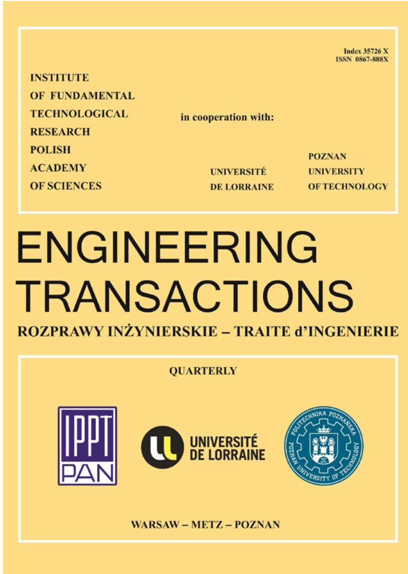

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On Frequency Dependence of Stability in Materials with Fractional Viscosity

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Material instability refers to the tendency of materials to undergo alterations in its properties in loading. The concept of instability is governed by the constitutive equation of solids. Our analysis uses the entire set of equations describing the motion of solids by adding the kinematical equation and Cauchy's equations of motion. Damping or rate-dependence plays a crucial role in stability. A potential generalization involves the utilization of fractional order derivatives of strain or stress tensors. The stability analysis primarily focuses on periodic perturbations. The way of the loss of stability on various parts of the stability boundary is under consideration.

Keywords: applied fractional calculus; material instability; dynamical systems.

1 Introduction

When the kinematical concept of stability is accepted, a state of the material is identified with a solution of the basic equations, and stability properties are studied as of the stability of the solution of differential equations (dynamical systems) [6]. Material instability (divergence or flutter) [39] happens well beyond elastic domain, thus the form of the constitutive equation might remind visco-elasticity but the physical background is different. The paper does not concentrates on various plasticity theories and does not treat forms of (even the existence of) flow rules, hardening models, which have various interpretations in [21]. These problems are well beyond the scope. A simplified constitutive equation is used, with an assumption that after unloading, the body does not regain its original shape.

In physical interpretation static bifurcation (or divergence instability) can be observed as necking or shear banding, being phenomena of strain localization. Flutter is the dynamic bifurcation and is considered here as the onset of an oscillating being observed in plastic flow theory [28, 29] visco-plasticity [14] smeared crack models [43] or serrated flow called the Portevin–Le Chatelier effect. The study is based on Kubin and Estrin work [20], which uses a semi-empirical constitutive law

$$\sigma = h\varepsilon + \bar{F}(\dot{\varepsilon})$$

with negative rate-dependence included in function $\bar{F}$. Here ε and $\dot{\varepsilon}$ denote strain and strain rate, while σ is stress in uniaxial case and h is work hardening rate. As a phenomenon, flutter instability means the existence of propagating deformation bands. Their work cites lots of papers based on experimental results on serrated flow, a few of them are listed: [8, 15, 24, 42, 36]. One of the main observations of them is that serrated flow appears at negative strain-rates. In this paper such model is generalized to a fractional rate.

Bifurcation analysis of a solution is a well-known and widely applied field in nonlinear dynamics. The first step of it starts with a linearization of the system of the basic equations at that solution, identified by the state of the material [7]. Then the critical non-trivial eigenspace of the operator is studied. As a further step, the non-linearities should be projected into that non-trivial eigenspace to classify the type of bifurcation and describe the postbifurcation behavior. Two key elements should be mentioned at that point. Firstly, the loss-of-stability should happen by crossing the imaginary axis of either a real eigenvalue, or a pair of complex eigenvalues as load (the bifurcation parameter) changes quasi-statically [47]. Second, at the critical value of the bifurcation parameter (zero real value of the eigenvalues) the critical eigenspace should be of finite dimensional.

Generally, such studies are of ordinary differential equations with integer order derivatives.

In material instability problems bifurcation describes the types of instability. These two phenomena are identified as static and dynamic bifurcations. In the static case the loss of (Lyapunov) stability is coupled with the change of the number of the solutions [25, 27, 30] while at dynamic bifurcation a self-sustained oscillation can be observed. In a large range of materials damping is described by fractional order derivatives.

