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On Frequency Dependence of Stability in Materials with Fractional Viscosity

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Material instability refers to the tendency of materials to undergo alterations in their properties in loading. The concept of instability is governed by the constitutive equation of solids. Our analysis uses the entire set of equations describing the motion of solids, including the kinematical equation and Cauchy's equations of motion. Damping, or rate-dependence, plays a crucial role in stability. A potential generalization involves the utilization of fractional-order derivatives of the strain or stress tensors. The stability analysis primarily focuses on periodic perturbations. The mechanism of loss of stability on various parts of the stability boundary is under consideration.

Keywords: applied fractional calculus, material instability, dynamical systems.

1. Introduction

When the kinematical concept of stability is adopted, a material state is identified with a solution of the basic equations, and stability properties are studied in terms the stability of solutions of differential equations (dynamical systems) [6]. Material instability (divergence or flutter) [39] occurs well beyond the elastic domain; thus, the form of the constitutive equation might remind viscoelasticity but the physical background is different. This paper does not concentrate on various plasticity theories and does not treat the formulations (or even the existence of) of flow rules or hardening models, which have various interpretations in [21]. These problems are well beyond the scope of this study. Instead, a simplified constitutive equation is used, under the assumption that, after unloading, the body does not regain its original shape.

In physical interpretation static bifurcation (or divergence instability) can be observed as necking or shear banding, which are phenomena of strain localization. Flutter is the dynamic bifurcation and is considered here as the onset of an oscillation, being observed in plastic flow theory [28, 29], viscoplasticity [14], smeared crack models [43] or serrated flow, called the Portevin–Le Chatelier effect. This study is based on the work of KUBIN and ESTRIN [20], which uses a semi-empirical constitutive law:

$$\sigma = h\varepsilon + \overline{F}\left(\dot{\varepsilon}\right),$$

with negative rate-dependence included in function $\overline{F}$. Here ε and $\dot{\varepsilon}$ denote strain and strain rate, respectively, while σ is the stress in the uniaxial case and h is the work hardening rate. As a phenomenon, flutter instability means the existence of propagating deformation bands. Their work cites many papers based on experimental results on serrated flow, such as [8, 15, 24, 36, 42]. One of the main observations in this studies is that serrated flow appears at negative strain-rates. In this paper, such a model is generalized to a fractional rate.

Bifurcation analysis of a solution is a well-known and widely applied field in nonlinear dynamics. The first step of it starts with a linearization of the system of basic equations at that solution, identified by the state of the material [7]. Then the critical non-trivial eigenspace of the operator is studied. As a further step, the nonlinearities should be projected onto that non-trivial eigenspace to classify the type of bifurcation and describe the post-bifurcation behavior. Two key elements should be mentioned at this point. Firstly, the loss of stability should happen by crossing the imaginary axis by either a real eigenvalue or a pair of complex eigenvalues as the load (the bifurcation parameter) changes quasi-statically [47]. Second, at the critical value of the bifurcation parameter (zero real value of the eigenvalues), the critical eigenspace should be finite-dimensional.

Generally, such studies concern ordinary differential equations with integer-order derivatives. In material instability problems, bifurcation describes the types of instability. These two phenomena are identified as static and dynamic bifurcations. In the static case the loss of (Lyapunov) stability is coupled with a change in the number of solutions [25, 27, 30] while in the dynamic bifurcation a self-sustained oscillation can be observed. In a large range of materials, damping is described by fractional-order derivatives.

The aim of this paper is to perform such an analysis for a set of fractional-order equations. A method is presented to find the material instability condition. Then, a method to calculate the critical eigenvalues leads to conditions for static and dynamic bifurcation, even for fractional dynamical systems.

While fractional calculus has produced many new results and has found more and more applications in mechanics, control, economics, and several fields of science, one might have the feeling that this topic is just a fashionable tool of recent years, with no deep physical necessity. However, its roots have already been present in solid mechanics for more than fifty years and can be tracked back to the birth of continuum field theory in the middle of the last century. The study of creep and relaxation in Rabotnov's hereditary mechanics [34] is based on integral operators in form of convolutions with a fractional-order kernel, which are equivalent to fractional derivatives [38]. An early application was published by Caputo [9] in viscoelasticity [2, 9, 23] and then even in viscoplasticity [44], as a kind of fractional viscosity or non-local time effect. When non-locality is studied in Eringen's approach [16], similar mathematical tools can be used. Furthermore, non-locality may be extended from non-local time to spatial non-locality using fractional (non-local) derivatives [3].

The appearance of fractional calculus goes back to the origin of calculus by Leibniz and Euler, as a possible generalization. Most of the definitions were given by Liouville, Riemann, and others [22]. Fractional derivatives can be easily deduced from Cauchy's repeated integral formula and its generalization. For the n-th (integer) order, it leads to:

$$ {}_aI_t^n f(t) = \frac{1}{(n-1)!} \int\limits_a^t f(\xi)(t-\xi)^{-n-1}\,\mathrm{d}\xi. $$

The α-th fractional-order generalization is the Riemann–Liouville integral operator ($\alpha < 1$):

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t f(\xi)(t-\xi)^{\alpha-1}\,\mathrm{d}\xi = \frac{1}{\Gamma(\alpha)} \int\limits_a^t \frac{f(\xi)}{(t-\xi)^{1-\alpha}}\,\mathrm{d}\xi. $$

By taking the derivative of the Riemann–Liouville integral operator:

$$ (1.1) \qquad {}_aD_t^\alpha f(t) = \frac{\mathrm{d}}{\mathrm{d}t} {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_a^t f(\xi)(t-\xi)^{-\alpha}\,\mathrm{d}\xi, $$

the Riemann–Liouville derivative is obtained for the interval $[a,t]$.

