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QUASI-STATIC MOTION OF COMPRESSIBLE HYPO-ELASTIC SPHERE AND CYLINDER

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The present paper deals with the quasi-static motion of compressible hypo-elastic bodies of grade zero and grade one. The initial-boundary problems of the raidal motion of a sphere and the plane-radial motion of a cylinder have been studied in detail, their exact solutions being obtained. It has been concluded that for any fixed moment the density of mass and the stress field of the body are homogeneous, while the velocity of the particle is in proportion to its distance from the centre of the sphere (or, from the axis of the cylinder).

1. Basic equations

The concept of hypo-elasticity was proposed by TRUESDELL [1, 2] in 1955. In the same year NOLL [3] proved that every elastic body in finite deformations is a hypo-elastic one. In 1960 BERNSTEIN [4, 5] gave the condition for a hypo-elastic body being, at the same time, Cauchy or Green elastic. Thus hypo-elasticity is a concept wider than elasticity. It has been indicated in [6] that the constitutive equation of a compressible isotropic homogeneous hypo-elastic body has the following form:

$$\begin{aligned}(1.1)\qquad \frac{D\mathbf{S}}{Dt} = {} & (\alpha_1 \operatorname{tr} \mathbf{D} + \alpha_2 \operatorname{tr} \mathbf{DS} + \alpha_3 \operatorname{tr} \mathbf{DS}^2 + \alpha_3 \operatorname{tr} \mathbf{DS}^2)\,\mathbf{I} + \\ & + (\alpha_4 \operatorname{tr} \mathbf{D} + \alpha_5 \operatorname{tr} \mathbf{DS} + \alpha_6 \operatorname{tr} \mathbf{DS}^2)\,\mathbf{S} + \\ & + (\alpha_7 \operatorname{tr} \mathbf{D} + \alpha_8 \operatorname{tr} \mathbf{DS} + \alpha_9 \operatorname{tr} \mathbf{DS}^2)\,\mathbf{S}^2 + \alpha_{10}\,\mathbf{D} + \\ & + \alpha_{11}(\mathbf{DS} + \mathbf{SD}) + \alpha_{12}(\mathbf{DS}^2 + \mathbf{S}^2\mathbf{D}),\end{aligned}$$

where $\mathbf{S}$, $\mathbf{D}$, $\mathbf{I}$ and t are the dimensionless Cauchy stress tensor.(1) the stretching tensor (i.e. the symmetric part of the velocity gradient), the unit tensor and time, respectively.

$$(1.2)\qquad \frac{D\mathbf{S}}{Dt} \overset{\mathrm{df}}{=} \dot{\mathbf{S}} - \mathbf{WS} + \mathbf{SW}$$

is the constitutive derivative(2) of the tensor $\mathbf{S}$ (cf. [7]) and $\alpha_1, \alpha_2, \ldots, \alpha_{12}$ are polynomials of the three invariants

$$(1.3)\qquad \operatorname{tr} \mathbf{S},\ \operatorname{tr} \mathbf{S}^2,\ \operatorname{tr} \mathbf{S}^3$$

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(1) The Cauchy stress tensor is $2\,\mu S$. μ and λ are the Lamé coefficients of classical elasticity.

(2) $(\dot{\ })$ denotes the material differentiation. $\mathbf{W}$ is the spin tensor-skew-symmetric part of the velocity gradient.

of the tensor S. When the right-hand side of Eq. (1.1) does not involve S, we then have a hypo-elastic body of grade zero:

$$\frac{D\mathbf{S}}{Dt}=\left(\frac{\lambda}{2\mu}\operatorname{tr}\mathbf{D}\right)I+\mathbf{D}. \tag{1.4}$$

For the grade-one body, the right-hand side is linearly dependent on S:

$$\frac{D\mathbf{S}}{Dt}=[(\beta_1+\beta_2\operatorname{tr}\mathbf{S})\operatorname{tr}\mathbf{D}+\beta_3\operatorname{tr}\mathbf{DS}]\,\mathbf{I}+(\beta_4\operatorname{tr}\mathbf{D})\,\mathbf{S}+ \\ +(\beta_5+\beta_6\operatorname{tr}\mathbf{S})\,\mathbf{D}+\beta_7\,(\mathbf{DS}+\mathbf{SD}), \tag{1.5}$$

where $\beta_1, \beta_2, ..., \beta_7$ are constants.