The aim of the paper is to perform such analysis for a set of fractional order equations. A method will be presented to find material instability condition. Then the way to calculate the critical eigenvalues leads to get conditions for static and dynamic bifurcation, even for fractional dynamical systems.

While fractional calculus has got lots of new results and gets more and more applications in mechanics, control, economics, and several fields in sciences, one might have the feeling that this topic is just a fashionable tool of the recent years, with no deep physical necessity. However, the roots have already been present in solid mechanics for more than fifty years and can be originated at the birth of continuum field theory in middle of the last century. The study of creep and relaxation in Rabotnov's hereditary mechanics [34] is based on integral operators in form of convolutions with a fractional order kernel, being equivalent to fractional derivatives [38]. The early application was published by Caputo [9] in viscoelasticity [2, 9, 23] and then even in viscoplasticity [44] as a kind of fractional viscosity or non-local time effect. When non-locality is studied in Eringen's approach [16], similar mathematical tools could be used. Furthermore, non-locality may be extended from non-local time to spatial non-locality using fractional (non-local) derivatives [3].

The appearance of fractional calculus goes back to the origin of calculus by Leibniz and Euler as a possible generalization. Most of the definitions were given by Liouville, Riemann, and others [22]. Fractional derivatives can easily be deduced from Cauchy's repeated integral formula and its generalization. For n^{th} (integer) order, it leads to

$$ {}_aI_t^n f(t) = \frac{1}{(n-1)!} \int_a^t f(\xi)(t-\xi)^{-n-1} d\xi. $$

The α^{th} fractional order generalization is the Riemann-Liouville integral operator, $(\alpha < 1)$:

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\xi)(t-\xi)^{\alpha-1} d\xi = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\xi)}{(t-\xi)^{1-\alpha}} d\xi. $$

By taking derivative of Riemann-Liouville integral operator

$$ {}_aD_t^{\alpha} f(t) = \frac{d}{dt} {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(\xi)(t-\xi)^{-\alpha} d\xi \tag{1} $$

the Riemann-Liouville derivative for interval $[a,t]$.

By changing operators of derivation an integration Caputo's derivative is defined:

$$ {}_a^C D_t^{\alpha} f(t) = {}_aI_t^{\alpha} \left(\frac{d}{d\xi} f \right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{df(\xi)}{d\xi} (t-\xi)^{-\alpha} d\xi \tag{2} $$

for interval $[a,t]$.

From [18] the connection of derivatives (1) and (2) is

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{f(a)}{(t-a)^{\alpha}} + {}_a^C D_t^{\alpha} f(t). \tag{3} $$

In several cases, called full-memory assumption in applications, the starting time is zero, $a = 0$, and the notations are simply $D_t^{\alpha} f(t)$ and ${}^C D_t^{\alpha} f(t)$. At this point only the most important definitions are given, more details can be found in several monographs [10, 12, 33, 40, 46]

The method is mainly analytic by using Fourier transformation. It is restrictive compared to numerical analysis [41, 27, 30], and excludes for example short-memory effects [45, 48]. However, it makes a deeper insight possible into the roots of unstable behavior, especially at dynamic bifurcation. For the same reason, only uniaxial case is studied. In 3D problems the orientation of shear bands is a key factor [29, 31, 39], which requires detailed investigation of the constitutive acoustic tensor, already at static bifurcation analysis. Then 3D fractional generalization of continuum mechanics is another wide field of research [13].

2 Rate dependence and material instability

This section explains, why rate independent constitutive equation is not suitable in material instability problems. Firstly, a rate-independent material, with constitutive equation

$$ F(\sigma, \varepsilon) = 0 \tag{4} $$

is studied to point out its singular behavior at instability. Here F is a general form of constitutive function. Assume that a uniaxial problem is studied and the linearized constitutive equation is simply in form

$$\sigma = C\varepsilon, \tag{5}$$

where

$$C := -\left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}$$

denotes tangent stiffness. Now the equation of motion, the kinematic equation and the so-called rate-form of Eq. (5) are:

$$\dot{v} = \frac{1}{\rho}\frac{\partial \sigma}{\partial x}, \tag{6}$$

$$\dot{\varepsilon} = \frac{\partial v}{\partial x}, \tag{7}$$

$$\dot{\sigma} = \left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\dot{\varepsilon}, \tag{8}$$

where ρ denotes mass density, and v is the velocity field.