By changing operators of differentiation and integration Caputo's derivative is defined:

$$ (1.2) \qquad {}_a^CD_t^\alpha f(t) = {}_aI_t^\alpha \left(\frac{\mathrm{d}}{\mathrm{d}\xi} f \right)(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_a^t \frac{\mathrm{d}f(\xi)}{\mathrm{d}\xi} (t-\xi)^{-\alpha}\,\mathrm{d}\xi, $$

for the interval $[a,t]$.

From [18], the connection between derivatives (1.1) and (1.2) is

$$ (1.3) \qquad {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{f(a)}{(t-a)^\alpha} + {}_a^CD_t^\alpha f(t). $$

In several cases, called the full-memory assumption in applications, the starting time is zero, $a = 0$, and the notations are simply $D_t^\alpha f(t)$ and ${}^C D_t^\alpha f(t)$. At this point only the most important definitions are given; more details can be found in several monographs [10, 12, 33, 40, 46].

The method is mainly analytic, using Fourier transformation. It is restrictive compared to numerical analysis [27, 30, 47], and excludes, for example, short-memory effects [45, 48]. However, it allows deeper insight into the roots of unstable behavior, especially at dynamic bifurcation. For the same reason, only the uniaxial case is studied. In 3D problems, the orientation of shear bands is a key factor [29, 31, 39], which requires detailed investigation of the constitutive acoustic tensor, already at the level of static bifurcation analysis. Then, the 3D fractional generalization of continuum mechanics constitutes another wide field of research [13].

2. Rate dependence and material instability

This section explains why a rate-independent constitutive equation is not suitable in material instability problems. Firstly, a rate-independent material, with the constitutive equation

$$F(\sigma, \varepsilon) = 0 \tag{2.1}$$

is studied to point out its singular behavior at instability. Here, F is a general form of the constitutive function. Assume that a uniaxial problem is studied, and the linearized constitutive equation is simply in the form:

$$\sigma = C\varepsilon, \tag{2.2}$$

where

$$C := -\left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}$$

denotes the tangent stiffness. Now, the equation of motion, the kinematic equation, and the so-called rate form of Eq. (2.2) are:

$$\dot{v} = \frac{1}{\rho}\frac{\partial \sigma}{\partial x}, \tag{2.3}$$

$$\dot{\varepsilon} = \frac{\partial v}{\partial x}, \tag{2.4}$$

$$\dot{\sigma} = \left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\dot{\varepsilon}, \tag{2.5}$$

where ρ denotes the mass density, and v is the velocity field.

By taking the time derivative of Eq. (2.3):

$$\ddot{v} = \frac{1}{\rho}\frac{\partial \dot{\sigma}}{\partial x}, \tag{2.6}$$

the gradient of Eq. (2.4):

$$\frac{\partial \dot{\varepsilon}}{\partial x} = \frac{\partial^2 v}{\partial x^2}, \tag{2.7}$$

and the gradient of Eq. (2.5):

$$\frac{\partial \dot{\sigma}}{\partial x} = \left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\frac{\partial \dot{\varepsilon}}{\partial x}. \tag{2.8}$$

By substituting Eq. (2.6) and Eq. (2.7) into Eq. (2.8), one obtains:

$$\rho \ddot{v} = \left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\frac{\partial^2 v}{\partial x^2}. \tag{2.9}$$

By introducing new variables:

$$y_1 = v, \tag{2.10}$$

$$y_2 = \dot{v}. \tag{2.11}$$

Equation (2.9) can be written formally as a dynamical system [7]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \left(\left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\frac{\partial^2}{\partial x^2}\right) & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{2.12}$$

and the stability of a material state is studied, Eq. (2.12) is applied to small perturbations of the form:

$$\begin{bmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{y}_{10} \\ \widetilde{y}_{20} \end{bmatrix} \exp\left(\omega x\right)\exp\left(\lambda t\right).$$

Then, Eq. (2.12) reads:

$$\lambda \begin{bmatrix} \widetilde{y}_{10} \\ \widetilde{y}_{20} \end{bmatrix} \exp\left(\omega x\right)\exp\left(\lambda t\right) = \begin{bmatrix} 0 & 1 \\ \left(\left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\omega^2\right) & 0 \end{bmatrix}\begin{bmatrix} \widetilde{y}_{10} \\ \widetilde{y}_{20} \end{bmatrix} \exp\left(\omega x\right)\exp\left(\lambda t\right).$$

Now, the characteristic equation of (2.12) has the form:

$$\begin{vmatrix} -\lambda & 1 \\ \left(\left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\omega^2\right) & -\lambda \end{vmatrix} = 0. \tag{2.13}$$

From Eq. (2.13):

$$\lambda^2 - \left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\omega^2 = 0. \tag{2.14}$$

Now, the stability condition is $\operatorname{Re}\lambda < 0$, for all solutions of Eq. (2.14). The two generic [1] instabilities are the static (at $\lambda = 0$) or the dynamic one (at $\lambda_{1,2} = \pm i\beta$), when a real eigenvalue, or a pair of imaginary eigenvalues reach the stability boundary. In non-linear studies, such cases are referred to as static and dynamic bifurcations.

