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METHOD FOR THE CALCULATION OF DISPLACEMENTS AND INTERNAL FORCES FOR A PLATE OF ARBITRARY SHAPE

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A method for the calculation of displacements and internal forces for a plate of arbitrary shape and under arbitrary support conditions, being under static loading, is presented. This problem has been solved on the basis of the classical linear theory of ideally elastic thin plates. A set of Fredholm's integral equations, which expressed the support conditions, was used to solve this problem.

The particular integral $W^*=r^2 \ln r^2/16\pi D$ of the differential equation $D\Delta\Delta W(r)=\delta(0)/r$, or a combination of the corresponding differentials was assumed to be the kernels of the integral equations. This particular integral expressing the deflection of an infinite plate under a simple force loading will be termed later as the fundamental solution.

The formulation of integral equations is independent of the static scheme of the plate being solved. Some examples of calculations using a computer program developed by the author are also presented.

NOTATION

T	the set of points of coordinates (x_T, X_T) for which the displacements and internal forces are calculated,
S	the set of points of coordinates (x_S, y_S) where the loading $p(S)$ is applied,
Q	the set of points of coordinates (x_Q, y_Q) belonging to the plate boundaries, $l(Q)$,
F	the set of points of coordinates (x_F, y_F) lying along the elastic supports $l(F)$ inside the plate area Ω,
$q(Q)$, $g(Q)$	the set of forces and moments applied along the plate boundaries $l(Q)$, and $f(F)$—the set of forces applied along the support lines $l(F)$ inside the plate area, which both make possible the fulfillment of the support conditions,
α, n, s	indexes expressing the direction (α perpendicular, tangent) or the derivatives of the fundamental function with respect to the first pair of variables (x_T, y_T),
β, n, s	indexes expressing the derivatives of the fundamental functions with respect to the second pair of variables (x_S, y_S),
$W^*(TS)$	the fundamental solution of the particular integral of the differential equation $D\Delta\Delta W(r)=1/r\delta(0)$, expressing the deflection at the point T caused by the action of the simple force at the point S,
φ^*, M^*, Q^*, V^*	fundamental functions expressing the angle of rotation of the deflection surface, the bending moment, the torsional moment, the shearing force, and the generalized shearing force, respectively,
w, φ, m, T, Q, V	displacements and internal forces in the arbitrary plate.

1. Introduction

Pucher [1] has shown that the deflection surface of a plate loaded with a single force may be described by the following equation:

$$W(x_T, y_T, x_s, y_s) = \frac{P}{8\pi D} r^2 \ln r + W_1(x_T, y_T, x_s, y_s), \tag{1.1}$$

where

$$r = \sqrt{(x_T - x_s)^2 + (y_T - y_s)^2}.$$

The first term of the sum expresses a singular solution for an infinite plate loaded with a single force. This particular integral does not fulfil, however, the boundary conditions. The function $W_1(x_T, y_T, x_S, y_S)$ is a general solution of the differential equation for the plate expressed in the form of a series composed of the harmonic functions; together with the particular integral it satisfies the boundary conditions. In another work [2], Pucher published a large number of the influence surface for both rectangular and cirular plates, as determined by this method.

In a recent monograph [3], Nowacki discusses a method for the determination of displacements and internal forces for plates of various static schemes by reducing the problem to a Fredholm integral equation of the 1st or 2nd kind. A number of other authors [4–8] reported the results obtained by applying this method to some particular static schemes. To arrive at the solution of this problem for different plates, the author had to know various influence functions for plates of the same shaps as those under study, but characterized by simplified support conditions. These functions, so-called fundamental functions, were the kernels of the integral equations and they made it possible to impose appropriate boundary conditions for each of the plates studied. The fundamental function was usually expressed as a trigonometric series, but the analytical solution of the entire problem was found only for few particular cases. Most often the integral equation was solved by approximate methods.

In this work a universal fundamental function independent of the static scheme of the plate has been assumed. A simple, closed form of the fundamental solution is an advantage of considerable consequences, for it makes numerical calculations of the problem much easier. In this case it is unnecessary to define the convergence, which is so cumbersome an operation for fundamental solutions in the form of a series. This paper presents a method to determine displacements and internal forces for plates of arbitrary static schemes; it uses a fundamental solution in the form $W^* = r^2 \ln r^2/16\pi D$ as well as combinations of its corresponding derivatives.

2. The fundamental function and its derivatives

The fundamental function has the following form in Cartesian coordinates:

$$W^*(x_T, y_T, x_S, y_S) = W^*(TS) = \frac{1}{16\pi D}[x_T - x_S)^2 + (y_T - y_S)^2] \ln[(x_T - x_S)^2 + (y_T - y_S)^2]. \tag{2.1}$$

In further considerations knowledge of the fundamental function derivatives and their combinations representing the internal forces will be necessary. Differentiating the fundamental function (2.1) and using the transforming formulae describing the derivatives in the arbitrary direction α:

$$W^*_{,\alpha}=W^*_{,x}\cos\alpha+W^*_{,y}\sin\alpha,$$

$$W^*_{,\alpha\alpha}=W^*_{,xx}\cos^2\alpha+W^*_{,xy}\sin 2\alpha+W^*_{,yy}\sin^2\alpha,$$

(2.2) $$W^*_{,\alpha\alpha+\frac{\pi}{2}}=W^*_{,xy}\cos 2\alpha+\frac{1}{2}(W^*_{,yy}-W^*_{,xx})\sin 2\alpha,$$

$$W^*_{,\alpha\alpha\alpha}=W^*_{,xxx}\cos^3\alpha+W^*_{,xxy}\sin 2\alpha\cos\alpha+W^*_{,xyy}\sin^2\alpha\cos\alpha+$$
$$+W^*_{,xxy}\cos^2\alpha\sin\alpha+W^*_{,xyy}\sin 2\alpha\sin\alpha+W^*_{,yyy}\sin^3\alpha,$$

$$W^*_{,\alpha\alpha+\frac{\pi}{2}\alpha+\frac{\pi}{2}}=W^*_{,xxx}\sin^2\alpha\cos\alpha-W^*_{,xxy}\sin 2\alpha\cos\alpha+W^*_{,xyy}\cos^3\alpha+$$
$$+W^*_{,xxy}\sin^3\alpha+W^*_{,xyy}\sin 2\alpha\sin\alpha+W^*_{,yyy}\cos^2\alpha\sin\alpha,$$

as well as the relations linking the internal forces with displacement:

$$\varphi^*_\alpha=W^*_{,\alpha},$$

$$M^*_\alpha=-D(W^*_{,\alpha\alpha}+\nu W^*_{,\alpha+\frac{\pi}{2}\alpha+\frac{\pi}{2}}),$$