By taking time derivative of Eq. (6):

$$\ddot{v} = \frac{1}{\rho}\frac{\partial \dot{\sigma}}{\partial x}, \tag{9}$$

the gradient of Eq. (7):

$$\frac{\partial \dot{\varepsilon}}{\partial x} = \frac{\partial^2 v}{\partial x^2}, \tag{10}$$

the gradient of Eq. (8):

$$\frac{\partial \dot{\sigma}}{\partial x} = \left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\frac{\partial \dot{\varepsilon}}{\partial x}. \tag{11}$$

By substituting Eqs. (9) and (10) into Eq. (11)

$$\rho\ddot{v} = \left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\frac{\partial^2 v}{\partial x^2}. \tag{12}$$

By introducing new variables:

$$y_1 = v, \tag{13}$$

$$y_2 = \dot{v}, \tag{14}$$

Eq. (12) can be written formally as a dynamical system [7]:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \left(\left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\frac{\partial^2}{\partial x^2}\right) & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{15}$$

and the stability of a state of the material is studied, Eq. (15) is applied to small perturbations in form:

$$\begin{bmatrix} \widetilde{y_1} \\ \widetilde{y_2} \end{bmatrix} = \begin{bmatrix} \widetilde{y_{10}} \\ \widetilde{y_{20}} \end{bmatrix} \exp(\omega x)\exp(\lambda t).$$

Then Eq. (15) reads

$$\lambda \begin{bmatrix} \widetilde{y_{10}} \\ \widetilde{y_{20}} \end{bmatrix} \exp(\omega x)\exp(\lambda t) =$$
$$\begin{bmatrix} 0 & 1 \\ \left(\left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\omega^2\right) & 0 \end{bmatrix}\begin{bmatrix} \widetilde{y_{10}} \\ \widetilde{y_{20}} \end{bmatrix} \exp(\omega x)\exp(\lambda t).$$

Now the characteristic equation of (15) has the form:

$$\begin{vmatrix} -\lambda & 1 \\ \left(\left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\omega^2\right) & -\lambda \end{vmatrix} = 0. \tag{16}$$

From Eq. (16):

$$\lambda^2 - \left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}\omega^2 = 0. \tag{17}$$

Now the stability condition is $\mathrm{Re}\lambda < 0$, for all solutions of (17). The two generic [1] instabilities are the static (at $\lambda = 0$) or the dynamic one (at $\lambda_{1,2} = \pm i\beta$), when a real eigenvalue, or a pair of imaginary eigenvalues reach the stability boundary. In non-linear studies such cases are referred as static and dynamic bifurcations.

In the first study tangent stiffness $c' := \left(\frac{\partial F}{\partial \varepsilon}\right)\left(\frac{\partial F}{\partial \sigma}\right)^{-1}$ acts as a bifurcation parameter. In case:

$$c' < 0, \tag{18}$$

Eq. (17) has a pair of pure imaginary roots:

$$\lambda_{1,2} = \pm i\omega\sqrt{-c'}.$$

When

$$c' > 0,$$

Eq. (17) has one positive and one negative real roots:

$$\lambda_{1,2} = \pm\omega\sqrt{c'},$$

while at $c' = 0$, a multiplicity two zero eigenvalue is obtained.

Such way of loss of stability of a dynamical system is a highly degenerate one. Firstly, at (Eq. (18)) no stability (by Lyapunov's definition) is present. In the theory of dynamical systems a situation like that is referred as stability boundary, or neutral state of the system. Thus for constitutive Eq. (5) no stable state can be found, which contradicts all real life experiences. Moreover, for such material model a co-existent degenerate static and dynamic bifurcation can be recognized and no critical eigenvector can be defined to the critical eigenvalues [7]. Thus material model (Eq. (5)) cannot be used in material instability analysis and rate dependent terms should be added [28], and new variables should appear in the constitutive function F in Eq. (4):

$$F(\sigma, \dot{\sigma}, \varepsilon, \dot{\varepsilon}) = 0.$$

For example, a linearized form:

$$\frac{\partial F}{\partial \dot{\sigma}}\dot{\sigma} + \frac{\partial F}{\partial \sigma}\sigma = \frac{\partial F}{\partial \dot{\varepsilon}}\dot{\varepsilon} + \frac{\partial F}{\partial \varepsilon}\varepsilon,$$

or simply

$$a_1\dot{\sigma} + a_2\sigma = a_3\dot{\varepsilon} + a_4\varepsilon \tag{19}$$

should be used in stability analysis, where coefficients a_1, a_2, a_3, a_4 denote the partial derivatives of the constitutive function.