In the first study, the tangent stiffness $c' := \left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}$ acts as a bifurcation parameter. In the case

$$c' < 0, \tag{2.15}$$

Eq. (2.14) has a pair of pure imaginary roots:

$$\lambda_{1,2} = \pm i\omega\sqrt{-c'}.$$

When $c' > 0$, Eq. (2.14) has one positive and one negative real roots:

$$\lambda_{1,2} = \pm\omega\sqrt{c'},$$

while at $c' = 0$, a double zero eigenvalue is obtained.

Such a way of loss of stability of a dynamical system is a highly degenerate one. Firstly, at Eq. (2.15), no stability (by Lyapunov's definition) is present. In the theory of dynamical systems a situation like this is referred to as the stability boundary, or neutral state of the system. Thus, for the constitutive Eq. (2.2), no stable state can be found, which contradicts all real-life experiences. Moreover, for such a material model, a co-existent degenerate static and dynamic bifurcation can be recognized and no critical eigenvector can be defined for the critical eigenvalues [7]. Thus, the material model (Eq. (2.2)) cannot be used in material instability analysis, and rate-dependent terms should be added [28], and new variables should appear in the constitutive function F in Eq. (2.1):

$$F(\sigma, \dot{\sigma}, \varepsilon, \dot{\varepsilon}) = 0.$$

For example, a linearized form

$$\frac{\partial F}{\partial \dot{\sigma}}\dot{\sigma} + \frac{\partial F}{\partial \sigma}\sigma = \frac{\partial F}{\partial \dot{\varepsilon}}\dot{\varepsilon} + \frac{\partial F}{\partial \varepsilon}\varepsilon,$$

or simply

$$a_1\dot{\sigma} + a_2\sigma = a_3\dot{\varepsilon} + a_4\varepsilon, \tag{2.16}$$

should be used in stability analysis, where the coefficients a_1, a_2, a_3, a_4 denote the partial derivatives of the constitutive function.

3. Material model with fractional derivatives

Several studies have dealt with connecting the hereditary approach of creep and relaxation [34] to rate-dependence [19, 37] and proved the equivalence of the two. When a 'fractional-order rate,' with the Riemann–Liouville or Caputo derivative D_t^α $0 < \alpha < 1$, is used:

$$\sum_{i=0}^{n} a_i D_t^{\alpha_i}\varepsilon = \sum_{j=0}^{n} b_j D_t^{\alpha_j}\sigma \tag{3.1}$$

is obtained instead of Eq. (2.16). Remark that such form of the constitutive equation is a generalization of Bagley's viscoelastic material [4, 5]. However, it is important to note that material instability is outside of the domain of elastic deformation.

For stability analysis, Eq. (2.3) and Eq. (2.4) should be transformed into the velocity field. In view of Eq. (2.16) and Eq. (3.1), assume that the constitutive equation is

$$\sigma = E_0\varepsilon + E_1 D_t^\alpha \varepsilon, \tag{3.2}$$

where E_0 is the tangent stiffness and E_1 is the fractional-rate sensitivity parameter. After differentiation

$$\dot{\sigma} = E_0\dot{\varepsilon} + E_1 D_t^\alpha \dot{\varepsilon}. \tag{3.3}$$

By taking its 'gradient' (derive with respect to x):

$$\frac{\partial \dot{\sigma}}{\partial x} = \left(E_0 + E_1 D_t^\alpha\right)\frac{\partial \dot{\varepsilon}}{\partial x}. \tag{3.4}$$

From Eq. (2.6) and Eq. (2.7):

$$\rho \ddot{v} = \left(E_0 + E_1 D_t^\alpha\right)\frac{\partial^2 v}{\partial x^2}. \tag{3.5}$$

By using the harmonic perturbation technique:

$$\widetilde{v} = \widetilde{v}_0 v_t(t) \exp(i\omega\, x),$$

for Eq. (3.5), and by using the notation D_t^2 for the second time derivative, Eq. (3.5) is equivalent to:

$$D_t^2 v_t + \frac{E_1}{\rho}\omega^2 D_t^\alpha v_t + \frac{E_0}{\rho}\omega^2 v_t = 0. \tag{3.6}$$

In Eq. (3.6), homogeneous perturbations are used, thus $v_t(0) = 0$. From Eq. (1.3), the notation

$$D_t^\alpha v_t := {}_0D_t^\alpha v_t = {}_0^C D_t^\alpha v_t$$

is justified, in Eq. (3.1) and Eq. (3.2) and later.

Stability analysis can be performed as in [11, 26, 32] and in Radwan's research [35]. By performing a Laplace transformation, the characteristic equation of (3.6) reads:

$$s^2 + \frac{E_1}{\rho}\omega^2 s^\alpha + \frac{E_0}{\rho}\omega^2 = 0. \tag{3.7}$$

Then, by following the method in [35], the transformation $W = s^{\frac{1}{m}}$ is used, where $\alpha = \frac{k}{m}$ is rational. Then, Eq. (3.7) takes the form:

$$W^{2m} + \frac{E_1}{\rho}\omega^2 W^k + \frac{E_0}{\rho}\omega^2 = 0. \tag{3.8}$$