(2.3) $$T^*_\alpha=-D(1-\nu)W^*_{,\alpha\alpha+\frac{\pi}{2}},$$

$$Q^*_\alpha=-D(W^*_{,\alpha\alpha\alpha}+W^*_{,\alpha\alpha+\frac{\pi}{2}\alpha+\frac{\pi}{2}}),$$

$$V^*_\alpha=-D[W^*_{,\alpha\alpha\alpha}+(2-\nu)W^*_{,\alpha\alpha+\frac{\pi}{2}\alpha+\frac{\pi}{2}}],$$

the displacements and the internal forces for an infinite plate loaded with a single force may be described by the following formulae:

(2.4) $$W^*(x_T,y_T,x_S,y_S)=W^*(TS)=W^*=\frac{1}{16\pi D}Z\ln zz,$$

(2.5) $$\varphi^*_\alpha=\frac{1}{8\pi D}(1+\ln z)(x\cos\alpha+y\sin\alpha),$$

(2.6) $$M^*_\alpha=-\frac{1}{8\pi}\{(1+\nu)(1+\ln z)+\frac{2}{z}[(x\cos\alpha+y\sin\alpha)^2+$$
$$+\nu(x\sin\alpha-y\cos\alpha)^2]\},$$

(2.7) $$T^*_\alpha=-\frac{1-\nu}{8\pi z}[2xy\cos 2\alpha+(y^2-x^2)\sin 2\alpha],$$

(2.8) $$Q^*_\alpha=\frac{1}{2\pi z}(x\cos\alpha+y\sin\alpha),$$

(2.9) $$V^*_{\alpha} = -\frac{1}{4\pi z}\left[\left(3-\frac{2x^2}{z}\right)xA_1+\left(1-\frac{2x^2}{z}\right)yA_2+\left(1-\frac{2y^2}{z}\right)xA_3+\right.$$
$$\left.+\left(3-\frac{2y^2}{z}\right)yA_4\right],$$

where

(2.10) $$A_1=\cos^3\alpha+(2-\nu)\sin^2\alpha\cos\alpha,$$
$$A_2=\frac{3}{2}\sin 2\alpha\cos\alpha+(2-\nu)(\sin^2\alpha-2\cos^2\alpha)\sin\alpha,$$
$$A_3=\frac{3}{2}\sin 2\alpha\sin\alpha+(2-\nu)(\cos^2\alpha-2\sin^2\alpha)\cos\alpha,$$
$$A_4=\sin^3\alpha+(2-\nu)\cos^2\alpha\sin\alpha,$$
$$x=x_T-x_S,\quad y=y_T-y_S,\quad z=x^2+y^2,\quad z\neq 0.$$

After appropriate differentiation of the formulae (2.4)–(2.10) with respect to variables x_S, y_S, the displacements and internal forces for the infinite plate at the point $T(x_T, y_T)$ caused by a simple moment acting at the point $S(x_S, y_S)$ in the direction β were obtained. They are given below:

(2.11) $$W^*_{,(\beta)}(x_T\, y_T\, x_S\, y_S)=W^*_{,(\beta)}(TS)=W^*_{,(\beta)}=-\frac{1}{8\pi D}(1+\ln z)(x\cos\beta+$$
$$+y\sin\beta),$$

(2.12) $$\varphi^*_{\alpha,(\beta)}=-\frac{1}{8\pi D}[1+\ln z)(\cos\alpha\cos\beta+\sin\alpha\sin\beta)+$$
$$+\frac{2}{z}(x\cos\alpha+y\sin\alpha)(x\cos\beta+y\sin\beta)],$$

(2.13) $$M^*_{\nu,(\beta)}=\frac{1}{4\pi z}\{(1+\nu)(x\cos\beta+y\sin\beta)-\frac{2}{z}[(x\cos\alpha+y\sin\alpha)^2+$$
$$+\nu(y\cos\alpha-x\sin\alpha)^2](x\cos\beta+y\sin\beta)+2(x+\cos\alpha+y\sin\alpha)\times$$
$$\times(\cos\alpha\cos\beta+\sin\alpha\sin\beta)+2\nu(x\sin\alpha-y\cos\alpha)\times$$
$$\times(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\},$$

(2.14) $$T^*_{\alpha,(\beta)}=-\frac{1-\nu}{4\pi z}\left\{\frac{1}{z}(x\cos\beta+y\sin\beta)[2xy\cos 2\alpha+(y^2-x^2)\sin 2\alpha]+\right.$$
$$\left.+(x\sin 2\alpha-y\cos 2\alpha)\cos\beta-(x\cos 2\alpha+y\sin 2\alpha)\sin\beta\right\},$$

