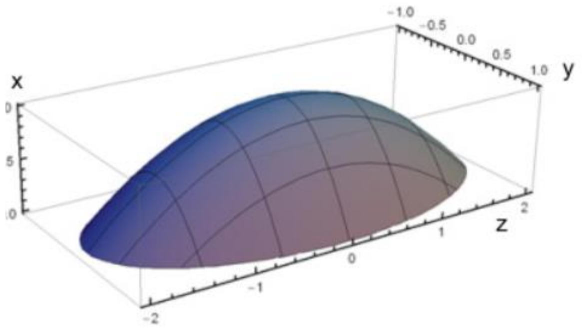

Fig. 4. Prandtl's membrane analogy for elliptic isotropic cross section, after Gil-Martin et al. [18].

The analysis of an isotropic material is based on a Poisson type partial differential equation that describes the torsional behavior of a shaft, thereby relating it to the membrane stress T. The following formula applies:

$$\nabla^2 \omega = -\frac{p}{T} \tag{1}$$

where p is the distributed pressure across the membrane, the analog of the torsional moment, and T is the equivalent torsional stiffness.

In the case of FG materials, the mathematical formulation is more complicated, since the material properties such as Young's modulus E, Poisson's ratio, and consequently the Kirchhoff modulus G must depend on the functions $p(y,z)$ and $T(y,z)$, which describe the change in material properties along the directions of gradation, according to the following formula:

$$\nabla^2 \omega = -\frac{p(y,z)}{T(y,z)} \tag{2}$$

where $p(y,z)$ and $T(y,z)$ are the functions describing local material properties.

The application of the above formulation allows the changes in mechanical properties to depend on the directions of length (x) and width (y). This approach is crucial for tailoring material behavior under mechanical stresses to specific directional requirements, thereby enhancing the design and functionality of advanced material systems such as functionally graded materials, as shown in Fig. 5.

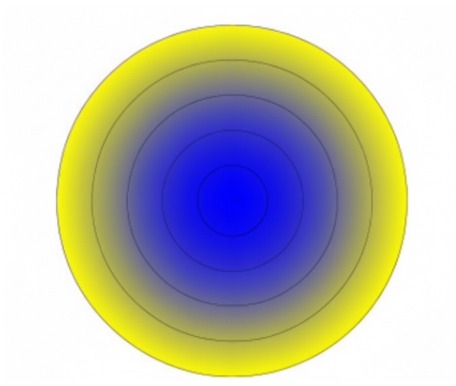

Fig. 5. Material gradation in circular cross section.

Numerous analogies for various functionally graded materials can be found, among others, in the work of Barretta and Luciano [10], who established a new analogy between the orthotropic FGM Saint-Venant beam and the Kirchhoff plate. These studies demonstrated the aforementioned relationship and expanded upon earlier assumptions by Timoshenko [19], Irschik [20, 21, 22, 23, 24, 25], Romano et al. [26], and Barretta and Marotti de Sciarra [27]. During the simulation of torsion processes in graded materials, a number of approximation issues arise. To address these, numerous theories are employed, typically assuming heterogeneity in one direction of the coordinate system. These will be presented in the following sections.

3. Analytical Torsion Modeling Methods

Modeling of functionally graded materials primarily involves applying variable material properties along a single direction. To describe behavior under a twisting moment, existing torsion models are predominantly used with extensions that account for the property gradient in one direction and the dependence of changes in specific properties such as Young's modulus, the Kirchhoff modulus, Poisson's ratio, and density on variations along the chosen direction. The most widely used developments in torsion modeling include the classical Saint-Venant torsion model, Prandtl's membrane analogy, and the Vlasov model. These models are described below in the context of their application to functionally graded materials.

3.1. Saint-Venant theorem

The beginnings of mathematical modeling of structures made from gradient materials are based on issues raised by Saint-Venant. The Saint-Venant principle, although originally developed for homogeneous and isotropic materials, can also be applied to the analysis of functionally graded materials. These materials are characterized by a gradual change in composition or structure, which leads to a change in their mechanical and thermal properties along a specified direction. The Saint-Venant formulation follows several assumptions. The shaft cross section rotates approximately as a rigid entity around a twist axis. This implies that during torsion the cross section retains its shape with minimal distortion, and every point on it moves in a circular trajectory around the twist axis. The shaft features a prismatic cross section, meaning it is consistent and uniform along its entire length. This uniformity simplifies analysis since the same geometric and material properties apply to every cross section. There is a warping of the cross section that remains constant across all sections along the shaft length. Warping refers to the out-of-plane displacement of points on the cross section, accounting for the fact that in real materials the cross section is not perfectly rigid. These restrictions impose several limitations on the torsion model itself, especially when considering nonhomogeneous materials such as FGMs. Overcoming some of these limitations allows for a more comprehensive simulation of shaft torsion.

In functionally graded materials the Saint-Venant principle is particularly useful because it allows simplification of stress analysis in regions far from the point of load application. Despite the variable characteristics of the material, this principle assumes that local effects of loads, such as stress concentration or detailed stress distribution around the points of force application, diminish quickly as one moves away from the source of the load. Most examples are based on torsion studied by Saint-Venant. The characterization of torsional behavior is challenging for FGM structures. The original theory of torsion is based on isotropic shafts with a constant angle of twist. When it is applied to FGM structures, it is crucial to account properly for the changing material properties across the volume. FGMs are designed so that their mechanical, thermal, or electrical properties change gradually in response to specific application requirements. The

Poisson-type equation is presented as follows [28, 29].

$$\nabla^2\Phi + \left(\frac{\text{grad}G}{G}\right) \cdot \nabla\Phi = -2G\theta \ . \tag{3}$$

The Poisson type torsion equation is a partial differential equation that, for describing changes in the Kirchhoff modulus G, must depend on variations in one direction. $\nabla^2\Phi$ denotes the Laplacian of the Prandtl function in the shaft cross section. $(\nabla G/G) \cdot \nabla\Phi$ represents the spatial variability of the Kirchhoff modulus G across the volume. $-2G\theta$ captures the effect of the Kirchhoff modulus and the boundary-condition terms. The Saint-Venant torsion formulation relies on the dependence of the torsion function on properties that vary in one or more directions. Because Saint-Venant theory is limited for FGM materials, it is necessary to improve and extend it to describe their torsional behavior more comprehensively and accurately. An important aspect is the proper treatment of material properties that vary with coordinate direction (usually a single direction), together with a fuller account of material inhomogeneity, which requires consideration of the equilibrium equations and boundary conditions. Another promising direction is the development of validation experiments to confirm and calibrate the theory.