The procedure is based on the fact that the imaginary axes of the s-plane are mapped onto lines:

$$|W_\theta| = \frac{\pi}{2m}, \tag{3.9}$$

where W_θ denotes the argument of W in the complex plane ($\arg(W)$). In Fig. 1, the stability map is presented, while Fig. 2 shows the location of the static bifurcation (at the origin) and the lines of dynamic bifurcation. The system will be stable, if and only if all roots of Eq. (3.8) in the W-plane lie in the region:

$$|W_\theta| > \frac{\pi}{2m}, \tag{3.10}$$

thus, the stability condition reads:

$$\min \arg(W) > \frac{\pi}{2m}.$$

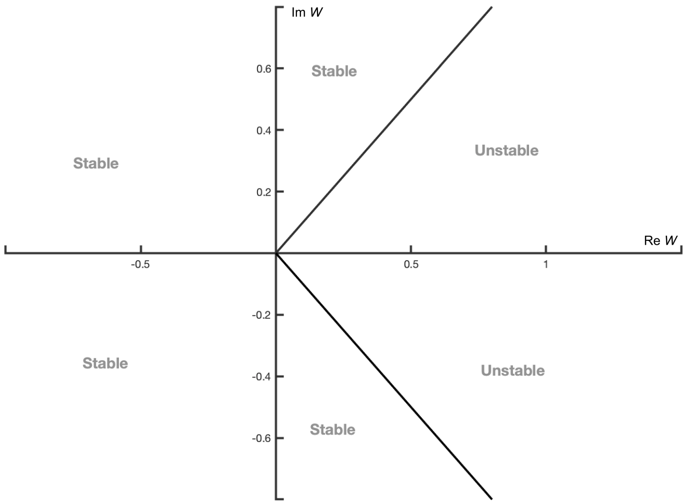


FIG. 1. Domains of stability.

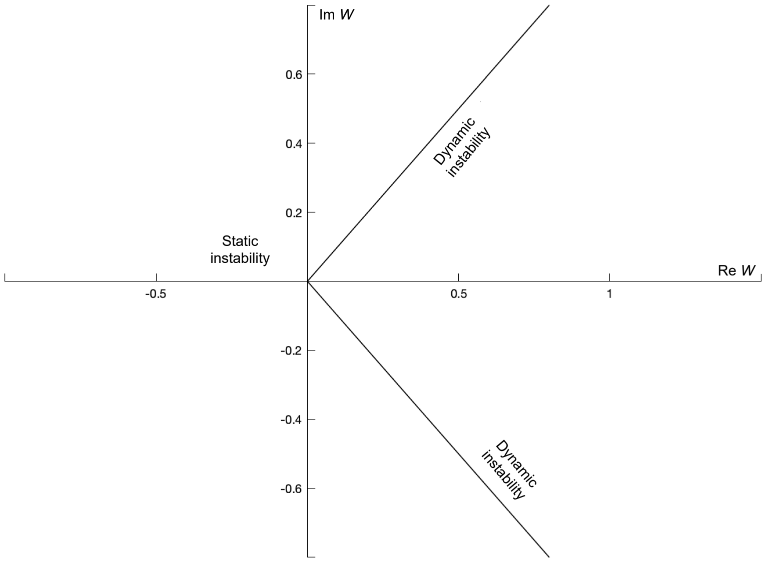


FIG. 2. Static and dynamic instability boundaries.

Now, static instability happens at $W_{cr\,s} = 0$, and its condition from Eq. (3.8) is

$$E_0 = 0. \tag{3.11}$$

Unfortunately, no critical eigenfunction can be attached to this zero eigenvalue from the periodic perturbation functions, so nonlinear analysis cannot be performed.

The dynamic instability condition can also be derived. At dynamic instability, the critical solution $W_{cr\,1,2}$ should lie on the stability boundary lines, thus Eq. (3.9) should be satisfied:

$$W(r) = r\left(\cos\left(\frac{\pi}{2\,m}\right) \pm \sin\left(\frac{\pi}{2\,m}\right) i\right), \quad r \geq 0. \tag{3.12}$$

Now, $W(r)$ from Eq. (3.12) should be substituted into the integer-order characteristic equation (Eq. (3.8)):

$$\left(r\left(\cos\left(\frac{\pi}{2\,m}\right) + \sin\left(\frac{\pi}{2\,m}\right) i\right)\right)^{2\,m} + a_0\,\omega^2 + a_1\,\omega^2\left(r\left(\cos\left(\frac{\pi}{2\,m}\right) + \sin\left(\frac{\pi}{2\,m}\right) i\right)\right)^{k} = 0,$$

where $a_0 = \frac{E_0}{\rho}$ and $a_1 = \frac{E_1}{\rho}$.

After proper rearrangements:

$$r^{2m}\cos\pi + a_0\,\omega^2 + a_1\,\omega^2\,r^k\left(\cos\left(\frac{\pi}{2\,m}\right) + \sin\left(\frac{\pi}{2\,m}\right) i\right)^k = 0. \tag{3.13}$$

From the imaginary part of Eq. (3.13) $a_1 = 0$ is obtained; thus, the dynamic instability condition is

$$E_1 = 0. \tag{3.14}$$

The critical radius at dynamic instability can be calculated from Eq. (3.13):

$$-r^{2m} + a_0\,\omega^2 = 0 \quad \Rightarrow r = \left(a_0\,\omega^2\right)^{\frac{1}{2m}}, \tag{3.15}$$

and is plotted in Fig. 3 (continuous line $m = 2$, dashed line $m = 4$, dotted line $m = 8$). From Eq. (3.15), we can see that by increasing the frequency ω of the perturbation, the radius of the critical eigenvalue increases. On the other hand, the results show that dynamic instability is a material instability, while condition Eq. (3.14) applies only to the material property E_1.

