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Technical Note

Complex Exponential Method for Solving Partial Differential Equations

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For constant-coefficient linear partial differential equations solvable by separation of variables, an alternative solution method is proposed. The method employs complex exponential functions to find exact analytical solutions. Examples include the heat conduction equation, homogenous and non-homogenous wave equations, and the beam vibration equation. The method can be effectively used for partial differential equations (PDEs) whose solutions can be expressed as a product of harmonic and/or exponential type series.

Keywords: complex functions; partial differential equations; heat transfer; wave motion; beam vibrations.

1. Introduction

PDEs frequently appear as mathematical models in many engineering and mathematical physics problems. One of the most common methods for solving engineering problems such as heat transfer, wave motion, and vibrations of continuous structures is the separation of variables method. In this method, a special solution is assumed, which is a multiplication of pure functions of independent variables. This assumption transforms the partial differential system into a set of ordinary differential system. Usually, the boundary conditions require series-type harmonic/exponential solutions for such problems. For several applications of the separation of variables method and the use of Fourier series in the solutions, see O'Neil [1]. For mechanical vibrational problems in continuous systems solved by separation of variables, see Tse *et al.* [2]. Davies and Radford [3] addressed diffusion-type problems by separation of variables. Turek [4] demonstrated that Fourier cosine series approximated well the solution obtained by separation of variables for the heat conduction equation with mixed boundary conditions. Motamedian and Rahmati [5] solved the homogenous problem of gaseous slip flow in a microchannel by separation of variables.

The diffusion equation appearing in drug delivery systems was also solved by separation of variables [6].

GOMATHY *et al.* [7] investigated the flow of a micropolar fluid past a sphere. PATIL and KADOLI [8] considered the control of a functionally graded beam problem with the aid of separation of variables. KOBAYASHI [9] presented a visual understanding of the solutions to heat conduction and wave equation. MATURI [10] solved the heat equation modeling the refrigeration of apple. ÇETINKAYA *et al.* [11] presented an analytical solution to the time- fractional diffusion equation.

In this introductory work, an alternative method to the separation of variables is discussed. The method involves complex exponential functions to seek analytical solutions. Fundamental problems such as heat conduction, the wave equation and beam vibration are treated to outline the details of the method. The application of boundary conditions necessitates series-type exponential/harmonic solutions. The contribution of the paper lies not in the specific problems considered, but rather in the way the method is presented.

2. Solution method

The solution method is discussed briefly in a general sense in this section, with applications to some physical equations given in the subsequent sections. Consider a PDE with a linear constant-coefficient differential operator L in two independent variables x and t

$$\mathcal{L}u(x,t) = h(x,t), \tag{2.1}$$

where $u(x,t)$ is the dependent variable. The equation may be homogenous ($h(x,t) = 0$) or non-homogenous ($h(x,t) \neq 0$). A simple particular solution may be added to the solution to remove the inhomogeneity in the equation (see Sec. 6). The boundary and initial conditions are linear and may or may not be homogenous (see Secs. 3–6). The method relies on satisfying an infinite series solution of the form:

$$u(x,t) = \sum_{n=0}^{\infty}(A_n e^{a_n t + i b_n x} + cc), \tag{2.2}$$

where the parameters a_n and b_n are determined by the relevant equation and the accompanying boundary and initial conditions. By removing any inhomogeneity in the equation, if present, the substitution of the solution into the PDE leads to a relationship between the parameters a_n and b_n. Using the relationship between a_n and b_n, the series solution is forced to satisfy the conditions one by one. The homogenous conditions are applied first, followed by the non-homogenous conditions at the end.

If there are more than two independent variables, solution (2.2) has to be adjusted accordingly. The method works well for linear partial differential systems with constant coefficients, which may have non-homogeneities either in the equation or in the boundary conditions. Problems solvable by the separation of variables can also be addressed with this alternative method. Compared to the separation of variables, this method is more straightforward, requires less algebra, and involves searching for a single solution at each step, rather than decomposing it into two or more solutions as in the case of the separation of variables approach.