3 The material model with fractional derivatives

Several studies have dealt with to connect hereditary approach of creep and relaxation [34] to rate dependence [19, 37] and proved the equivalence of them. When 'fractional order rate' with Riemann–Liouville or Caputo derivative D_t^{α} $\quad 0 < \alpha < 1$ is used:

$$\sum_{i=0}^{n} a_i D_t^{\alpha_i}\varepsilon = \sum_{j=0}^{n} b_j D_t^{\alpha_j}\sigma \tag{20}$$

is obtained instead of Eq. (19). Remark that such form of constitutive equation is a generalization of Bagley's visco-elastic material [4, 5]. However, an important fact that material instability is outside of the domain of elastic deformation.

For stability analysis Eqs. (6) and (7) should be transformed into the velocity field. In view of Eqs. (19) and (20) assume that the constitutive equation is

$$\sigma = E_0\varepsilon + E_1 D_t^{\alpha}\varepsilon, \tag{21}$$

where E_0 is tangent stiffness and E_1 is the fractional rate sensitivity parameter. After derivation:

$$\dot{\sigma} = E_0\dot{\varepsilon} + E_1 D_t^{\alpha}\dot{\varepsilon}. \tag{22}$$

By taking its 'gradient' (derive with respect to x):

$$\frac{\partial\dot{\sigma}}{\partial x} = \left(E_0 + E_1 D_t^{\alpha}\right)\frac{\partial\dot{\varepsilon}}{\partial x}. \tag{23}$$

From Eqs. (9) and (10):

$$\rho\ddot{v} = \left(E_0 + E_1 D_t^{\alpha}\right)\frac{\partial^2 v}{\partial x^2}. \tag{24}$$

By using harmonic perturbation technique:

$$\widetilde{v} = \widetilde{v}_0 v_t\left(t\right)\exp\left(i\omega x\right),$$

for Eq. (24) and by using notation D_t^2 for second time derivative, Eq. (24) is equivalent to:

$$D_t^2 v_t + \frac{E_1}{\rho}\omega^2 D_t^{\alpha} v_t + \frac{E_0}{\rho}\omega^2 v_t = 0. \tag{25}$$

In Eq. (25) homogeneous perturbations are used, thus $v_t(0) = 0$. From (3), notation

$$D_t^{\alpha} v_t := {}_0D_t^{\alpha} v_t = {}_0^C D_t^{\alpha} v_t$$

is justifed, in (20), (21) an later on.

Stability analysis can be performed as in [11, 26, 32] and in Radwan's research [35]. By performing Laplace transformation, the characteristic equation of Eq. (25) reads

$$s^2 + \frac{E_1}{\rho}\omega^2 s^{\alpha} + \frac{E_0}{\rho}\omega^2 = 0. \tag{26}$$

Then by following the method in [35], transformation $W = s^{\frac{1}{m}}$ is used, where $\alpha = \frac{k}{m}$ is rational. Then Eq. (26) gets the form:

$$W^{2m} + \frac{E_1}{\rho}\omega^2 W^k + \frac{E_0}{\rho}\omega^2 = 0. \tag{27}$$

The procedure is based on the fact that imaginary axes of plane s are mapped onto lines:

$$|W_\theta| = \frac{\pi}{2m} \tag{28}$$

where W_θ denotes the argument of W in the complex plane ($\arg(W)$). In Fig. 1 the stability map is presented, while Fig. 2 shows the location of static bifurcation (at the origin) and the lines of dynamic bifurcation. The system will be stable, if and only if all roots of Eq. (27) in the W-plane lie in the region:

$$|W_\theta| > \frac{\pi}{2m}, \tag{29}$$

thus the stability condition reads:

$$\min \arg(W) > \frac{\pi}{2m}.$$

Now, static instability happens at $W_{crs} = 0$, and its condition from Eq. (27) is

$$E_0 = 0. \tag{30}$$

Unfortunately, no critical eigenfunction can be attached to that zero eigenvalue from the periodic perturbation functions, thus nonlinear analysis cannot be performed.