Moreover, dynamic instability can be treated as a generic bifurcation, which means that it is different from static instability and for the critical eigenfunction:

$$v(x) = \exp\left(i\omega x\right),$$

the critical eigenvalue is

$$W_{cr\,1,2} = \left(a_0\,\omega^2\right)^{\frac{1}{2m}}\left(\cos\left(\frac{\pi}{2\,m}\right) \pm \sin\left(\frac{\pi}{2\,m}\right) i\right).$$

In such a case, a non-linear stability analysis is possible by projecting into the non-trivial critical eigenspace. This result differs from the static instability case. We might state that the constitutive Eq. (3.2) can be used in dynamic bifurcation analysis, but not in static bifurcation analysis.

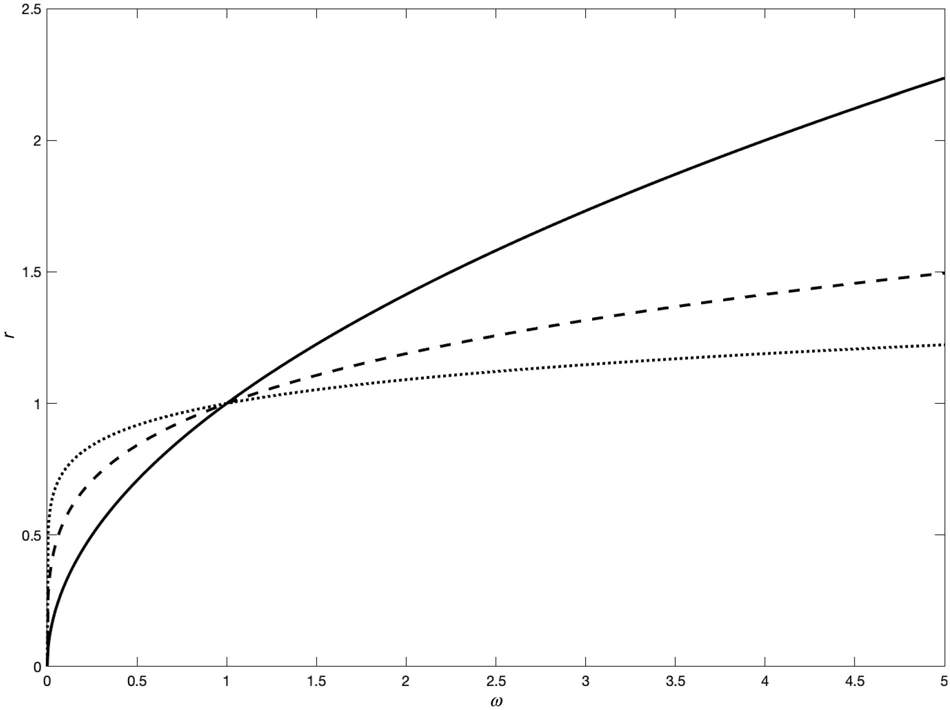


FIG. 3. Critical radius at dynamic instability for $m = 2, 4, 8$.

4. Eigenvalue distribution plots

To demonstrate the results, by solving Eq. (3.8) numerically for W, the solutions are plotted in Fig. 4 to Fig. 6. In all figures, 8 eigenvalues are marked with dots in the $\operatorname{Re} W$, $\operatorname{Im} W$ plane because the order of the derivative was selected as $\alpha = 0.25$. Two periodic perturbation frequencies ($\omega = 0.3$ and $\omega = 0.8$) are selected in each figures.

In Fig. 4, both parameters are positive, $a_0 > 0$, $a_1 > 0$; consequently, all eigenvalues are in the stability domain for both frequencies. Here, the radii of the eigenvalues increase as ω gets larger, but this has no significant effect on their location. The same observation holds for Fig. 5 at $a_0 > 0$, $a_1 < 0$, but here the material is in an unstable state, which can also be detected from the presence of a pair of eigenvalues in the unstable region.

In Fig. 6, the eigenvalue distributions are plotted at the loss of stability parameters. In Fig. 6a, the material parameter $a_0 = 0$, which shows static-type instability. Then, all eigenvalues are in the stable domain except one zero eigenvalue. In Fig. 6b, one pair of eigenvalues lies on the stability boundary under the $a_1 = 0$ dynamic instability condition. Figure 7 shows two types of unstable cases called static and dynamic post-bifurcations. Here, the situations 'after' the