3.2. Prandtl's membrane analogy

Prandtl's torsion model addresses the torsion of prismatic shafts and primarily characterizes the distribution of shear stresses in prismatic shafts subjected to a twisting moment. Initially, the membrane analogy was applied only to isotropic and homogeneous materials, but over time it has been extended to nonhomogeneous materials such as FGMs. Prandtl's membrane analogy relates the torsion of a prismatic rod to a thin elastic membrane that is hypothetically stretched and conformed to the given cross section subjected to torsion. In the case of FGMs, applying this theory requires accounting for material dependence along the directions of gradation, x and y. For isotropic and homogeneous shafts, the Poisson type equation is given by [30, 31].

$$\nabla^2\Phi = -2G\theta \tag{4}$$

where Φ is the Prandtl function, G is the shear modulus (constant for homogeneous materials), and θ is the angle of twist per unit length.

In the case of FGM materials, to obtain the correct Prandtl function it is necessary to treat the Kirchhoff modulus as dependent on the spatially varying material properties, $G(x, y)$. This requires modifying the classical Prandtl equation by allowing G to depend on the x and y directions. After this modification, the Poisson-type equation for functionally graded materials is as follows:

$$\nabla \cdot [G(x, y)\nabla\Phi] = -2G(x, y)\theta \ . \tag{5}$$

After modifying the differential equation and making it dependent on the derivatives of the material properties in the x and y directions, the equation is as follows [31]

$$\frac{\partial}{\partial x}\left[G(x, y)\frac{\partial\Phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[G(x, y)\frac{\partial\Phi}{\partial y}\right] = -2G(x, y)\theta \ . \tag{6}$$

The membrane analogy offers several advantages. It enables relatively straightforward determination of shear stress distributions in cross sections of rods subjected to torsion and can be visualized clearly. It is applicable to various cross-sectional shapes, which makes it highly useful. For functionally graded materials, however, applying the membrane analogy introduces computational complexity. The use of complex material models can make analytical solutions difficult to obtain and, in some cases, unattainable.

3.3. Vlasov's torsion model

Another commonly used torsion model is Vlasov's torsion model. It extends the Saint-Venant model by incorporating the effects of warping restraint. In this model warping is not negligible, which introduces additional dependencies required to obtain accurate results. The theory is mainly applied to thin-walled beam elements, where the warping effect is particularly evident and significant.

Vlasov's torsion theory primarily accounts for warping effects in cross sections and the interaction between cross-sectional torsion and bending deformation. Another key concept is the shear center, defined as the point where shear stresses do not induce additional torsional effects [32, 33]. The torsion equation for a thin-walled beam with cross-sectional warping presented by Vlasov is given by [33, 34, 35].

$$\frac{\partial}{\partial x}\left(G(x) J_t \frac{\partial \theta}{\partial x}\right) + E(x) I_w \frac{\partial^3 \theta}{\partial x^3} = 0 \tag{7}$$

where $G(x)$ is the shear modulus dependent on the direction of property changes, J_t is the polar moment of inertia of the cross section, $E(x)$ is Young's modulus dependent on the direction of material property changes, I_w is the warping constant (warping moment of inertia) for the characteristic cross section, and θ is the angle of twist of the given cross section. Many studies address the torsion problem for functionally graded thin-walled beams. A few representative works are outlined below.

Addessi et al. [35] presented a comparison of the impact of warping effects on various thin-walled cross sections according to the Vlasov and Benscoter theories. They developed numerical models and compared the resulting predictions. Additional contributions include three works by Mur'in et al. [36, 37, 38], which present a series of extensions on the application of thin-walled theory to FGM beams, the role of warping during torsion, and the influence of graded property variation on the distribution of mode shapes, bimoment, and shear stresses in thin beam cross sections.

4. Modeling methods

Due to the complexity of calculations and the sophisticated material models required to represent variable material properties, functionally graded materials (FGM) are of great interest in the contemporary scientific community. The potential for continuous improvements and the development of stiffer, stronger materials that can be applied across diverse industrial environments motivates researchers to conduct new experiments and studies on modeling the mechanics of various FGM structures.

The most commonly used methods for modeling structures with graded properties are the finite element method (FEM) and the finite difference method (FDM). With advanced numerical models, researchers can represent the behavior of FGM beams under loads, isolate the effects of cross-sectional warping, and analyze shear stress distributions during torsion, shear deformations, and mode shapes with reasonable accuracy. Every numerical model involves some degree of approximation and assumptions, so it is never fully consistent with reality. FGMs, and the simulation of property variation in at least one direction, require homogenization and property approximation using established cross-sectional homogenization formulas. The accuracy of models and simulations may be questioned because many publications do not include validation or calibration of the models. Below, issues related to these problems and the current state of knowledge in the scientific community are presented.

The internal structure of each FGM can be designed in different ways. Owing to variation in properties and their spatial distribution within a given structure, one can distinguish materials with different gradient architectures. Achieving a specific gradient form depends on the manufacturing method used for the FGM. The following types of gradients are distinguished: discontinuous with interface-a, continuous with interface-b, composition gradient-c, orientation gradient-d, and fraction gradient-e, as shown in Fig. 6.