The dynamic instability condition can also be derived. At dynamic instability, critical solution $W_{cr1,2}$ should fit in stability boundary lines, thus Eq. (28) should be satisfied:

$$W(r) = r\left(\cos\left(\frac{\pi}{2m}\right) \pm \sin\left(\frac{\pi}{2m}\right) \mathrm{i}\right), \quad r \geq 0. \tag{31}$$

Now $W(r)$ from Eq. (31) should be substituted into the integer order characteristic equation (Eq. (27)):

$$\left(r\left(\cos\left(\frac{\pi}{2m}\right) + \sin\left(\frac{\pi}{2m}\right) \mathrm{i}\right)\right)^{2m} + a_0\,\omega^2 + a_1\,\omega^2 \left(r\left(\cos\left(\frac{\pi}{2m}\right) + \sin\left(\frac{\pi}{2m}\right) \mathrm{i}\right)\right)^{k} = 0,$$

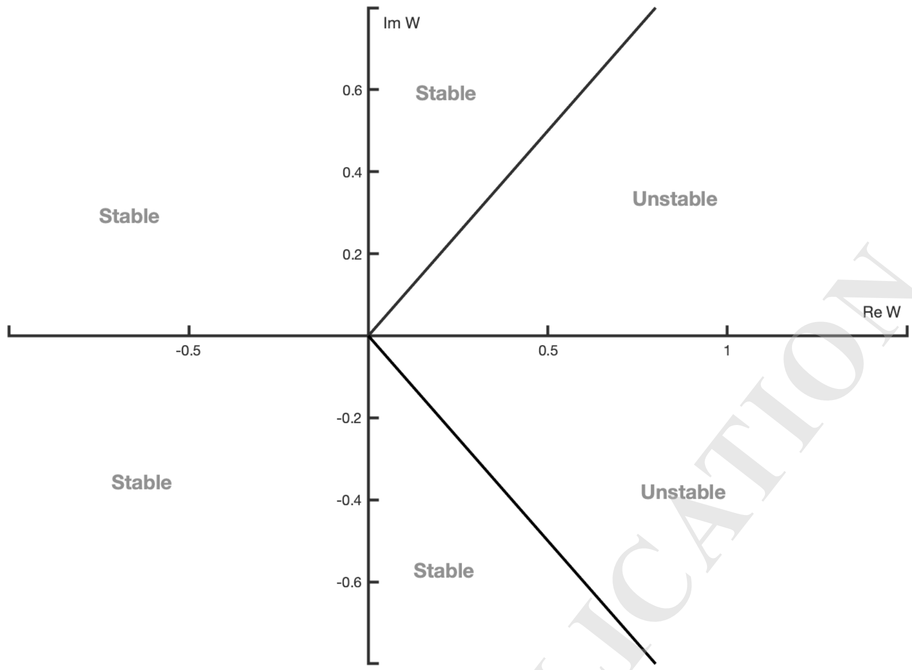


Figure 1: Domains of stability.