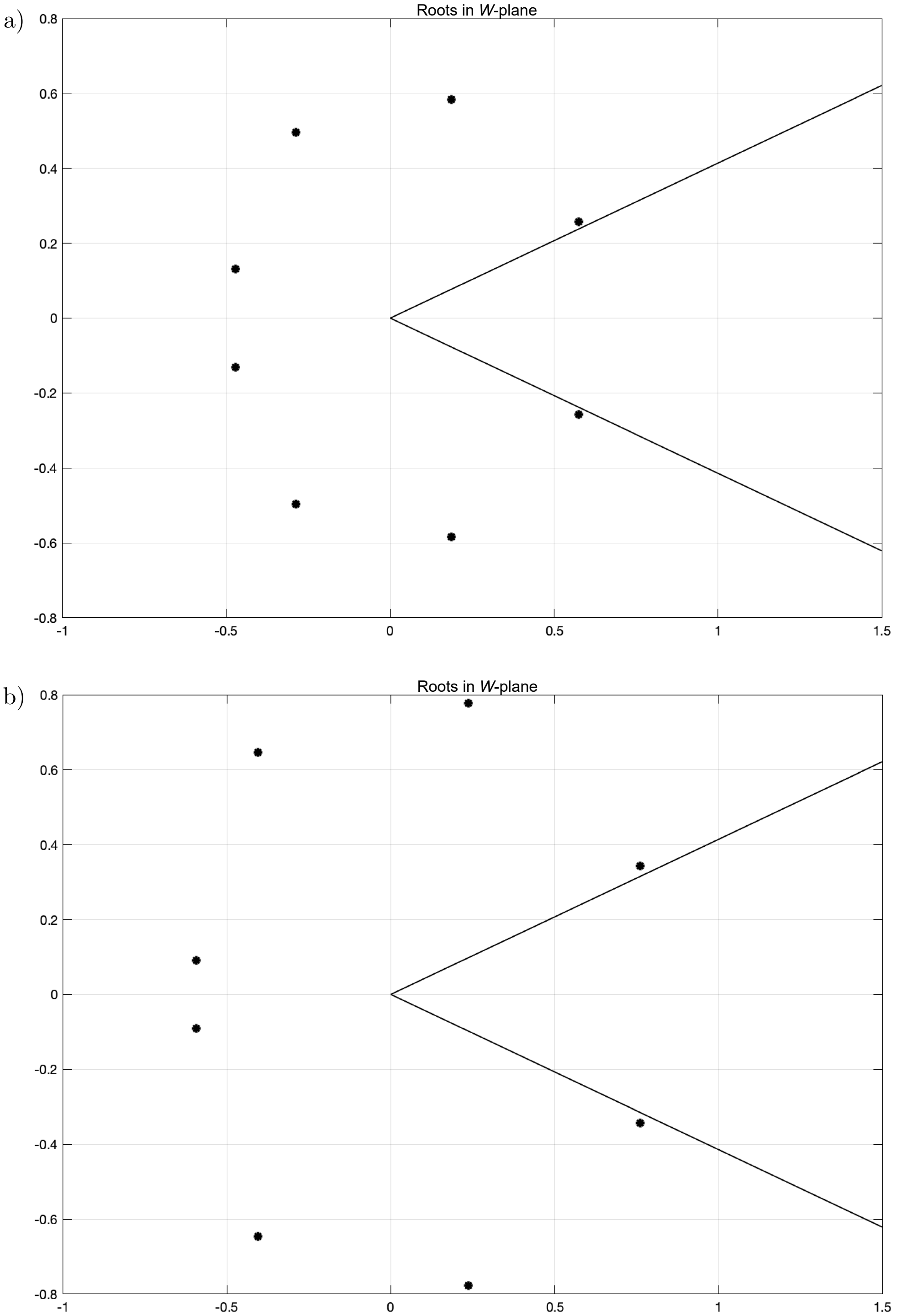
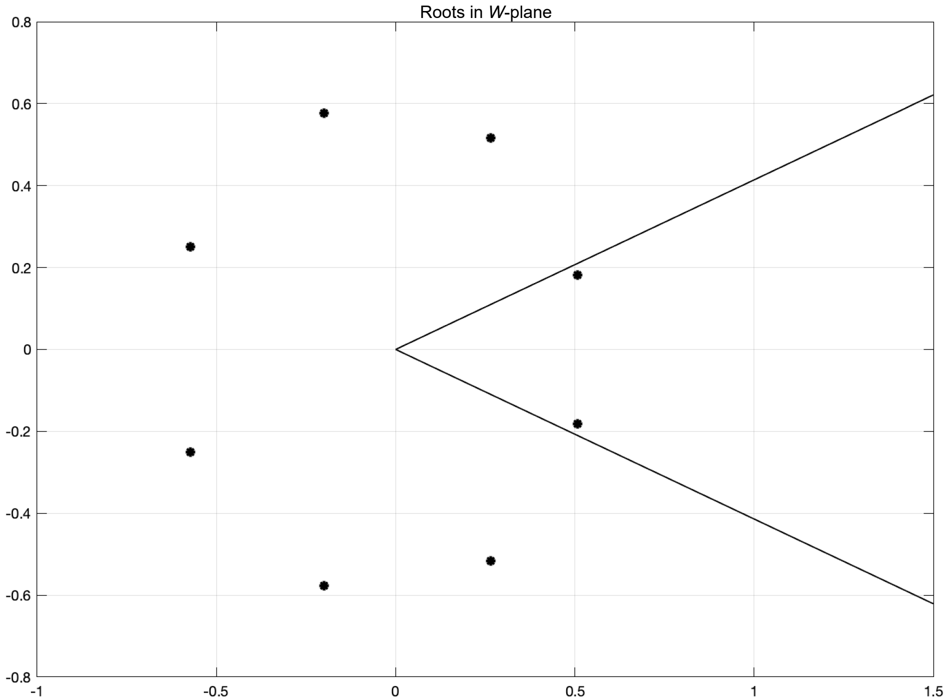


Fig. 4. Eigenvalue distribution in stable state, $E_0 > 0$, $E_1 > 0$: a) $\omega = 0.3$, b) $\omega = 0.8$.

a)



b)