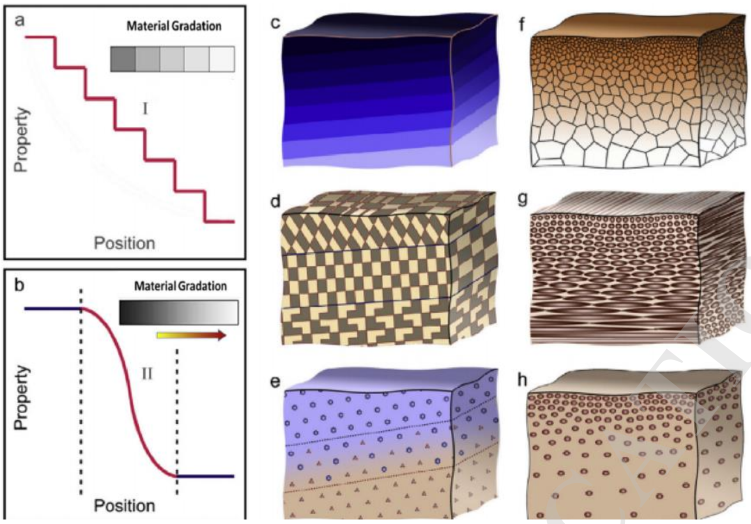

Fig. 6. FGM with various form of gradient, after El-Galy et al. and Zhang et al. [53, 54].

FGMs can be categorized into discontinuous and continuous types, as illustrated schematically in Figs. 6a and 6b. In discontinuous FGMs, the compositions and microstructures change in a stepwise manner, typically with an interface. In continuous FGMs, the compositions and microstructures vary gradually and continuously with position. Figure 6 schematically depicts various types of FGMs. Additionally, graded structures may occur throughout the entire material or within specific localized regions [54].

4.1. FEM modeling

Modeling of FGMs can be performed using two different approaches to finite element formulation. The first is the classical method, which models finite elements that are homogeneous and isoparametric, following the principles used in commercial software such as Ansys. This approach is based on the formulation of isoparametric finite elements, which are defined by a prescribed shape function and then used to approximate both the element geometry and the unknown field. In this method, the displacement vector and the coordinate vector are expressed as functions [39, 40, 42].

$$\boldsymbol{u}_i^{\mathrm{e}} = \sum_{i=1}^{n} \boldsymbol{N}_i \boldsymbol{u}^{\mathrm{e}}, \quad \boldsymbol{x} = \sum_{i=1}^{n} \boldsymbol{N}_i \boldsymbol{x}_i \tag{8}$$

where N_i is equal a shape function, u_i is a nodal displacement, m is a quantity of nodal points of element. Below are examples of shape functions for a triangular element

$$N_1 = 1 - \xi - \eta \qquad N_2 = \xi \qquad N_3 = \eta \tag{9}$$

where ξ and η are the natural coordinates within the triangular element.

The constitutive relation between stress tensor $\boldsymbol{\sigma}^{\mathrm{e}}$ and strain is equal to [40, 42]

$$\boldsymbol{\sigma}^{\mathrm{e}} = \boldsymbol{D}^{\mathrm{e}} \boldsymbol{\varepsilon}^{\mathrm{e}} \tag{10}$$

where $\boldsymbol{D}^{\mathrm{e}}$ is a constitutive matrix and ε^{e} is a strain gained from displacement. Thus, $\boldsymbol{\varepsilon}^{\mathrm{e}}$ can be formulated as [40]

$$\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{B}^{\mathrm{e}} \boldsymbol{u}^{\mathrm{e}} \tag{11}$$

where $\boldsymbol{B}^{e}$ is a strain-displacement matrix of shape function, $\boldsymbol{u}^{e}$ is a nodal displacement vector.

The main static equation based on the principle of virtual work is equal to [40, 42]

$$\boldsymbol{F}^{e} = \boldsymbol{k}^{e}\boldsymbol{u}^{e} \tag{12}$$

where $\boldsymbol{F}^{e}$ is a force vector described with integration formula [40, 42]

$$\boldsymbol{k}^{e} = \int_{\Omega^{e}} (\boldsymbol{B}^{e})^{T}\boldsymbol{D}^{e}\boldsymbol{B}^{e}\mathrm{d}\Omega^{e} \tag{13}$$

superscript ^{T} describes a transpose and Ω^{e} domain of element (e). This type of classical formulation provides constant material properties, thus stiffness matrix has constant properties. Divided into segments, elements with different material properties maintain continuity between finite elements, which ensures the consistency of properties at the Gauss integration points [39].

The next method of numerical representation is the method based on the works of Kim and Paulino [41] and called isoparametric graded finite elements. This method involves interpolating material properties at each integration point from the material properties at each node using isoparametric shape functions, which have identical properties in the given coordinate system (x, y)

$$x = \sum_{i=1}^{n} N_i x_i, \qquad y = \sum_{i=1}^{n} N_i y_i \tag{14}$$

and for the displacements

$$u = \sum_{i=1}^{n} N_i u_i, \qquad v = \sum_{i=1}^{n} N_i v_i \tag{15}$$

When using the above properties, the Young's modulus $E = E(x)$ and Poisson's ratio $\nu = \nu(x)$ functions can be interpolated using the isoparametric concept

$$E = \sum_{i=1}^{n} N_i E_i, \qquad \nu = \sum_{i=1}^{n} N_i \nu_i \tag{16}$$

what is shown by Kim and Paulino in Fig. 7 below.

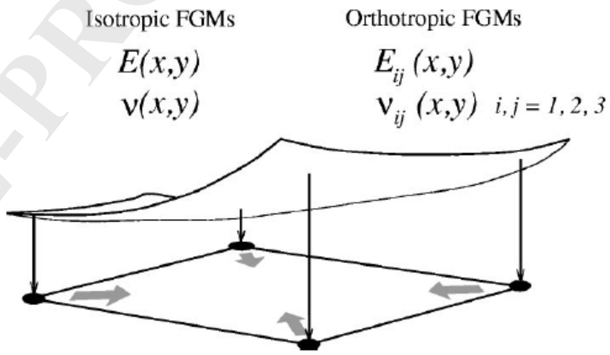

Fig. 7. Isoparametric formulation for isotropic and orthotropic FGMs, after Kim and Paulino [41].