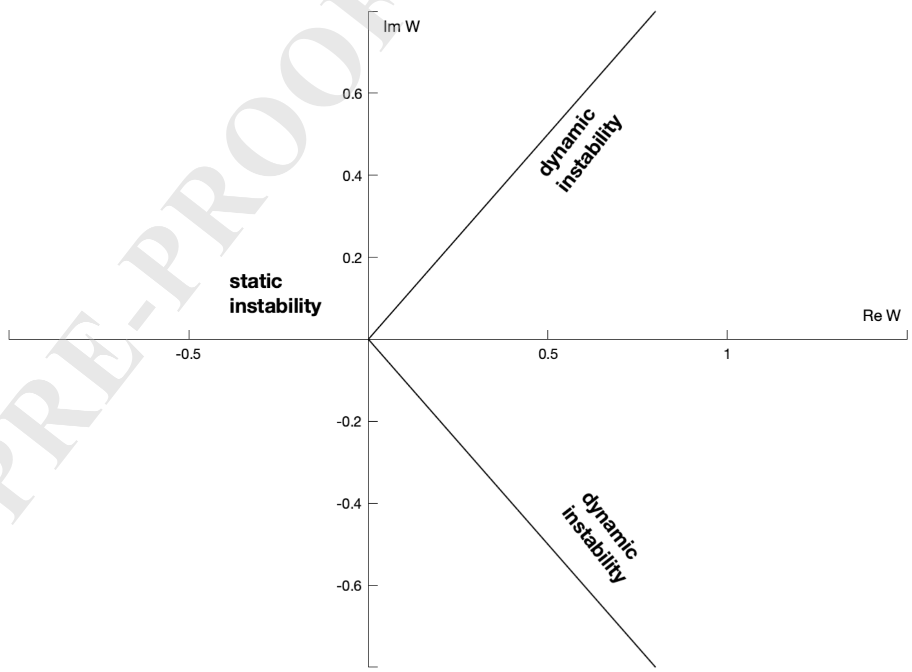


Figure 2: Static and dynamic instability boundaries.

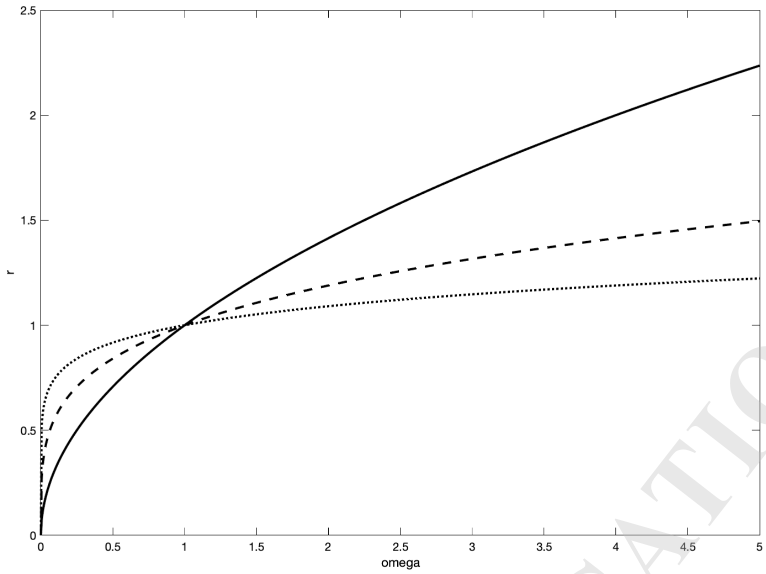


Figure 3: Critical radius at dynamic instability at $m = 2, 4, 8$.

where $a_0 = \dfrac{E_0}{\rho}$ and $a_1 = \dfrac{E_1}{\rho}$.

After proper rearrangements:

$$r^{2m}\cos\pi + a_0\,\omega^2 + a_1\,\omega^2\,r^k\left(\cos\left(\frac{\pi}{2m}\right) + \sin\left(\frac{\pi}{2m}\right)\mathrm{i}\right)^k = 0\,. \tag{32}$$

From the imaginary part of Eq. (32) $a_1 = 0$ is obtained, thus the dynamic instability condition is

$$E_1 = 0. \tag{33}$$

The critical radius at dynamic instability can be calculated from Eq. (32):

$$-r^{2m} + a_0\,\omega^2 = 0 \quad \Rightarrow r = \left(a_0\,\omega^2\right)^{\frac{1}{2m}}\,, \tag{34}$$

which is plotted in Fig. 3 (continuous line $m = 2$, dashed line $m = 4$, dotted line $m = 8$). From Eq. (34) we can see that by increasing frequency ω of the perturbation, the radius of the critical eigenvalue gets larger values. On the other hand, the results show that dynamic instability is material instability, while condition Eq. (33) only applies to material property E_1.