(2.15) $$Q^*_{\alpha,(\beta)}=-\frac{1}{2\pi z}\left[(\cos\alpha\cos\beta+\sin\alpha\sin\beta)+\frac{2}{z}(x\cos\alpha+y\sin\alpha)\times\right.$$
$$\left.\times(x\cos\beta+y\sin\beta)\right],$$

$$(2.16)\quad V^*_{\alpha,(\beta)} = -\frac{1}{4\pi z}\Bigg[\left(\frac{12x^2}{z}-\frac{8x^4}{z^2}-3\right)A_1\cos\beta+\left(\frac{12y^2}{z}-\frac{8y^4}{z^2}-3\right)A_4\sin\beta+$$

$$+\frac{2}{z}xy\left(3-\frac{4x^2}{z}\right)\left(A_2\cos\beta+A_1\sin\beta\right)+\left(1-\frac{8x^2y^2}{z^2}\right)\times$$

$$\left(A_3\cos\beta+A_2\sin\beta\right)+\frac{2}{z}xy\left(3-\frac{4y^2}{z}\right)\left(A_4\cos\beta+A_3\sin\beta\right)\Bigg],$$

where A_1, A_2, A_3, A_4 are expressed by the formulae (2.10), $z\neq 0$. The formulae (2.4)–(2.9) and (2.11)–(2.16) hold for the entire surface of the plate except for the attachement point of the acting force. At this point, if $z=0$, undefined values would be obtained since the z value is in the denominator. The limits of these relations may be calculated for $z\to 0$. The fundamental function and its first-order derivatives will assume then finite values. The internal forces expressed by higher order derivatives of the fundamental function will approach infinity. In further considerations the fundamental function and its derivatives will appear in the kernels of the integral equations. Instead of determining the displacement and internal force values at the point of attachment of the concentrated loading, the generalized simple force was substituted by the equivalent, uniformly distributed loading along a sector of length Δ. The values of displacements and internal forces caused by this loading

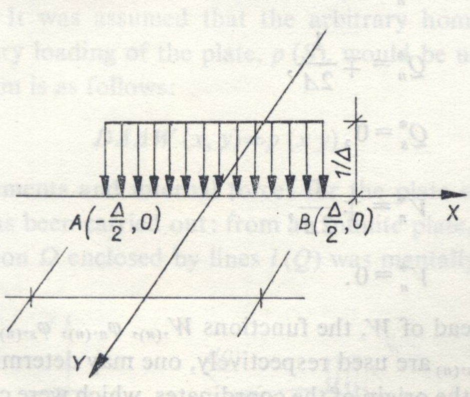

FIG. 1.

were calculated at the mid-point of this sector. In most cases the obtained values of displacements and internal forces at the point of attachment of the concentrated loading do not depend on the position of this point. In order to simplify the calculations, it was assumed that the generalized concentrated force is attached at the origin of the coordinates. This force was replaced by the equivalent loading uniformly distributed along the sector $|AB|=\Delta$ having coordinates of the ends: $A(-\Delta/2, 0)$, $B(\Delta/2, 0)$, as was shown in Fig. 1. With a loading so attached, the values of displacements and internal forces were determined at the origin of the coordinates.

In the case of the plate loaded with a force, the deflection may be determined form the following relationship:

$$W^*(0,0,0,0) \cong \frac{1}{\Delta} \lim_{y_T \to 0+} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} W^*(0, y_T, y_S, 0)\, dx_S. \tag{2.17}$$

To determine the rotation angle of the deflection surface as well as the internal forces, the function W should be appropriately substituted by one of the following functions: φ_n, φ_s, M_n, M_s, T_n, Q_n, Q_s, V_n, V_s. After integration one obtains

$$W^*(0,0,0,0) = W^* = \frac{1}{16\pi D} \cdot \frac{\Delta^2}{6}\left(\frac{1}{2}\ln\frac{\Delta^2}{4} - \frac{1}{3}\right), \tag{2.18}$$

$$\varphi_n^* = 0, \tag{2.19}$$

$$\varphi_s^* = 0, \tag{2.20}$$

$$M_n^* = \frac{1}{8\pi}\left[1 - \nu - (1+\nu)\ln\frac{\Delta^2}{4}\right], \tag{2.21}$$

$$M_s^* = -\frac{1}{8\pi}\left[1 - \nu + (1+\nu)\ln\frac{\Delta^2}{4}\right], \tag{2.22}$$

$$T_n^* = 0, \tag{2.23}$$

$$Q_n^* = \mp\frac{1}{2\Delta}, \tag{2.24}$$

$$Q_s^* = 0, \tag{2.25}$$

$$V_n^* = \mp\frac{1}{2\Delta}, \tag{2.26}$$

$$V_n^* = 0. \tag{2.27}$$

In the case when instead of W, the functions $W_{,(n)}$, $\varphi_{n,(n)}$, $\varphi_{s,(n)}$, $M_{n,(n)}$, $M_{s,(n)}$, $T_{n,(n)}$, $Q_{n,(n)}$, $Q_{s,(n)}$, $V_{n,(n)}$, $V_{s,(n)}$ are used respectively, one may determine the displacements and internal forces at the origin of the coordinates, which were caused by the moment uniformly distributed along the sector $|AB| = \Delta$. They may be expressed by the following equations:

$$W^*_{,(n)} = 0, \tag{2.28}$$

$$\varphi^*_{n,(n)} = -\frac{1}{8\pi D}\left(\ln\frac{\Delta^2}{4} - 1\right), \tag{2.29}$$

$$\varphi^*_{s,(n)} = 0, \tag{2.30}$$

$$M^*_{n,(n)} = \pm\frac{1}{2\Delta}, \tag{2.31}$$

$$M^*_{s,(n)} = \pm \frac{\nu}{2\Delta}, \tag{2.32}$$

$$T^*_{n,(n)} = 0, \tag{2.33}$$

$$Q^*_{n,(n)} = -\frac{2}{\pi \Delta^2}, \tag{2.34}$$

$$Q^*_{s,(n)} = 0, \tag{2.35}$$

$$V^*_{n,(n)} = -\frac{1+\nu}{\pi \Delta^2}, \tag{2.36}$$

$$V^*_{s,(n)} = 0. \tag{2.37}$$

In Eqs. (2.24), (2.26), (2,31) and (2.32) the left- and right-hand limits differ by their signs. The upper sign applies to the right-hand limit. In further considerations only the right-hand one will be used since during subsequent integration the normal axis will be towards the interior of the plate.