The above consideration presents the isoparametric formulation for an isotropic material. For an orthotropic material, it involves formulating four elastic parameters: Young's moduli $E_{11} = E_{11}(x)$, $E_{12} = E_{12}(x)$, shear modulus $G_{12} = G_{12}(x)$, and Poisson's ratio $\nu_{12} = \nu_{12}(x)$. It has been presented in Fig. 7. The final formulation is equal [41, 43]

$$\begin{aligned} E_{11} &= \sum_{i=1}^{m} N_i (E_{11})_i, \quad E_{22} = \sum_{i=1}^{m} N_i (E_{22})_i \\ G_{12} &= \sum_{i=1}^{m} N_i (G_{12})_i, \quad \nu_{12} = \sum_{i=1}^{n} N_i (\nu_{12})_i \ . \end{aligned} \tag{17}$$

Another considered model is one that varies depending on the volume fraction V and the material phase p. For this type of model, the isoparametric formulation is carried out in the classical manner according to a given function with exponent V_i^p for it all values V^p are the values in nodal points [41, 43]

$$V^p = \sum_{i=1}^{m} N_i V_i^p \ . \tag{18}$$

4.2. Homogenization rules

Due to the variable internal structure in terms of mechanical properties such as Young's modulus, Poisson's ratio, and the Kirchhoff modulus in torsion, each object must be approximated according to the spatially varying properties. This is necessary to obtain accurate analyses and results. Material approximations or FGM models that represent property variation are described in various ways. To model material behavior under torsion correctly, it is essential to idealize mathematically the internal substructures of the materials and, consequently, the variation in the volume fractions of their constituents within each unit length or volume of the FGM. The material model may be represented as a particulate model with defined mixing phases, or as a multilayer model in which each layer follows a gradient approximation of properties. The first model presented is the Voigt material model [47], which describes changes in material properties in terms of changes in volume fraction. The expression for Young's modulus is given by the following equation [47]:

$$\overline{E}^{\mathrm{V}} = E_1 V_1 + E_2 V_2 \tag{19}$$

where V_1 and V_2 are the volume fractions of the two materials. Thus, the sum of the fractions must satisfy

$$V_{\mathrm{f}} = 1 - V_1 \ . \tag{20}$$

According to Voigt, the fraction V_{f} can be written as

$$V_{\mathrm{f}} = \left(0.5 + \frac{z}{h}\right)^k \tag{21}$$

where k is a positive power-law coefficient and z/h is a dimensionless coordinate ratio. The resulting relation for Young's modulus is

$$\overline{E}^{\mathrm{V}}(z) = E_2 + (E_1 - E_2)\left[1 - \left(0.5 + \frac{z}{h}\right)^k\right] \ . \tag{22}$$

Accordingly, the dependence of the Kirchhoff modulus on the phases is [47]

$$\overline{G}^{\mathrm{V}}(V_{\mathrm{f}}) = G_1 V_{\mathrm{f}} + G_2(1 - V_{\mathrm{f}}) \tag{23}$$

where G_1 and G_2 are the Kirchhoff moduli of the two constituent materials.

The next model is the Reuss model [44, 45, 46], which assumes a uniform stress distribution throughout the material. The Young's modulus $E = E(V_{\mathrm{f}})$ is given by

$$\overline{E}^{\mathrm{R}}(V_{\mathrm{f}}) = \frac{E_1 E_2}{E_1 V_{\mathrm{f}} + E_2(1 - V_{\mathrm{f}})} \tag{24}$$

where E_1 and E_2 are the Young's moduli of the two materials, and V_{f} is the same volume-fraction function as in the Voigt model. The corresponding Kirchhoff modulus is [46]

$$\overline{G}^{\mathrm{R}}(V_{\mathrm{f}}) = \frac{G_1 G_2}{G_1 V_{\mathrm{f}} + G_2(1 - V_{\mathrm{f}})} \ . \tag{25}$$

The next model considered is the Mori–Tanaka material model. In this approach, the homogenized composite consists of two phases: inclusions that are uniformly distributed and assumed spherical, and a matrix that is randomly distributed. The influence of Poisson's ratio on overall behavior is typically considered negligible and taken as constant. The Kirchhoff modulus according to Mori–Tanaka is [48, 49]

$$\overline{G}^{\mathrm{MT}}(V_{\mathrm{f}}) = G_1 + \frac{V_{\mathrm{f}}(G_2 - G_1)G_1}{(1 - V_{\mathrm{f}})(G_2 - G_1)\beta_1 + G_1} \tag{26}$$

where

$$\beta_1 = \frac{6(K_1 + 2G_1)}{5(3K_1 + 4G_1)} \tag{27}$$

and K is the bulk modulus.

Hashin and Shtrikman proposed narrower bounds using the principle of minimum potential energy and polarization concepts, thereby defining rigorous upper and lower bounds for homogenized properties. The upper bound of the Kirchhoff modulus is [50, 51]

$$\overline{G}^{\mathrm{HS+}}(V_{\mathrm{f}}) = G_2 + \frac{(G_1 - G_2)V_{\mathrm{f}}}{1 + \zeta(1 - V_{\mathrm{f}})(G_1/G_2 - 1)} \tag{28}$$

where

$$\zeta = \frac{1 + \nu}{3(1 - \nu)} . \tag{29}$$

The lower bound is [50, 51]

$$\overline{G}^{\mathrm{HS-}}(V_{\mathrm{f}}) = G_1 + \frac{(G_2 - G_1)(1 - V_{\mathrm{f}})}{1 + \zeta V_{\mathrm{f}}(G_2/G_1 - 1)} . \tag{30}$$

Each of these theories specifies the Kirchhoff modulus as a function of the volume fraction. As shown in Fig. 8, the Voigt and Reuss models yield substantially wider bounds than the Hashin–Shtrikman bounds, which provide tighter constraints. These bounds depend solely on the volume fractions of the phases and are therefore scale-independent.