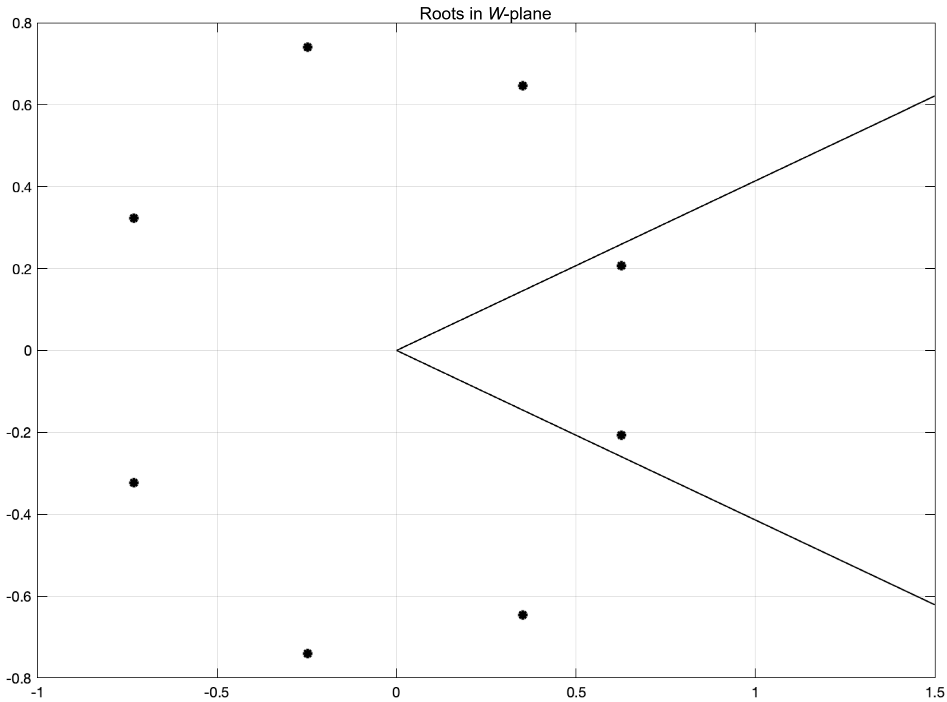
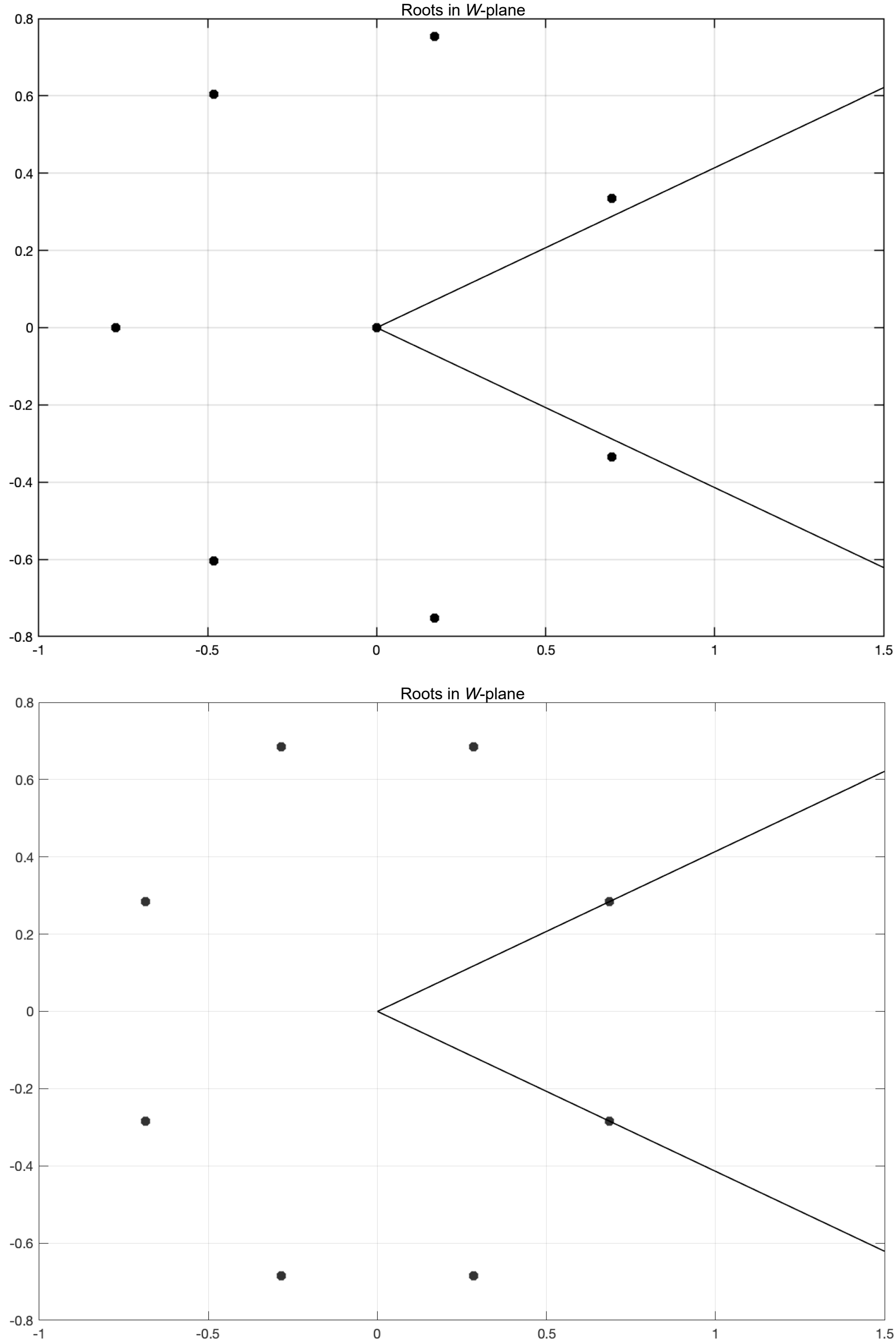


FIG. 5. Eigenvalue distribution in unstable state, $E_0 > 0$, $E_1 < 0$: a) $\omega = 0.3$, b) $\omega = 0.8$.