3. Description of the method

In Fig. 2 the static scheme of a plate of arbitrary shape with the boundary $l(Q)$ is shown. The elastic supports can be distributed in the inner region of the plate along the lines $l(F)$. It was assumed that the arbitrary homogeneous boundary conditions and arbitrary loading of the plate, $p(S)$, would be used. The differential equation of the problem is as follows:

$$D\Delta\Delta W(x, y) = p(x\, y). \tag{3.1}$$

To determine displacements and internal forces for the plate shown in Fig. 2, the following reasoning has been carried out: from an infinite plate, such as that shown also in Fig. 2, the region Ω enclosed by lines $l(Q)$ was mentally singled out. Along

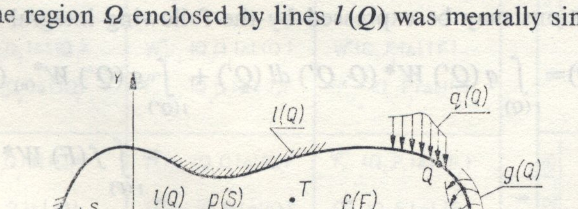

FIG. 2.

these lines a force $q(Q)$ and a moment $g(Q)$ acting perpendicularly to these lines were distributed. Inside the region Ω, along the lines $l(F)$ the force $f(F)$ was also distributed. An arbitrary loading $p(S)$ onto this infinite plate was also applied. By applying the principle of superposition, a deflection of the plate under the given loading may be described as follows:

$$W(T)=W^0(T)+\int\limits_{l(Q)} q(Q)\,W^*(T,Q)\,dl(Q)+\int\limits_{l(Q)} g(Q)\,W^*_{)(n)}(T,Q)\,dl(Q)+\int\limits_{l(F)} f(F)\,W^*(T,F)\,dl(F), \tag{3.2}$$

where

$$W^0(T)=\int\limits_{\Omega(s)} p(S)\,W^*(T,S)\,d\Omega(S).$$

One may choose a system of generalized forces $q(Q)$, $g(Q)$, $f(F)$, such that together with $p(S)$ it would satisfy arbitrary, already assumed support conditions along the lines $l(Q)$, $l(F)$. Furthermore, depending on the support conditions at the points Q or F, the following relationships, shown in Table 1, should be satisfied:

Table 1

Q	$W(Q)=0\ ,\ M_n(Q)=0$
Q	$W(Q)=0\ ,\ \varphi_n(Q)=0$
Q	$\varphi_n(Q)=0\ ,\ Q_n(Q)=0$
Q	$M_n(Q)=0\ ,\ V_n(Q)=0$
F, k	$W(F)=-k(F)\times f(F)$

These conditions may be expressed by the following integral equations:

$$W^0(Q)=\int\limits_{l(Q)} q(Q')\,W^*(Q,Q')\,dl(Q')+\int\limits_{l(Q')} g(Q')\,W^*_{,(n)}(Q,Q')\,dl(Q')+\int\limits_{l(F)} f(F)\,W^*(Q,F)\,dl(F)=0, \tag{3.3}$$

$$\varphi_n^0(Q)+\int\limits_{l(Q')} q(Q')\,\varphi_n^*(Q\,Q')\,dl(Q')+\int\limits_{l(Q')} g(Q')\,\varphi^*_{n,(n)}(Q,Q')\,dl(Q')+\int\limits_{l(F)} f(F)\,\varphi_n^*(Q,F)\,dl(F)=0, \tag{3.4}$$

$$M_n^0(Q)+\int\limits_{l(Q')} q(Q')\,M_n^*(Q,Q')\,dl(Q')+\int\limits_{l(Q')} g(Q')\,M^*_{n,(n)}(Q,Q')\,dl(Q')+\int\limits_{l(F)} f(F)\,M_n^*(Q,F)\,dl(F)=0, \tag{3.5}$$

$$(3.6)\qquad V_n^0(Q)+\int\limits_{l(Q')} q(Q')\,V_n^*(Q,Q')\,dl(Q')+\int\limits_{l(Q')} g(Q')\,V_{n,(n)}^*(Q,Q')\,dl(Q')+ + \int\limits_{l(F)} f(F)\,V_n^*(Q,F)\,dl(F)=0,$$

$$(3.7)\qquad Q_n^0(Q)+\int\limits_{l(Q')} q(Q')\,Q_n^*(Q,Q')\,dl(Q')+\int\limits_{l(Q')} g(Q')\,Q_{n,(n)}^*(Q,Q')\,dl(Q')+ + \int\limits_{l(F)} f(F)\,Q_n^*(Q,Q')\,dl(Q')=0,$$

$$(3.8)\qquad W^0(F)+\int\limits_{l(Q)} q(Q)\,W^*(F,Q)\,dl(Q)+\int\limits_{l(Q)} g(Q)\,W_{,(n)}^*(FQ)\,dl(Q)+ + \int\limits_{l(E')} f(F')\,W^*(F,F')\,dl(F')=-k(F)*f(F).$$

By using Eqs. (3.3)–(3.8) one can formulate a set of Fredholm's integral equations of the first and second kind which express the cupport conditions for an arbitrary plate, e.g. such as that shown in Fig. 2. After the functions $q(Q)$, $g(Q)$, $f(F)$ have been found, the deflection of the plate may be determined from Eq. (3.2). This formula gives the deflection of an infinite plate loaded with forces $p(S)$, $q(Q)$, $g(Q)$, $f(F)$. Within the area Ω, the solution for an infinite plate is exactly the same as for the plate shown in Fig. 2. The solution for a region other than Ω will not be considered here. The internal forces may be determined by appropriate differentiation of the deflection function, see Eq. (3.2), and by using a suitable combination of its derivatives.