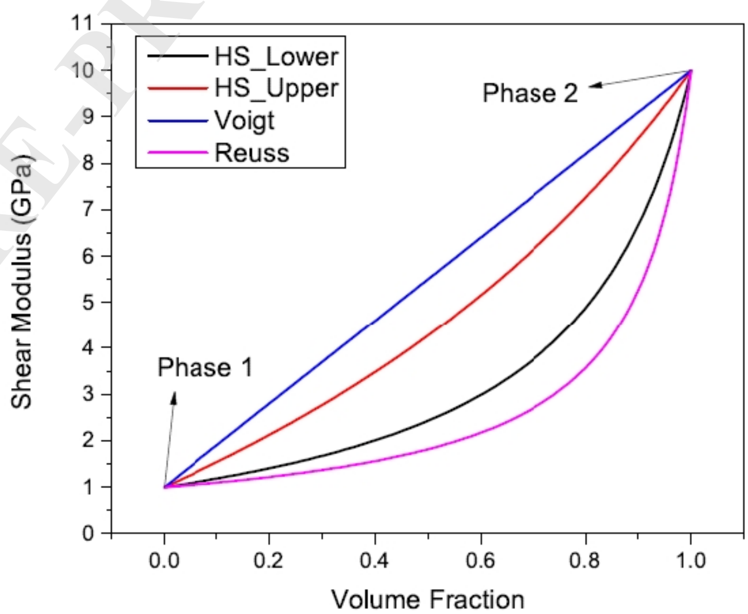

Fig. 8. Shear modulus between two phases as a function of volume fraction, after Saharan et al. [52].

Another material model is the Exponential Material Model. In this formulation, properties vary according to an exponential function, producing smooth transitions through the FGM thickness. This approximation can be applied to properties such as Young's modulus, the Kirchhoff modulus, or thermal conductivity. If the property variation through the thickness is exponential, the Kirchhoff modulus can be approximated as [53]

$$\overline{G}^{\mathrm{EMM}}(z) = G_2 \exp(\beta z) \tag{31}$$

where $\beta = \frac{1}{h} \ln\left(\frac{G_1}{G_2}\right)$, G_1 and G_2 are the Kirchhoff moduli of the two materials, h is the thickness, and z is the coordinate varying from 0 to h.

The Power-Law Material Model is also widely used for FGMs. The volume-fraction variation of an FGM layer can be represented as

$$V_{\mathrm{f}} = f(z) = \left(\frac{z + h/2}{h}\right)^n. \tag{32}$$

Here, $f(z)$ denotes the volume fraction of one constituent; accordingly, the other constituent has fraction $1 - V_{\mathrm{f}}$. Once the local volume fraction is known, pointwise properties follow from the rule of mixtures. In particular, the Kirchhoff modulus $G(z)$ within the layer is [53]

$$\overline{G}^{\mathrm{PLMM}}(z) = G_1 V_{\mathrm{f}} + G_2(1 - V_{\mathrm{f}}) \tag{33}$$

where G_1 and G_2 are the Kirchhoff moduli of the two materials.

Another important model is the Linear Variation Model. Here, Poisson's ratio $\nu(x)$ is taken as constant, while Young's modulus and the Kirchhoff modulus vary linearly with the gradation. The elastic modulus $E(x)$ and the Kirchhoff modulus $G(x)$ are expressed as [41]

$$\overline{E}^{\mathrm{LVM}}(x) = E + \gamma x \qquad \overline{G}^{\mathrm{LVM}}(x) = G + \gamma x \tag{34}$$

where x is the graded coordinate and γ is a parameter of nonhomogeneity defined by [41]

$$\gamma = \frac{G(W) - G(0)}{W} \qquad \gamma = \frac{E(W) - E(0)}{W} \tag{35}$$

with W denoting the width of the graded region.

The final model is the Sigmoid Material Model. It uses two functions to represent the change in properties with gradation, effectively partitioning the beam into two regions and describing the behavior in each. For one coordinate direction, changes are defined on the domains $-h/2$ to 0 and 0 to $h/2$, where h is the graded height. The expressions for Young's modulus are

$$\overline{E}^{\mathrm{SMM}}(z) = E_2 + (E_1 - E_2)\left[1 - \frac{1}{2}\left(\frac{z}{h} + \frac{1}{2}\right)^k\right] \tag{36}$$

for $-h/2 \leq z \leq 0$, and

$$\overline{E}^{\mathrm{SMM}}(z) = E_1 + \frac{E_1 - E_2}{2}\left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{37}$$

for $0 \leq z \leq h/2$.

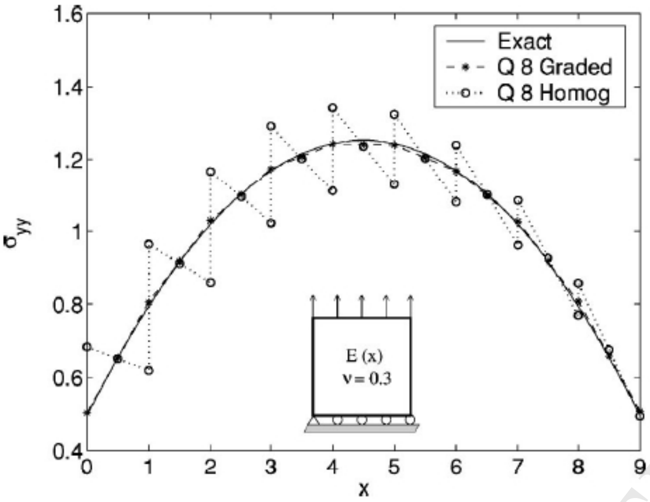

Fig. 9. Stress-distribution differences among exact, graded, and homogeneous elements, after Kim and Paulino [41].

4.3. Accuracy of results

Modeling FGM properties typically uses either homogeneous or heterogeneous finite elements. Kim and Paulino [41] were the first to model FGMs with heterogeneous elements, moving beyond classical homogeneous elements for which only midpoint responses match the FGM solution.

Partitioning the domain into equal segments and assigning homogeneous elements does not accurately represent heterogeneous behavior, as shown by Hernik [39]. Using heterogeneous elements that account for spatial variation in Young's modulus and Kirchhoff modulus at each point along the graded layer leads to markedly improved simulation outcomes in FGM regions. The figure below shows x-direction displacements and differences between homogeneous and heterogeneous finite elements.