3. The heat conduction equation

Consider the dimensionless heat conduction equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{3.1}$$

with the boundary and initial conditions:

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = f(x). \tag{3.2}$$

Assume an exponential complex function solution of the form:

$$u(x,t) = \sum_{n=0}^{\infty} (A_n e^{a_n t + i b_n x} + cc), \tag{3.3}$$

where cc stands for the complex conjugate of the preceding terms and A_n are complex constants. Substituting (3.3) into (3.1) and canceling the complex exponential terms yields:

$$a_n = -b_n^2, \tag{3.4}$$

where $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$ and $b_n > 0$. Condition (3.4) is necessary for satisfying the equation. Next, the boundary conditions should be satisfied. First, consider the homogenous boundary condition:

$$u(0,t) = 0 = \sum_{n=0}^{\infty} (A_n e^{a_n t} + \overline{A}_n e^{a_n t}), \tag{3.5}$$

which requires

$$A_n + \overline{A}_n = 0 \rightarrow A_n = i c_n. \tag{3.6}$$

where $c_n \in \mathbb{R}$ and $c_n \neq 0$.

Thus, the solution is

$$u(x,t) = \sum_{n=0}^{\infty} (ic_n e^{a_n t + ib_n x} + cc). \tag{3.7}$$

The next condition:

$$u(1,t) = \sum_{n=0}^{\infty} \left(ic_n e^{a_n t} (e^{ib_n} - e^{-ib_n}) \right) = 0, \tag{3.8}$$

requires

$$ic_n (e^{ib_n} - e^{-ib_n}) = 0. \tag{3.9}$$

Employing Euler's formula $e^{ib_n} = \cos b_n + i \sin b_n$ we obtain:

$$2c_n \sin b_n = 0. \tag{3.10}$$

For nontrivial solutions, where $c_n \neq 0$, it follows that

$$b_n = n\pi, \qquad n = 1, 2, 3, ... \tag{3.11}$$

From (3.4) we obtain:

$$a_n = -n^2 \pi^2. \tag{3.12}$$

Therefore, the solution in Eq. (3.7) takes the special form:

$$u(x,t) = \sum_{n=0}^{\infty} \left[ic_n e^{-n^2\pi^2 t} (e^{in\pi x} - e^{-in\pi x}) \right]. \tag{3.13}$$

Employing Euler's formula we obtain:

$$u(x,t) = -\sum_{n=0}^{\infty} \left(2c_n e^{-n^2\pi^2 t} \sin{(n\pi x)} \right). \tag{3.14}$$

The last condition requires

$$u(x,0) = f(x) = -2 \sum_{n=0}^{\infty} c_n \sin{(n\pi x)}\,. \tag{3.15}$$

To find the coefficients c_n, multiply both sides of Eq. (3.15) by $\sin(m\pi x)$, and integrate over the domain. Using the orthogonality condition:

$$\int_0^1 \sin{(n\pi x)} \sin{(m\pi x)}\, \mathrm{d}x = \frac{\delta_{mn}}{2} = \begin{cases} 0 & \text{if} \;\; m \neq n, \\ \frac{1}{2} & \text{if} \;\; m = n, \end{cases} \tag{3.16}$$

we obtain:

$$c_n = -\int_0^1 f(x) \sin(n\pi x)\ \mathrm{d}x. \tag{3.17}$$

Hence, the final solution satisfying the equation and conditions is:

$$u(x,t) = \sum_{n=0}^{\infty} 2\left[\left(\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x\right) e^{-n^2\pi^2 t}\sin(n\pi x)\right]. \tag{3.18}$$

4. The wave equation

Consider the dimensionless homogenous wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{4.1}$$

with the boundary and initial conditions:

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = f(x), \qquad \dot{u}(x,0) = 0, \tag{4.2}$$

where the dot denotes differentiation with respect to time.