Table 2

	A: $j=1,2,\ldots,NB$	A: $j=1,2,\ldots,NB$	A: $j=1,2,\ldots,NS$	* X	= B
if Q_i ⇒	$W^*(Q_i,Q_j)\Delta l(Q_j)$ $M_n^*(Q_i,Q_j)\Delta l(Q_j)$	$W_{,(n)}^*(Q_i,Q_j)\Delta l(Q_j)$ $M_{n,(n)}^*(Q_i,Q_j)\Delta l(Q_j)$	$W^*(Q_i,F_j)\Delta l(F_j)$ $M_n^*(Q_i,F_j)\Delta l(F_j)$	$j=1,2,\ldots,NB$ $q(Q_j)$	$W^0(Q_i)$ $M_r^0(Q_i)$
if Q_i ⇒	$W^*(Q_i,Q_j)\Delta l(Q_j)$ $\varphi_n^*(Q_i,Q_j)\Delta l(Q_j)$	$W_{,(n)}^*(Q_i,Q_j)\Delta l(Q_j)$ $\varphi_{n,(n)}^*(Q_i,Q_j)\Delta l(Q_j)$	$W^*(Q_i,F_j)\Delta l(F_j)$ $\varphi_n^*(Q_i,F_j)\Delta l(F_j)$		$W^0(Q_i)$ $\varphi_r(Q_i)$
if Q_i ⇒	$\varphi_n^*(Q_i,Q_j)\Delta l(Q_j)$ $Q_n^*(Q_i,Q_j)\Delta l(Q_j)$	$\varphi_{n,(n)}^*(Q_i,Q_j)\Delta l(Q_j)$ $Q_{n,(n)}^*(Q_i,Q_j)\Delta l(Q_j)$	$\varphi_r^*(Q_i,F_j)\Delta l(F_j)$ $Q_n^*(Q_i,F_j)\Delta l(F_j)$	$j=1,2,\ldots,NB$ $g(Q_j)$	$\varphi_r^0(Q_i)$ $Q_n^0(Q_i)$
if Q_i ⇒	$M_r^*(Q_i,Q_j)\Delta l(Q_j)$ $V_r^*(Q_i,Q_j)\Delta l(Q_j)$	$M_{n,(n)}^*(Q_i,Q_j)\Delta l(Q_j)$ $V_{n,(n)}^*(Q_i,Q_j)\Delta l(Q_j)$	$M_n^*(Q_i,F_j)\Delta l(F_j)$ $V_n^*(Q_i,F_j)\Delta l(F_j)$		$M_n^0(Q_i)$ $V_n^0(Q_i)$
if F_i, K_i ⇒	$W^*(F_i,Q_j)\Delta l(Q_j)$	$W_{,(n)}^*(F_i,Q_j)\Delta l(Q_j)$	$i=j \Rightarrow k_i+W^*(F_i,F_j)\Delta l(F_j)$ $i\neq j \Rightarrow W^*(F_i,F_j)\Delta l(F_j)$	$j=1,2,\ldots,NS$ $f(F_j)$	$W^0(F_i)$

$$X=A^{-1}*B$$

Unfortunately, it is not possible to find the analytical solution for this set of integral equations—it may be found using approximate methods only, e.g. by the colocation method. The problem can be reduced, therefore, to solving a set of inhomogeneous linear algebraic equations. The procedure for calculating these coefficients is shown in Table 2.

The vector $[Xq(Q_i), g(Q_i), f(F_j)]^T$, where $i=1, 2, ..., NB$, and $j=1, 2, ..., NS$, is the sought solution of the algebraic equations. This expresses the discrete fulfillment of the support conditions. The displacements and the internal forces for the plate shown in Fig. 2 may be as follows:

$$(3.9) \qquad W(T)=W^0(T)+\sum_{i=1}^{NB} q(Q_i)\,W^*(T,Q_i)\,\Delta l(Q_i)+ \sum_{i=1}^{NB} g(Q_i)\,W^*_{)(n)}(T,Q_i)\,\Delta l(Q_i)+\sum_{i=1}^{NS} f(F_i)\,W^*(T,F_i)\,\Delta l(F_i),$$

$$(3.10) \qquad \varphi_\alpha(T)=\varphi^0_\alpha(T)+\sum_{i=1}^{NB} q(Q_i)\,\varphi^*_\alpha(T,Q_i)\,\Delta l(Q_i)+ \sum_{i=1}^{NB} g(Q_i)\,\varphi^*_{\alpha)(n)}(T,Q_i)\,\Delta l(Q_i)+\sum_{i=1}^{NS} f(F_i)\,M^*_\alpha(T,F_i)\,\Delta l(F_i),$$

$$(3.11) \qquad M_\alpha(T)=M^0_\alpha(T)+\sum_{i=1}^{NB} q(Q_i)\,M^*_\alpha(T,Q_i)\,\Delta l(Q_i)+ \sum_{i=1}^{NB} g(Q_i)\,M^*_{\alpha)(n)}(T,Q_i)\,\Delta l(Q_i)+\sum_{i=1}^{NS} f(F_i)\,M^*_\alpha(T,F_i)\,\Delta l(F_i),$$

$$(3.12) \qquad T_\alpha(T)=T^0_\alpha(T)+\sum_{i=1}^{NB} q(Q_i)\,T^*_\alpha(T,Q_i)\,\Delta l(Q_i)+ \sum_{i=1}^{NB} g(Q_i)\,T^*_{\alpha)(n)}(T,Q_i)\,\Delta l(Q_i)+\sum_{i=1}^{NS} f(F_i)\,T^*_\alpha(T,F_i)\,\Delta l(F_i),$$

$$(3.13) \qquad Q_\alpha(T)=Q^0_\alpha(T)+\sum_{i=1}^{NB} q(Q_i)\,Q^*_\alpha(T,Q_i)\,\Delta l(Q_i)+ \sum_{i=1}^{NB} g(Q_i)\,Q^*_{\alpha)(n)}(T,Q_i)\,\Delta l(Q_i)+\sum_{i=1}^{NS} f(F_i)\,Q^*_\alpha(T,F_i)\,\Delta l(F_i),$$

$$(3.14) \qquad V_\alpha(T)=V^0_\alpha(T)+\sum_{i=1}^{NB} q(Q_i)\,V^*_\alpha(T,Q_i)\,\Delta l(Q_i)+ \sum_{i=1}^{NB} g(Q_i)\,V^*_{\alpha)(n)}(T,Q_i)\,\Delta l(Q_i)+\sum_{i=1}^{NS} f(F_i)\,V^*_\alpha(T,F_i)\,\Delta l(F_i),$$

where, for example,

$$M^0_\alpha(T)=\sum_{i=1}^{NP} p(S_i)\, M^*_\alpha(T,S_i)\, \Delta\Omega(S_i).$$