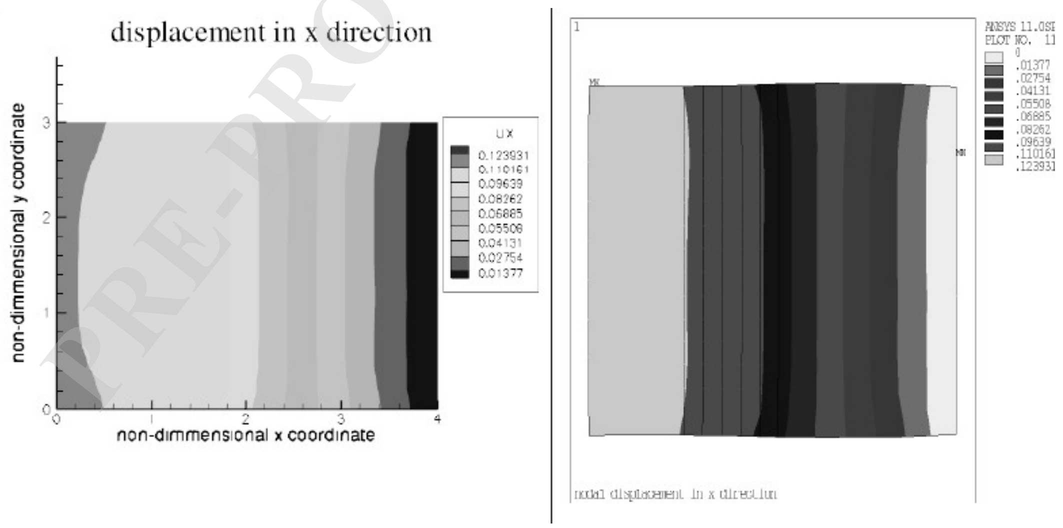

Fig. 10. Differences in x-direction displacement for homogeneous versus heterogeneous finite elements, after Hernik [39].

According to Kim and Paulino [41], employing heterogeneous elements yields more accurate stresses and deformations in tensile tests. Mesh refinement reduces discrepancies between homogeneous and heterogeneous formulations for all considered cases [41].

5. Comparison of achievements

5.1. Achievements

The torsional mechanics of structures made from functionally graded materials presents a range of challenges across various formulations, including homogenization of the material model, mixing rules for individual graded phases, and approaches to homogenizing the gradation direction itself. Many studies model gradation along the longitudinal direction for beams or rods, while others focus on homogenization of the cross section. Over time, approaches to analyzing FGMs have evolved. Numerous limitations have been identified when using homogeneous elements in the finite element method, including inaccuracies in representing gradient variations. The comparison presented in the table below summarizes advances in the torsion of FGMs developed over the past several decades, with the aim of compiling these results in a concise form for comparison.

5.2. Comparison and view

The torsional response of structures and bars made from functionally graded materials has been extensively studied, with the literature emphasizing their importance for understanding material behavior under complex loading conditions. Foundational work by Rooney and Ferrari [5] made a significant contribution through the analysis of functionally graded shafts with rectangular cross sections. They examined torsion and bending of bars with variable shear modulus, determined stiffness bounds, and presented solutions for specific cases such as laminates and cylindrical bars. The results were used to evaluate graded material properties. Their subsequent paper [56] focused on the torsion of bars with inhomogeneous shear modulus and arbitrary geometry, where the modulus varies across the section as a function of coordinates with the additional assumption of constancy at the boundary. Solutions were also presented for the torsion of a circular cylindrical bar with angular symmetry.

Further studies by Horgan and Chan [4] extended the analysis to isotropic, linearly elastic bars with functional gradients, providing deeper insight into stress and strain fields. Their additional works [8, 57] addressed rotating bodies and yielded exact solutions for power-law variation of Young's modulus, showing that stress distributions differ from homogeneous cases and that maximum stresses are not always located at the center. Ting, Chen, and Li [58] investigated the design of neutral cylinders under torsion, considering cylinders with multiple coatings or graded shear modulus in the cross section. A multilayer cylinder with piecewise-varying shear modulus was generalized to a graded cylinder with continuous radial variation of the shear modulus. The warping field of the neutral graded cylinder is governed by a second-order differential equation, with solutions obtained using the Frobenius method. Singh, Rokne, and Dhaliwal [59] studied torsional vibrations of functionally graded finite cylinders and demonstrated resonance behavior influenced by material gradation, improving understanding of vibrational responses in graded structures.

Batra [6] provided exact solutions for torsional behavior in functionally graded cylinders, refining theoretical models, while Sofiyev and Schnack [60] examined the stability of functionally graded cylindrical shells under dynamic torsional loads and highlighted the effects of transient stresses. Li, Weng, and Duan [61] analyzed a cylindrical crack in a functionally graded interlayer between two coaxial elastic cylinders under torsional impact. The shear modulus and density of the FGM layer vary continuously. The problem was solved numerically and the dynamic stress intensity factor was computed, showing that increasing the FGM gradient can significantly reduce this factor.

Continued contributions by Horgan [4, 8, 57] on anisotropic, linearly elastic bars with functional gradients, together with the study by Gholami Bazehhour and Rezaeepazhand [64] on

Table 1. Comparison of key findings in torsion of FG materials over decades.