Assuming a similar complex exponential solution as in the previous section, we propose

$$u(x,t) = \sum_{n=0}^{\infty}(A_n e^{a_n t + i b_n x} + cc), \tag{4.3}$$

and substituting this into Eq. (4.1) yields:

$$a_n^2 = -b_n^2. \tag{4.4}$$

Since b_n is real, a_n turns out to be imaginary:

$$a_n = \mp i b_n, \tag{4.5}$$

where $a_n \in \mathbb{C}$, $b_n \in \mathbb{R}$ and $b_n > 0$. The two solutions are distinguished from each other by employing different coefficients:

$$u(x,t) = \sum_{n=0}^{\infty}(A_n e^{i b_n (x+t)} + B_n e^{i b_n (x-t)} + cc), \tag{4.6}$$

where A_n and B_n are complex constants. In this expression, the first term represents a travelling wave solution moving to the left, and the second term represents a travelling wave solution moving to the right.

The first boundary condition is:

$$u(0,t) = 0 = \sum_{n=0}^{\infty}(A_n e^{ib_n t} + B_n e^{-ib_n t} + \overline{A}_n e^{-ib_n t} + \overline{B}_n e^{ib_n t}), \tag{4.7}$$

which requires

$$B_n = -\overline{A}_n, \tag{4.8}$$

yielding the solution:

$$u(x,t) = \sum_{n=0}^{\infty}(A_n e^{ib_n(x+t)} - \overline{A}_n e^{ib_n(x-t)} + cc). \tag{4.9}$$

The last condition is:

$$\dot{u}(x,0) = 0 = \sum_{n=0}^{\infty}(ib_n A_n e^{ib_n x} + ib_n \overline{A}_n e^{ib_n x} + cc = 0), \tag{4.10}$$

which requires:

$$A_n + \overline{A}_n = 0 \;\rightarrow\; A_n = ic_n, \tag{4.11}$$

where $c_n \in \mathbb{R}$ and $c_n \neq 0$.

Thus, the solution becomes:

$$u(x,t) = \sum_{n=0}^{\infty}(ic_n e^{ib_n(x+t)} + ic_n e^{ib_n(x-t)} + cc), \tag{4.12}$$

which reduces to

$$u(x,t) = \sum_{n=0}^{\infty}\left(2ic_n \cos(b_n t)\,(e^{ib_n x} - e^{-ib_n x})\right), \tag{4.13}$$

by employing Euler's relation.

Applying the second boundary condition:

$$u(1,t) = 0 = \sum_{n=0}^{\infty}\left[2ic_n \cos(b_n t)\,(e^{ib_n} - e^{-ib_n})\right], \tag{4.14}$$

requires

$$e^{ib_n} - e^{-ib_n} = 0, \tag{4.15}$$

or expanding the complex exponential terms, we obtain:

$$\sin b_n = 0. \tag{4.16}$$

Hence,

$$b_n = n\pi, \qquad n = 1, 2, 3, ... \tag{4.17}$$

The solution can now be expressed as:

$$u(x,t) = \sum_{n=0}^{\infty} \left[2ic_n \cos(n\pi t)\left(e^{in\pi x} - e^{-in\pi x}\right)\right], \tag{4.18}$$

or employing Euler's formula, the solution is:

$$u(x,t) = -\sum_{n=0}^{\infty} [4c_n \cos(n\pi t) \sin(n\pi x)]. \tag{4.19}$$

Finally, the non-homogenous condition is applied

$$u(x,0) = f(x) = -\sum_{n=0}^{\infty} [4c_n \sin(n\pi x)]. \tag{4.20}$$

The coefficients are calculated by multiplying both sides of the initial condition equation by $\sin(m\pi x)$ and integrating over the domain. Using the orthogonality conditions given in (3.16), the coefficients are:

$$c_n = -\frac{1}{2}\int_0^1 f(x) \sin(n\pi x)\, \mathrm{d}x. \tag{4.21}$$

Hence, the final solution satisfying the equation and the given conditions is:

$$u(x,t) = \sum_{n=0}^{\infty} 2\left[\left(\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x\right)\cos(n\pi t)\sin(n\pi x)\right]. \tag{4.22}$$

In the context of vibrations, the spatial variations $\sin(n\pi x)$ are called the mode shapes and $n\pi$ are the frequencies of vibration.