Moreover, dynamic instability can be treated as a generic bifurcation, which means that it is different from static instability and for critical eigenfunction:

$$v(x) = \exp(i\omega x),$$

the critical eigenvalue is

$$W_{cr\,1,2} = \left(a_0\,\omega^2\right)^{\frac{1}{2m}} \left(\cos\left(\frac{\pi}{2m}\right) \pm \sin\left(\frac{\pi}{2m}\right) \mathrm{i}\right).$$

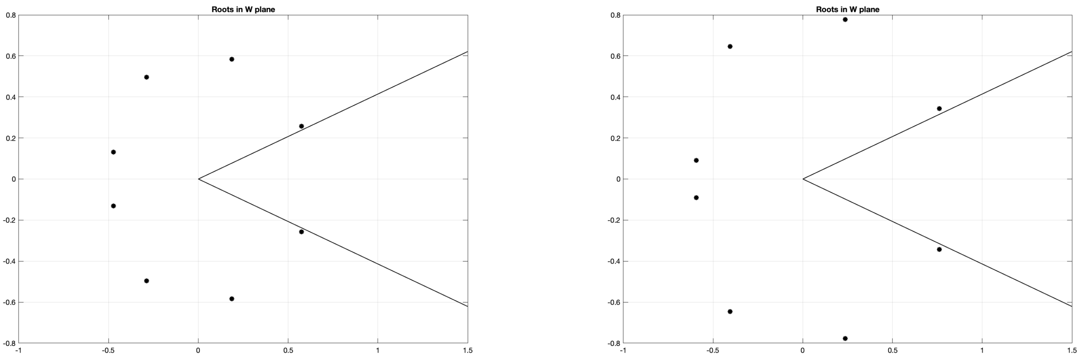
In such case a non-linear stability analysis is possible, by projecting into the non-trivial critical eigenspace. This result differs from the static instability case. We might state that constitutive Eq. (21) can be used in dynamic bifurcation analysis, but not in static bifurcation analysis.

4 Eigenvalue distribution plots

To demonstrate the results, by solving Eq. (27) numerically for W, the solutions are plotted in Figs. 4– 6. In all figures 8 eigenvalues are marked with dots in plane Re W, Im W, because the order of the derivative was selected to be $\alpha = 0.25$. Two periodic perturbation frequencies are selected ($\omega = 0.3$ and $\omega = 0.8$) at each figures.

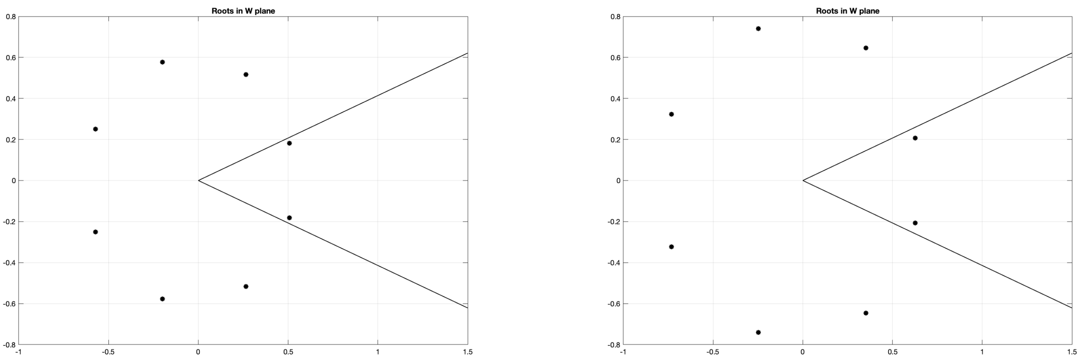
In Fig. 4 both parameters are positive, $a_0 > 0$, $a_1 > 0$, consequently all eigenvalues are in the stability domain for both frequencies. Here the radii of the eigenvalues are increased as ω gets larger, but it has no significant effect on the location of them. The same observation holds for Fig. 5 at $a_0 > 0$ $a_1 < 0$, but here the material is in unstable state, which can also be detected from the existence of a pair of eigenvalues in the unstable region.