Nr	Scientists	Landmark Achievement
1	Rooney & Ferrari [55]	Investigation on the torsion of a FGM shaft with a rectangular cross-section.
2	Rooney & Ferrari [56]	Torsional behavior analysis in various classes of functionally graded shafts.
3	Rooney & Ferrari [5]	Combining torsion and flexure in inhomogeneous elements, exploring the relationship between material properties and structural response.
4	Horgan & Chan [4]	Analysing of isotropic linearly elastic bars with a functional gradient, showing stress distribution across this bars.
5	Horgan & Chan [57]	Presenting stress response in rotating isotropic FGM disks, considering the affect of the gradient.
6	Ting et al. [58]	Demonstrates that neutrality occurs when the geometric mean of the cylinder's shear moduli equals the shear modulus of the shaft and outlines criteria for preserving rigidity with embedded cylinders.
7	Singh et al. [59]	Torsional vibrations of graded cylinders were analyzed, considering shear moduli and densities as functions of radius and axis.
8	Batra [6]	Examines the torsion of cylindrical bars with material moduli varying along the axis.
9	Sofiyev & Schnack [60]	Study on the stability of functionally graded cylindrical shells under dynamic torsional loading as a linear function of time.
10	Li et al. [61]	Examines a cylindrical crack in a graded layer between coaxial cylinders subjected to dynamic torsional loading.
11	Hematiyan & Estakhrian [62]	An approximate analytical method for analyzing the torsion of functionally graded open-section members with uniform thickness.
12	Arghavan & Hematiyan [63]	Study on the torsion of hollow tubes with a functional gradient, identifying the impact of the gradient on stiffness and torsional resistance.
13	Gholami & Rezaeepazhand [64]	Analysis of the torsion of multilayered tubes with non-circular cross-sections, focusing on the influence of material properties.
14	Vasiliev [65]	Analyzes the torsion of a circular punch on a half-space with a graded coating, reducing the problem to integral equations and deriving an explicit solution.
15	Wang et al. [66]	Solution for torsional vibrations in functionally graded finite hollow cylinders, focusing on the analysis of transient behavior and vibration characteristics..
16	Shen et al. [67]	A size-dependent gradient shaft model was developed to study the effects of microstructure and material scale on torsional wave propagation, free vibration, and static torsion, considering the radial variation of material properties.
17	Bayat & Toussi [68]	Study focuses on the elasto-plastic torsion of functionally graded material shafts with a ceramic-metal structure, where the plastic zone may develop on the surfaces or within the thickness of the shaft, depending on material inhomogeneity and thickness.

18	Huaiwei et al. [69]	The elasto-plastic buckling of cylindrical functionally graded material shells under axial and torsional loads was analyzed using the TTO model and the Ritz method.
19	Tsiatas & Babouskos [70]	Presents a new solution to the elasto-plastic torsion problem of functionally graded material bars using an iterative numerical method based on BEM and AEM.
20	Murín et al. [71]	Showes an elastostatic analysis of spatial beam structures made of FGM, considering smoothly varying material properties along the longitudinal direction and symmetric variations in the transverse and lateral directions.
21	Barretta et al. [12]	Formulates the elastostatic problem of functionally graded circular nanobeams under torsion, incorporating nonlocal elastic behavior based on Eringen's theory.
22	Liaghat et al. [72]	Investigates material tailoring in functionally graded hollow rods with arbitrary cross-sections under torsion.
23	Barretta et al. [73]	Analyzes the torsion of linearly elastic, isotropic beams with inhomogeneities in the cross-section and along the axis.
24	Bayat et al. [74]	Presents a torsion problem of hollow cylinders made of FGM, considering arbitrary variations of Young's modulus and Poisson's ratio in the radial direction.
25	Rizov [75]	An analysis was conducted on a cylindrical surface crack in circular shafts under torsional loading, considering the nonlinear behavior of the material.
26	Rahaeifard [76]	Examines the size-dependent behavior of functionally graded microbars based on the modified couple stress theory, defining two length scale parameters to describe their mechanical properties.
27	Aminbaghai et al. [77]	Analyzes the impact of torsional warping and secondary deformations on the deformation and stress state of thin-walled FGM beams with longitudinally varying properties.
28	Murín et al. [78]	Warping torsion in functionally graded material beams with spatially varying properties, analyzing torsional behavior.
29	Murín et al. [38]	Investigation of torsional warping eigenmodes in functionally graded material beams with longitudinally varying properties, determining modal characteristics.
30	Guendouz et al. [79]	Torsional-bending in functionally graded material beams using 3D Saint-Venant refined beam theory.
31	Murín et al. [80]	Investigates the effect of longitudinal variation in material properties on the deformation and stresses in thin-walled FGM beams with non-uniform torsion.
32	Guendouz et al. [81]	The static bending-torsion behavior of functionally graded cantilever beams is analyzed using an advanced 1D/3D beam theory.

33	Murín et al. [36]	Extends the 3D FGM Timoshenko finite element to include the warping torsion effect for non-uniform torsion.
34	Barati et al. [105]	Solution for static torsion in a microtube composed of bi-directional functionally graded materials (BDFGMs).
35	Naghibi et al. [83]	Determining the defects in hollow cylinders coated with functionally graded materials under torsion, identifying the impact on stress results.
36	Singh et al. [84]	Analyzes the shear stresses developed in functionally graded material bars under pure torsional loading, considering different cross-sectional shapes (circular, square, triangular) and varying thicknesses.
37	Zhang et al. [85]	Studies the buckling of functionally graded material cylindrical shells under torsional impact load using the symplectic method, considering torsional stress waves.
38	Noroozi et al. [86]	Investigates multiple cylindrical interface cracks between a homogeneous circular cylinder and its FGM coating under torsional impact loading.
39	Li & Hu [87]	Torsional statics of two-dimensionally functionally graded microtubes, focusing on stress distribution and material properties.
40	Karaca & Alyavuz [88]	Examines the torsional behavior of beams with one- and two-directional gradation under large displacements and angular deformations, considering power law and sinusoidal functions.
41	Barretta et al. [89]	Focuses on the nonlocal strain gradient theory of elasticity, combining Eringen's nonlocal integral convolution and Lam's strain gradient model through a variational approach.
42	Soltani & Asgarian [90]	Conducts a lateral buckling analysis of simply supported web- and/or flange-tapered I-beams made of axially functionally graded materials under uniformly distributed loads.
43	Murín et al. [91]	Extends previous research by investigating the effect of spatially varying material properties on the torsional eigenvibrations of FGM beams.
44	Hajhashemkhani & Hematiyan [92]	Analysis of problem of inflation, extension, and torsion in hyperelastic rods and tubes, focusing on rubber-like materials and soft biological tissues.
45	Murín et al. [93]	Investigates the impact of torsional warping on the elastostatic behavior of thin-walled twisted functionally graded material beams, considering longitudinal material property variations described by a polynomial.
46	Nie et al. [94]	Presents analytical solutions for the torsion of bi-directional functionally graded linearly elastic truncated conical cylinders, considering six different functional forms of shear modulus variations in both radial and axial directions.
47	Baksa [95]	Study on analytical solution for Saint-Venant's torsion of a circular bar with a slit extending radially from the boundary to the axis.
48	Rizov [96]	Analyzes cylindrical delamination in a multilayered functionally graded circular shaft under torsional loading, based on the Ramberg-Osgood equation.