5. The beam vibration equation

Consider the dimensionless Euler-Bernoulli beam vibration problem:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0, \tag{5.1}$$

with the initial and boundary conditions:

$$u(0,t) = u''(0,t) = u(1,t) = u''(1,t) = 0, \quad u(x,0) = 0, \quad \dot{u}(x,0) = f(x), \tag{5.2}$$

where the prime denotes differentiation with respect to x, and the dot denotes differentiation with respect to t.

Assuming a complex exponential solution of the form:

$$u(x,t) = \sum_{n=0}^{\infty}(A_n e^{a_n t + i b_n x} + cc), \tag{5.3}$$

and substituting this into (5.1) yields

$$a_n^2 + b_n^4 = 0. \tag{5.4}$$

Since b_n is real, a_n turns out to be imaginary:

$$a_n = \mp i b_n^2, \tag{5.5}$$

where $a_n \in \mathbb{C}$, $b_n \in \mathbb{R}$ and $b_n > 0$. This suggests a solution of the form:

$$u(x,t) = \sum_{n=0}^{\infty}(A_n e^{i b_n^2 t + i b_n x} + B_n e^{-i b_n^2 t + i b_n x} + cc), \tag{5.6}$$

where A_n and B_n are complex constants, and cc stands for the complex conjugates of the proceeding terms.

Applying all the conditions to the above solution, we start with the first boundary condition:

$$u(0,t) = 0 = \sum_{n=0}^{\infty}(A_n e^{i b_n^2 t} + B_n e^{-i b_n^2 t} + \overline{A}_n e^{-i b_n^2 t} + \overline{B}_n e^{i b_n^2 t}), \tag{5.7}$$

which requires:

$$B_n = -\overline{A}_n, \tag{5.8}$$

yielding the solution:

$$u(x,t) = \sum_{n=0}^{\infty}(A_n e^{i b_n^2 t + i b_n x} - \overline{A}_n e^{-i b_n^2 t + i b_n x} + cc). \tag{5.9}$$

For the second boundary condition:

$$u''(0,t) = 0 = \sum_{n=0}^{\infty} \left[-b_n^2(A_n e^{ib_n^2 t} - \overline{A}_n e^{-ib_n^2 t}) + cc\right] = 0. \tag{5.10}$$

This condition is satisfied identically when the complex conjugates are added to the expression.

The third boundary condition requires:

$$\begin{aligned} u(1,t) &= 0 \\ &= \sum_{n=0}^{\infty} \left[A_n e^{ib_n^2 t + ib_n} - \overline{A}_n e^{-ib_n^2 t + ib_n} + \overline{A}_n e^{-ib_n^2 t - ib_n} - A_n e^{ib_n^2 t - ib_n}\right], \end{aligned} \tag{5.11}$$

which simplifies to:

$$u(1,t) = 0 = \sum_{n=0}^{\infty} \left[A_n e^{ib_n^2 t}(e^{ib_n} - e^{-ib_n}) - \overline{A}_n e^{-ib_n^2 t}(e^{ib_n} - e^{-ib_n})\right], \tag{5.12}$$

which can be satisfied if

$$e^{ib_n} - e^{-ib_n} = 0. \tag{5.13}$$

Expanding the exponential terms:

$$2i \sin b_n = 0, \tag{5.14}$$

yields:

$$b_n = n\pi, \qquad n = 1, 2, 3, ... \tag{5.15}$$

The solution can now be expressed as:

$$u(x,t) = \sum_{n=0}^{\infty} [A_n e^{in^2\pi^2 t + in\pi x} - \overline{A}_n e^{-in^2\pi^2 t + in\pi x} + cc]. \tag{5.16}$$

The above solution satisfies the condition $u''(1,t) = 0$.