In Fig. 6 the eigenvalue distributions are plotted at the loss of stability parameters. In Fig. 6a material parameter $a_0 = 0$, which shows the static type instability. Then all the eigenvalues are in the stable domain except one zero eigenvalue. While in Fig. 6b one pair of eigenvalues is on the stability boundary at the $a_1 = 0$ dynamic instability condition. Fig. 7 shows two types of unstable cases called static and dynamic post-bifurcations. Here situations 'after' loss of stability are presented, that is, in plot (a) material parameter a_0 is infinitesimally less then zero, while in plot (b) material parameter a_1 is infinitesimally less then zero. In both cases the state is unstable, but plot (a) might be connected to shear banding or necking instabilities [39], while



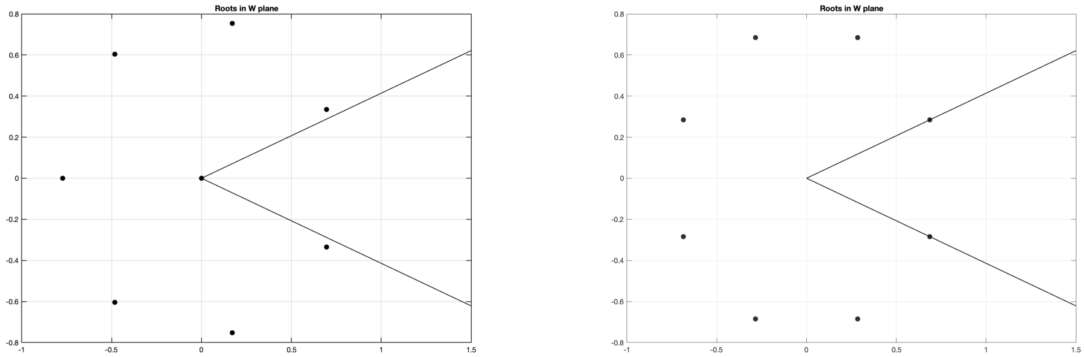
(a) $\omega = 0.3$ (b) $\omega = 0.8$

Figure 4: Eigenvalue distribution in stable state, $E_0 > 0$, $E_1 > 0$.



(a) $\omega = 0.3$ (b) $\omega = 0.8$

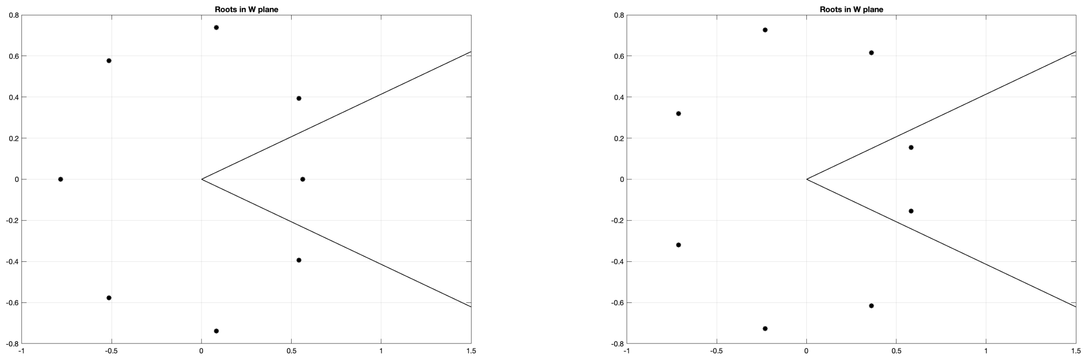
Figure 5: Eigenvalue distribution in unstable state, $E_0 > 0$ $E_1 < 0$.



(a) $a_0 = 0$ static bifurcation

(b) $a_1 = 0$ dynamic bifurcation

Figure 6: Eigenvalues at loss of stability.



(a) static post-bifurcation

(b) dynamic post-bifurcation

Figure 7: Eigenvalues after loss of stability.

plot (b) describes propagative material instabilities [17] as in Portevin–Le Chatelier effect [20].

5 Conclusion

Fractional derivatives can be and are already used to describe non-conventional rate dependence. When periodic perturbations are applied to stability investigations, they have no effect on stability conditions, which are determined by the material parameters only. This result is the same as in classical case. Of course it is what should be expected, while the way of approximation should not effect the outcome of material instability investigation. Frequency acts on the absolute value of eigenvalues, which has no consequences on qualitative behavior. The most important result achieved is that at dynamic instability, the frequency defines critical eigenfunctions to the eigenvalues at the stability boundary. Thus a non-linear study can be performed by projecting the equations to the non-trivial critical null-space spanned by such critical eigenfunctions.

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