multilayered tubes with non-circular cross sections, provided additional depth for complex geometries. Vasiliev [65] presented an analytical solution for the torsion of a circular punch on a transversely isotropic elastic half space with a functionally graded coating. Wang, Liu, and Ding [66] provided exact solutions for transient torsional responses of a finitely long functionally graded hollow cylinder under free–free, free–fixed, and fixed–fixed boundary conditions. Shen, Chen, and Li [67] developed a size-dependent shaft model within nonlocal strain gradient theory, accounting for radial power-law variation in a two-constituent FGM and investigating small-scale effects on static and dynamic torsion, including material length scale and nonlocal parameters.

Huaiwei, Zhang, and Han [69] proposed a method for elastoplastic buckling of cylindrical shells made of FG materials under axial and torsional loads, with properties varying by a power law. The Ritz method and stress-state analysis were used to determine the critical condition and the location of the elastoplastic interface. Tsiatas and Babouskos [70] provided solutions for the elastic-plastic torsion problem of functionally graded bars with arbitrary cross sections and property variation across the section, introducing a simplified nonlinear procedure using the Boundary Element Method and the Analog Equation Method. Elastostatic analyses by Murín, Aminbaghai, and Hrabovský [71], as well as the study by Barretta, Feo, and Luciano [73] on torsion in nonlocal viscoelastic nanobeams, highlighted novel modeling techniques for FGMs. Liaghat, Hematiyan, and Khosravifard [72] examined material tailoring for functionally graded rods under torsion, and Barretta and colleagues [73] presented closed-form solutions for torsion of linearly elastic isotropic beams with axial and cross-sectional inhomogeneities. New solutions were derived by analyzing axial distributions of longitudinal and shear moduli, and the effects of warping and shear modulus variation on the torsional behavior of elliptic and equilateral triangular beams were discussed.

Ongoing efforts by Bayat, Alizadeh, and Bayat [74] on generalized solutions for hollow cylinders, Rizov's [75] elastic-plastic fracture analysis, and Rahaeifard's [76] studies on size-dependent torsion illustrate substantial advances in predicting FGM behavior under diverse conditions. Aminbaghai [77] analyzed the influence of torsional warping on the elastostatic behavior of thin-walled twisted FGM beams with longitudinal material variation and secondary deformations due to twist angle. Murín et al. [78] examined the effect of spatially varying properties on the warping torsion of I-section FGM beams using the Reference Beam Method and the FGM WT finite element. Extended stress equations accounting for secondary torsional moment and warping were applied, reinforcing the importance of torsional analysis for the development of these materials.

Future research directions include expanding understanding and application of FGMs under torsional loads. Promising areas are the development of multifunctional FGMs that combine torsional resistance with enhanced thermal or electrical performance, and the advancement of modeling techniques to capture interactions among torsion, bending, and axial loads in complex geometries. Experimental validation and improved materials characterization are essential for representing gradients accurately in practice. Experimental torsion testing of FGM shafts faces significant reliability challenges. Residual stresses arising from manufacturing (e.g., AM, sintering, deposition) and non-ideal material interfaces across graded transitions can distort the measured response. Maintaining perfect coaxiality and geometric tolerances is difficult, even small deviations introduce parasitic bending, compromising data quality. In thin walled specimens, warping effects are pronounced and often amplified by gradation further obscuring a pure shear state. Finally, the absence of standardized torsion test protocols for FGMs hinders cross-laboratory validation and benchmarking. Investigation of fatigue and long-term durability under cyclic torsional loading is also important, with direct implications for aerospace, automotive, and civil engineering. While significant progress has been achieved, substantial opportunities remain to improve design and application of FGMs across engineering disciplines.

6. Conclusion and future research

Torsion in functionally graded materials (FGMs) can be described reliably within classical theories, provided they are generalized to account for the spatial variability of elastic parameters. It is essential to introduce $G(x, y)$, $E(x, y)$, and $\nu(x, y)$ explicitly into the torsion equations and to model gradation consistently. Below are concise conclusions and brief research pointers:

- Generalized Classical Theories in the Context of Torsion.

  Saint-Venant, Prandtl, and Vlasov formulations remain applicable when the spatial variability of G, E, and ν is explicitly incorporated and appropriate boundary conditions are enforced.

- Gradation modeling.

  Voigt, Reuss, Hashin-Shtrikman, Mori-Tanaka schemes and power-law or exponential profiles are useful; in practice, the tighter Hashin–Shtrikman bounds and Mori-Tanaka estimates typically yield more stable predictions of torsional stiffness.

- Numerical methods.

  Isoparametric graded finite elements (per Kim–Paulino) capture heterogeneity more faithfully than classical piecewise-homogeneous meshes and provide a robust basis for torsion analyses of FGMs.

- Future research.

  A promising direction is to implement torsion modeling in commercial solvers such as Ansys, moving beyond the prevailing APDL like studies. Representing material gradation with isoparametric graded finite elements offers a more rigorous, scalable path for complex geometries. Next steps include coupling torsion models with additive manufacturing process simulations to quantify porosity and residual-stress effects, and expanding experimental validation, especially for FGM shafts and thin-walled members where warping complicates pure shear. Future work should also assess fatigue and creep under torsion, develop multifunctional FGMs [97] that add thermal, damping, or conductive capabilities, and advance modeling [98, 99] that can capture coupled torsion–bending–axial behavior with explicit material, geometric nonlinearities and anisotropy.

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