In some simpler examples of the loading scheme, the displacements and the internal forces caused by the loading $p(S)$ may be replaced by the particular integral of the differential equation for the plate: $D\Delta\Delta W(x,y)=p(x,y)$. If the plate is uniformly loaded over its entire surface, the displacements and internal forces will be as follows:

$$W^0(T)=W^0(x_T\, y_T)=\frac{p}{64D}(x_T^2+y_T^2)^2, \tag{3.15}$$

$$\varphi^0_\alpha(T)=\frac{p}{16D}(x_T^2+y_T^2)(x_T\cos\alpha+y_T\sin\alpha), \tag{3.16}$$

$$M^0_\alpha(T)=-\frac{p}{16}[(3x_T^2+y_T^2)(\cos^2\alpha+\nu\sin^2\alpha)+ +2(1-\nu)x_T\, y_T\sin 2\alpha+(x_T^2+3y_T^2)(\sin^2\alpha+\nu\cos^2\alpha)], \tag{3.17}$$

$$T^0_\alpha(T)=-\frac{p(1-\nu)}{8}\left[\frac{1}{2}(y_T^2-x_T^2)\sin 2\alpha+x_T\, y_T\cos 2\alpha\right], \tag{3.18}$$

$$Q^0_\alpha(T)=-\frac{p}{2}(x_T\cos\alpha+y_T\sin\alpha), \tag{3.19}$$

$$V^0_\alpha(T)=-\frac{p(5-\nu)}{8}\left[x_T\cos^3\alpha+y_T\sin^3\alpha+\frac{1}{2}(x_T\sin\alpha+y_T\cos\alpha)\sin 2\alpha\right]. \tag{3.20}$$

In the case when the number of terms of the particular sums in Eqs. (3.9)–(3.14) approaches infinity, the solution becomes the exact one. It has already been mentioned that the accurate analytical solution of this problem is not possible so far, thus one has to assume the finite number of points satisfying the support conditions. Hence one obtains a finite number of terms for the particular sums in Eqs. (3.9)–(3.14). However, the greater the number of points satisfying the support conditions, the more accurate the solution of the problem.

The approximate solution of the set of integral equations leads to a replacement of generalized forces $q(Q)$, $g(Q)$, $f(F)$ distributed along lines $l(Q)$ and $l(F)$ by the equivalent forces $q(Q_i)$, $g(Q_i)$, $f(F_j)$, where $i=1, 2, ..., NB$, and $j=1, 2, ..., NB$, which are uniformly distributed along the sectors $l(Q_i)$ and $l(F_j)$. According to the Saint–Vennant principle, such a weakenig of support conditions will cause the most pronounced inaccuracies to appear along the support lines $l(Q)$ and $l(F)$. These inaccuracies will rapidly disappear at a distance greater than max $[\Delta l(Q_i), l(F_j)]$ from the support lines $l(Q)$ and $l(F)$.

4. Examples

The computer program in the Algol 1900 programming language that can be run on the computers of an Odra 1300 series has been developed. This program allows for the calculation of the displacements and internal forces in a plate of arbitrary shape and support conditions under the arbitrary static loading. The program was tested for a great number of exemplary plates. The results obtained subsequently compared with solutions from the literature that were arrived at with different methods. Very good agreement of our results with those known from the literature was obtained—the relative errors were in the range from 0 to 1%, and for more complicated static schemes less than 4%.

Of course, the accuracy of the solution depends on the number of points assumed for which the support conditions should be fulfilled. At the most, 60 support points were assumed in the examples. Using the standard operational memory of an Odra

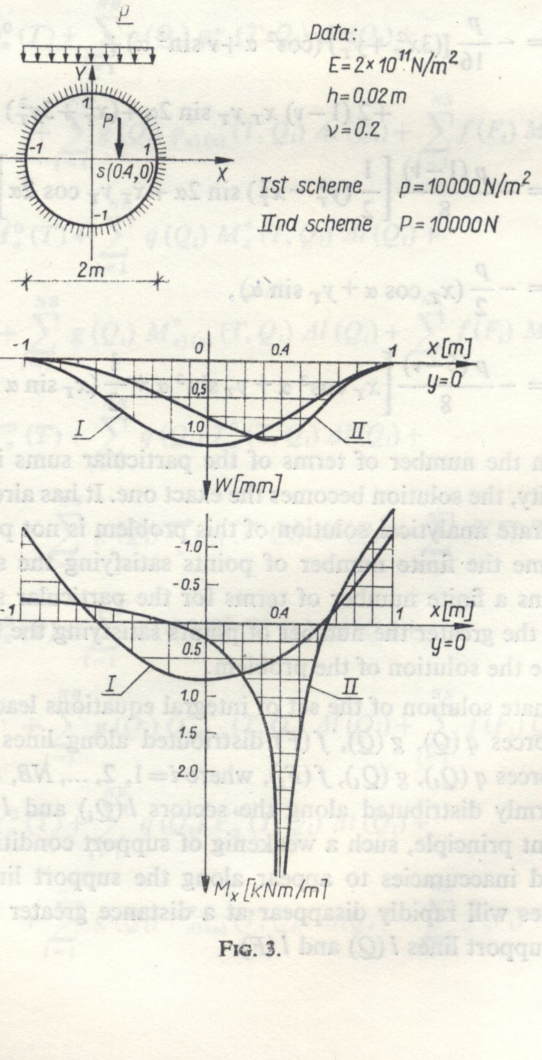

Fig. 3.

1325 computer with a capacity of 64 K bytes, it is possible to satisfy the support conditions at ca. 240 points. It would enhance the accuracy of the solution, but would inconveniently extend the calculating time.

Some examples of calculations for plates with various static schemes are presented below.