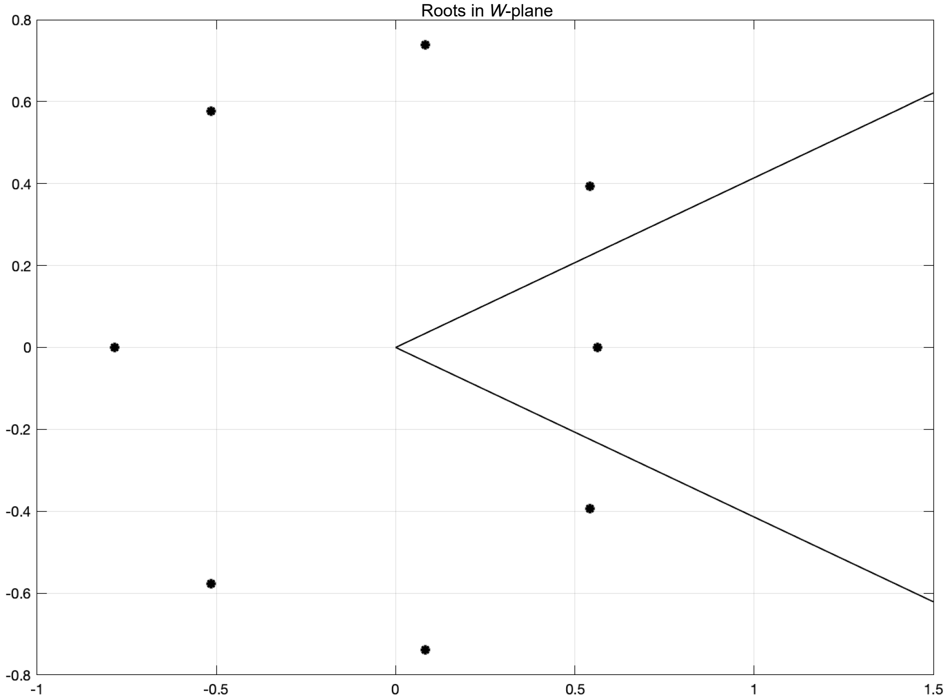
a)



b)

Fig. 6. Eigenvalues at loss of stability:
a) $a_0 = 0$ static bifurcation, b) $a_1 = 0$ dynamic bifurcation.

a)



b)

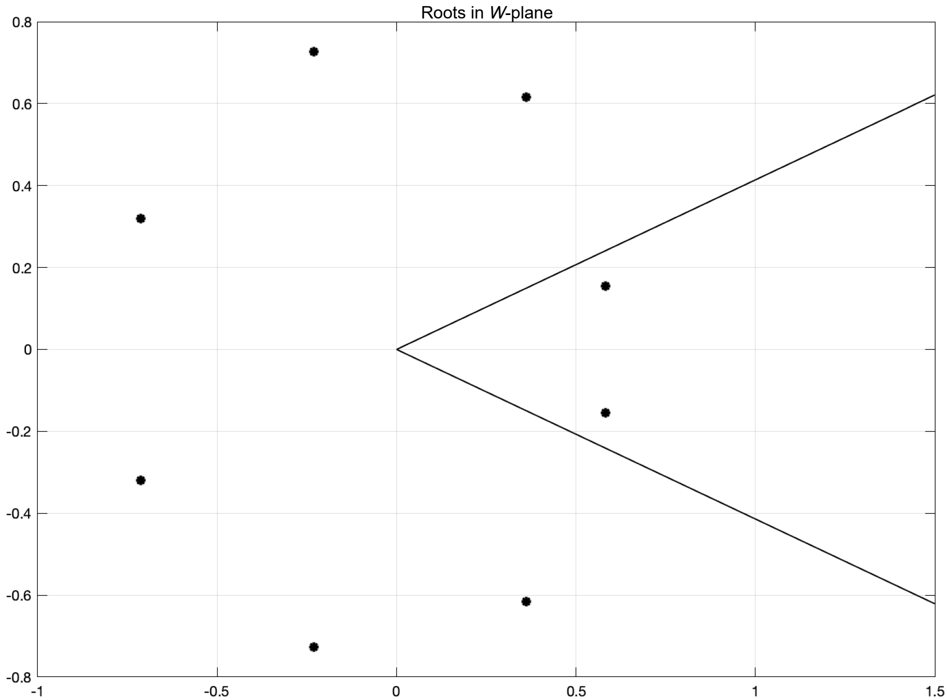


Fig. 7. Eigenvalues after loss of stability:
a) static post-bifurcation, b) dynamic post-bifurcation.

loss of stability are presented, that is, in plot (a) the material parameter a_0 is infinitesimally less than zero, while in plot (b) the material parameter a_1 is infinitesimally less than zero. In both cases, the state is unstable, but plot (a) may correspond to shear banding or necking instabilities [39], while plot (b) describes propagative material instabilities [17] as in Portevin–Le Chatelier effect [20].

5. Conclusion

Fractional derivatives can be and are already used to describe non-conventional rate dependence. When periodic perturbations are applied to stability investigations, they have no effect on stability conditions, which are determined by the material parameters only. This result is the same as in classical case. This result is in line with what is expected from classical theory, while the way of approximation should not affect the outcome of material instability investigation. Frequency acts on the absolute value of eigenvalues, which has no consequences on qualitative behavior. The most important result achieved is that at dynamic instability, the frequency defines critical eigenfunctions to the eigenvalues at the stability boundary. Thus a non-linear study can be performed by projecting the equations to the non-trivial critical null-space spanned by such critical eigenfunctions.

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Conflict of interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author's contribution

The author reviewed and approved the final manuscript.

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