Applying the first initial condition:

$$u(x,0) = 0 = \sum_{n=0}^{\infty} (A_n e^{in\pi x} - \overline{A}_n e^{in\pi x} + cc) \tag{5.17}$$

requires

$$A_n = \overline{A}_n. \tag{5.18}$$

Hence, A_n should be real

$$A_n = c_n, \tag{5.19}$$

where $c_n \in \mathbb{R}$ and $c_n \neq 0$.

The solution is

$$\begin{aligned} u(x,t) = \sum_{n=0}^{\infty} \Big[c_n (& e^{in^2\pi^2 t + in\pi x} - e^{-in^2\pi^2 t + in\pi x} \\ & + e^{-in^2\pi^2 t - in\pi x} - e^{in^2\pi^2 t - in\pi x}) \Big], \end{aligned} \tag{5.20}$$

which simplifies to

$$u(x,t) = -\sum_{n=0}^{\infty} \left(4c_n \sin(n\pi x) \sin\left(n^2\pi^2 t\right) \right), \tag{5.21}$$

by employing Euler's formulas for both variables.

Finally, the non-homogenous condition is applied

$$\dot{u}(x,0) = f(x) = \sum_{n=0}^{\infty} -4c_n n^2\pi^2 \sin(n\pi x). \tag{5.22}$$

The coefficients are calculated by multiplying both sides of the equation by $\sin(m\pi x)$, integrating over the domain, and using the orthogonality conditions (3.16). This results in:

$$c_n = -\frac{1}{2n^2\pi^2} \int_0^1 f(x) \sin(n\pi x)\, \mathrm{d}x. \tag{5.23}$$

Hence, the final solution satisfying the equation and conditions is

$$u(x,t) = \sum_{n=0}^{\infty} \left[\frac{2}{n^2\pi^2} \left(\int_0^1 f(x) \sin(n\pi x)\, \mathrm{d}x \right) \sin(n\pi x) \sin\left(n^2\pi^2 t\right) \right]. \tag{5.24}$$

6. The non-homogenous wave equation

Consider the dimensionless non-homogenous wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + kx, \tag{6.1}$$

with the homogenous boundary and initial conditions:

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = 0, \qquad \dot{u}(x,0) = 0, \tag{6.2}$$

and $k \in \mathbb{R}$.

Since there is inhomogeneity in the equation, we suggest a solution of the form:

$$u(x,t) = Y(x) + \sum_{n=0}^{\infty} (A_n e^{a_n t + i b_n x} + cc). \tag{6.3}$$

Substituting this into (6.1) yields

$$a_n = \mp i b_n, \qquad Y''(x) + kx = 0. \tag{6.4}$$

Thus, $a_n \in \mathbb{C}$, $b_n \in \mathbb{R}$ and $b_n > 0$.

Hence, the solution takes the form:

$$u(x,t) = -\frac{k}{6}x^3 + c_1 x + c_2 + \sum_{n=0}^{\infty} (A_n e^{i b_n (x+t)} + B_n e^{i b_n (x-t)} + cc), \tag{6.5}$$

where c_1, $c_2 \in \mathbb{R}$.