4.1. Analysis of solution accuracy and computing time depending on the number of points satisfying the boundary conditions

Figure 3 shows the plots of the deflection and the bending moment for a circular plate rigidly supported at its boundary and loaded with a concentrated force and

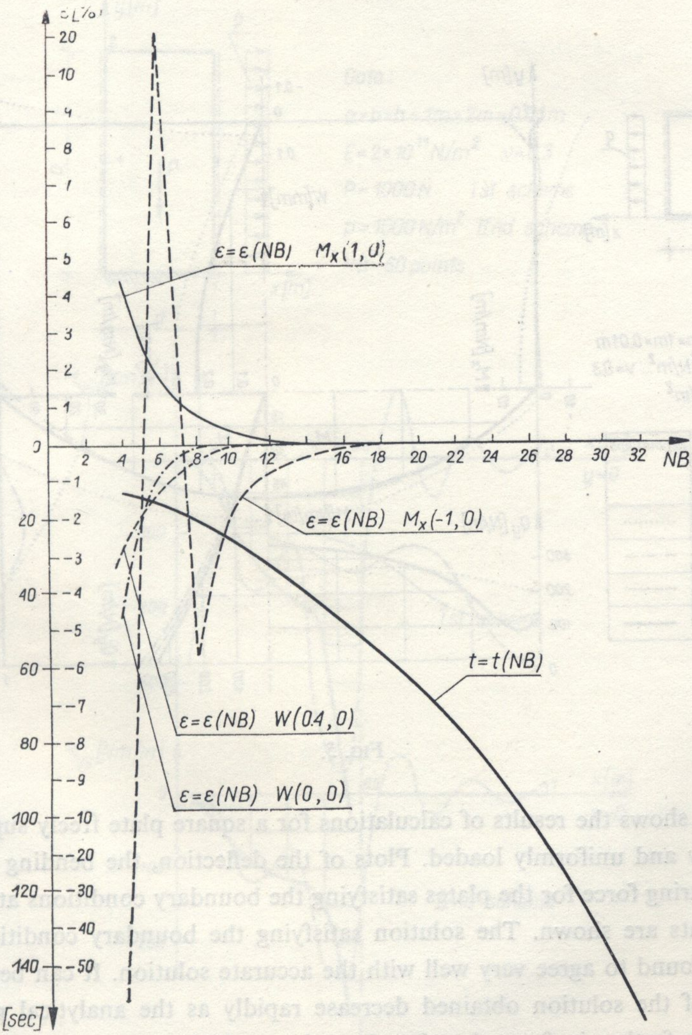

FIG. 4.

uniformly distributed loading. The plate satisfies the boundary conditions at 24 points, and the presented accuracy of the solution is bettere than 0,01 %. Figure 4 shows tha relationship between the relative errors for the deflection and moment values at selected points as well as the computing time, and the number of points satisfying the boundary conditions, for a circular plate loaded with a concentrated force (Fig. 3). It can be seen that the solution rapidly approaches the accurate one, but the computing time increases considerably with an increasing number of points satifsying the boundary conditions.

4.2. Analysis of solution accuracy depending on the position of analytical point on the plate

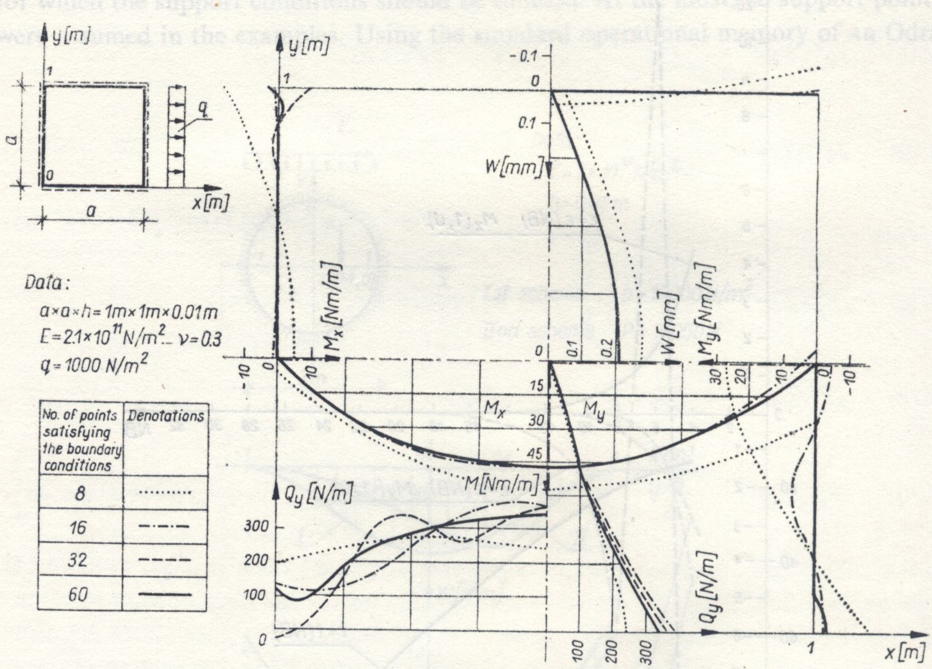

FIG. 5.

Figure 5 shows the results of calculations for a square plate freely supported at its boundary and uniformly loaded. Plots of the deflection, the bending moments, and the shearing force for the plates satisfying the boundary conditions at 8, 16, 32, and 60 points are shown. The solution satisfying the boundary conditions at 60 points was found to agree very well with the accurate solution. It can be seen that the errors of the solution obtained decrease rapidly as the analytical points are being chosen farther in from the plate boundary, which is in agreement with the Saint–Vennant principle.

4.3. Plate with discontinuous boundary conditions

Figure 6 shows the bending moments in a rectangular plate with discontinuous boundary conditions under two different loadings. The solid lines illustrate the bending moments determined only at the points exactly satisfying the boundary conditions, and the deshed line is drawn also by points where the largest errors appear, i.e. at mid-points between the points satisfying the boundary conditions. The obtained solutions (solid lines in Fig. 6) were compared with those reported by Nowacki [3, 9]. The differences between these two solutions were found to be less than 4%. This may confirm the usefulness of the presented method for solving plates with discontinuous boundary conditions.