The first boundary condition is:

$$u(0,t) = 0 = c_2 + \sum_{n=0}^{\infty} (A_n e^{i b_n t} + B_n e^{-i b_n t} + \overline{A}_n e^{-i b_n t} + \overline{B}_n e^{i b_n t}). \tag{6.6}$$

This condition requires:

$$B_n = -\overline{A}_n, \qquad c_2 = 0, \tag{6.7}$$

yielding the solution:

$$u(x,t) = -\frac{k}{6}x^3 + c_1 x + \sum_{n=0}^{\infty} \Big[A_n e^{i b_n (x+t)} - \overline{A}_n e^{i b_n (x-t)} + cc \Big]. \tag{6.8}$$

The initial condition is:

$$\dot{u}(x,0) = 0 = \sum_{n=0}^{\infty} \Big[i b_n A_n e^{i b_n x} + i b_n \overline{A}_n e^{i b_n x} + cc \Big]. \tag{6.9}$$

This condition requires:

$$A_n + \overline{A}_n = 0 \;\rightarrow\; A_n = i d_n, \tag{6.10}$$

where $d_n \in \mathbb{R}$ and $d_n \neq 0$.

The solution is

$$u(x,t) = -\frac{k}{6}x^3 + c_1 x + \sum_{n=0}^{\infty} \Big[i d_n (e^{i b_n (x+t)} + e^{i b_n (x-t)}) + cc \Big]. \tag{6.11}$$

Applying the boundary condition:

$$u(1,t) = 0 = -\frac{k}{6} + c_1 + \sum_{n=0}^{\infty} \left[i d_n (e^{ib_n(1+t)} + e^{ib_n(1-t)} - e^{-ib_n(1+t)} - e^{-ib_n(1-t)}) \right], \tag{6.12}$$

or rearranging terms:

$$0 = -\frac{k}{6} + c_1 + \sum_{n=0}^{\infty} i d_n \left[e^{ib_n t}(e^{ib_n} - e^{-ib_n}) + e^{-ib_n t}(e^{ib_n} - e^{-ib_n}) \right] \tag{6.13}$$

requires

$$e^{ib_n} - e^{-ib_n} = 0, \qquad -\frac{k}{6} + c_1 = 0, \tag{6.14}$$

or $\sin b_n = 0$; hence

$$b_n = n\pi, \qquad n = 1, 2, 3, ..., , \qquad c_1 = \frac{k}{6}. \tag{6.15}$$

The solution (6.11) can now be expressed as:

$$u(x,t) = \frac{k}{6} x(1 - x^2) - \sum_{n=0}^{\infty} \left[4 d_n \cos(n\pi t) \sin(n\pi x) \right]. \tag{6.16}$$

Finally, the last initial condition is applied:

$$u(x,0) = 0 = \frac{k}{6} x(1 - x^2) - \sum_{n=0}^{\infty} 4 d_n \sin(n\pi x). \tag{6.17}$$

The coefficients are calculated by multiplying both sides of the equation by $\sin(m\pi x)$, integrating over the domain, and using the orthogonality conditions (3.16):

$$d_n = \frac{k}{12} \int_0^1 x(1 - x^2) \sin(n\pi x)\, \mathrm{d}x. \tag{6.18}$$

Hence, the final solution is

$$u(x,t) = \frac{k}{6} x(1 - x^2) + \sum_{n=0}^{\infty} \frac{k}{3} \left(\int_0^1 x(x^2 - 1) \sin(n\pi x)\, \mathrm{d}x \right) \cos(n\pi t) \sin(n\pi x). \tag{6.19}$$

7. Concluding remarks

The complex exponential method for solving PDEs has been outlined. The method can be applied to a wide variety of PDEs that are solvable by the separation of variables method. The method works particularly well for linear PDE systems with constant coefficients. There might be inhomogeneities in the equation or in the boundary conditions. Although, only problems with a single inhomogeneity are considered in the worked examples, problems with multiple inhomogeneities may also be solved using this method. As a general rule, homogenous conditions must always be satisfied first and then non-homogenous conditions can be addressed. Problems with two independent variables are considered here. The form of the exponential solution has to be adjusted accordingly for three or more independent variables. The assumed complex exponential form of solution can be modified for specific problems, and variants of the method can be developed for more complex cases.

Ethical approval

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Availability of supporting data

There is no additional data associated with the paper.

Competing interests

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