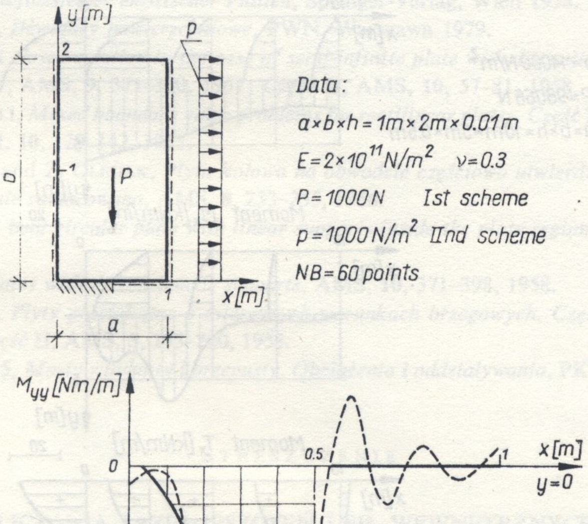

FIG. 6.

4.4. Bridge plate

In Fig. 7 plots of the deflection and the moments for a dead-loaded bridge plate, additionally loaded with a standard *K*–80 crawler [10], are shown.

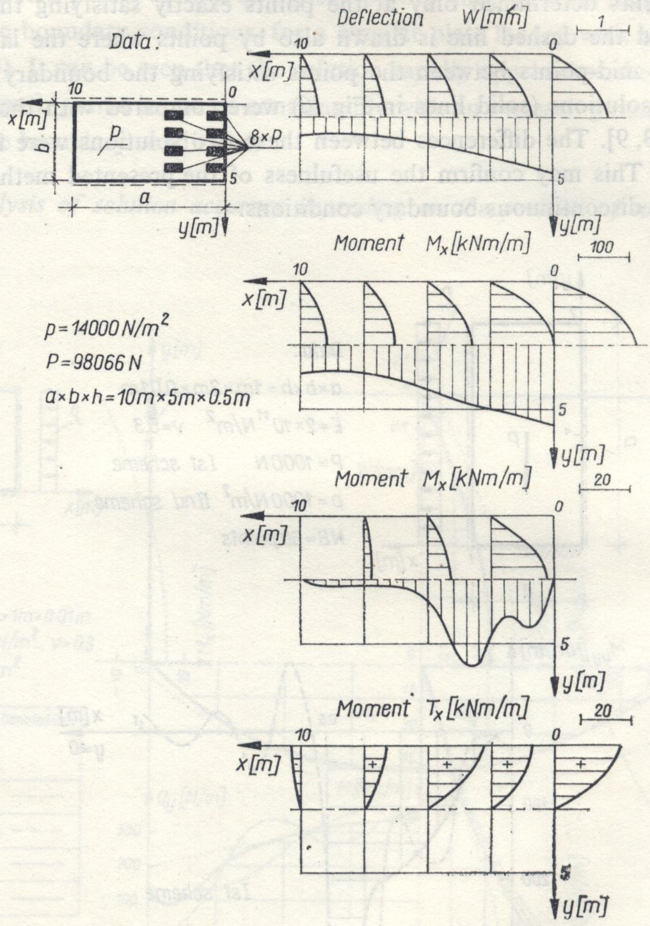

Fig. 7.

5. Conclusions

1. The method presented for the calculation of arbitrary plates has been verified for a great number of examples. This method allows one to obtain the correct results for plates with various static schemes.

2. The application of the particular integral $W=r^2 \ln r^2/16\pi D$ of the differential equation $D\Delta\Delta W=\delta(0)/r$ as the kernel of the set of Fredholm's integral equations is a new element of the method presented here. It makes it possible to eliminate the influence of the plate's shape, boundary conditions, and loading on the method of formulation of the set of integral equations of this problem. Hence it is possible to formulate the algorithm of the solution for an arbitrary plate.

3. The analytical solution for an arbitrary plate cannot be found. In order to find the approximate solution for the set of integral equations, e.g. by the colocation method, one has to discretize only the plate boundary, and not the entire region of the plate as is usually done in the method of finite elements, or finite differences. The presented solution satisfies the differential equation of the problem inside the inner region of the plate which is difficult to attain by other approximate methods.

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Streszczenie

METODA OBLICZANIA PRZEMIESZCZEŃ I SIŁ WEWNĘTRZNYCH W PŁYTACH DOWOLNEGO KSZTAŁTU

Przedstawiono metodę obliczania przemieszczeń i sił wewnętrznych w płytach dowolnego kształtu, o dowolnych warunkach podparcia, dowolnie obciążonych. Problem rozwiązano na podstawie założeń klasycznej, liniowej teorii płyt idealnie sprężystych, wyrażając warunki podparcia przez układ równań całkowych Fredholma I i II rodzaju. Jako jądra równań całkowych przyjęto całkę szczególną $W=r^2 \ln r^2/16\pi D$ równania różniczkowego płyty $D\Delta\Delta W(r)=\delta(0)/r$, zwaną dalej funkcją podstawową. Takie przyjęcie rozwiązania podstawowego eliminuje wpływ kształtu płyty, warunków brzegowych i obciążenia na sposób formułowania równań całkowych problemu. Przedstawiono kilka przykładów obliczeń wykonanych według opracowanego programu na komputer.

Резюме

МЕТОД РАСЧЕТА ПЕРЕМЕЩЕНИЙ И ВНУТРЕННИХ СИЛ В ПЛИТАХ ПРОИЗВОЛЬНОЙ ФОРМЫ

Представлен метод расчета перемещений и внутренних сил в плитах произвольной формы, с произвольными условиями опирания и произвольно нагруженных. Проблема решена на основе предположений классической, линейной теории идеально упругих плит, выражая условия опирания через систему интегральных уравнений Фредгольма I и II родов. Как ядро интегральных уравнений принят частный интеграл $W^*=r^2 \ln r^2/16\pi D$ дифференциального уравнения плиты $D\Delta\Delta W(r)=\delta(0)/r$, называемый далее фундаментальной функцией. Такое принятие фундаментального решения исключает влияние формы плиты, граничных условий и нагрузки на способ формулировки интегральных уравнений проблемы. Представлено несколько примеров расчетов проведенных согласно разработанной программе для ЭВЦМ.

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