The dynamical solutions of hypo-elasticity are very few in number, among them some can be mentioned as below: [8] treats the simple extension and [9, 10] make a study of radial motion of an incompressible sphere and cylinder. TRUESDELL [2] has discussed the accelerationless motion of the homogeneous stress state of a hypo-elastic body. In the present paper, neglecting the influence of inertia, the quasi-static motion of a compressible hypo-elastic body will be studied, namely, the motion of a sphere and cylinder of material of grade zero and grade 1, under the restriction

$$3\beta_6+2\beta_7=0. \tag{1.6}$$

The constitutive equation (1.4) or (1.5), equation of continuity and equation of momentum (in the absence of volume force)

$$\dot{\rho}+\rho\operatorname{div}\mathbf{v}=0, \tag{1.7}$$

$$\operatorname{div}\mathbf{S}=0 \tag{1.8}$$

constitute the closed system of equations for the quasi-static motion of a compressible hypo-elastic body where the mass density ρ, velocity v and stress S are unknown. The initial boundary problem can be investigated with the following initial conditions:

$$\rho|_{t=0}=\rho_0, \quad \mathbf{v}|_{t=0}=\mathbf{v}_0, \quad \mathbf{S}|_{t=0}=\mathbf{S}_0. \tag{1.9}$$

In accordance with the geometrical shape of the considered body, two orthogonal curvilinear coordinate systems will be used: the spherical system $\{r, \vartheta, \varphi\}$ and the cylindrical system $\{r, \vartheta, z\}$. In these coordinate systems, the physical components of velocity v and stress S are denoted by

$$\mathbf{v}=\begin{Vmatrix} v_r \\ v_\vartheta \\ v_\varphi \end{Vmatrix}, \quad \mathbf{S}=\begin{Vmatrix} S_{rr} & S_{r\vartheta} & S_{\varphi r} \\ \cdot & S_{\vartheta\vartheta} & S_{\vartheta\varphi} \\ \cdot & \cdot & S_{\varphi\varphi} \end{Vmatrix} \tag{1.10}$$

and

$$\mathbf{v}=\begin{Vmatrix} v_r \\ v_\vartheta \\ v_z \end{Vmatrix}, \quad \mathbf{S}=\begin{Vmatrix} S_{rr} & S_{r\vartheta} & S_{zr} \\ \cdot & S_{\vartheta\vartheta} & S_{\vartheta z} \\ \cdot & \cdot & S_{zz} \end{Vmatrix}, \tag{1.11}$$

respectively.

2. Radial motion of a sphere

Let the origin of the spherical system $\{r, \vartheta, \varphi\}$ coincide with the centre of the sphere. In the case of radial motion, among the unknows there remain only 4 non-zero physical components:

$$\rho(r,t)\equiv\rho, \quad v_r(r,t)\equiv v, \tag{2.1}$$
$$s_{rr}(r,t)\equiv s_r, \quad s_{\vartheta\vartheta}(r,t)=s_{\varphi\varphi}(r,t)\equiv s_t;$$

the continuity equation (1.7) and momentum equation (1.8) reduce to

$$\frac{\partial\rho}{\partial t}+v\frac{\partial\rho}{\partial r}+\rho\left(\frac{\partial v}{\partial r}+\frac{2v}{r}\right)=0, \tag{2.2}$$

$$\frac{\partial s_r}{\partial r}+\frac{2}{r}(s_r-s_t)=0. \tag{2.3}$$

The constitutive equations (1.4) and (1.5) take the form

$$\frac{\partial s_r}{\partial t}+v\frac{\partial s_r}{\partial r}=\frac{\lambda}{2\mu}\left(\frac{\partial v}{\partial r}+\frac{2v}{r}\right)+\frac{\partial v}{\partial r}, \tag{2.4}$$
$$\frac{\partial s_t}{\partial t}+v\frac{\partial s_t}{\partial r}=\frac{\lambda}{2\mu}\left(\frac{\partial v}{\partial r}+\frac{2v}{r}\right)+\frac{v}{r};$$

$$\frac{\partial s_r}{\partial t}+v\frac{\partial s_r}{\partial r}=[\beta_1+\beta_2(s_r+2s_t)]\left(\frac{\partial v}{\partial r}+\frac{2v}{r}\right)+\beta_3\left(s_r\frac{\partial v}{\partial r}+2s_t\frac{v}{r}\right)+$$
$$+\beta_4 s_r\left(\frac{\partial v}{\partial r}+\frac{2v}{r}\right)+[\beta_5\, 4\beta_6(s_r+2s_t)]\frac{\partial v}{\partial r}+2\beta_7 s_r\frac{\partial v}{\partial r}, \tag{2.5}$$
$$\frac{\partial s_t}{\partial t}+v\frac{\partial s_t}{\partial r}=[\beta_1+\beta_2(s_r+2s_t)]\left(\frac{\partial v}{\partial r}+\frac{2v}{r}\right)+\beta_3\left(s_r\frac{\partial v}{\partial r}+2s_t\frac{v}{r}\right)+$$
$$+\beta_4 s_t\left(\frac{\partial v}{\partial r}+\frac{2v}{r}\right)+[\beta_5+\beta_6(s_r+2s_t)]\frac{v}{r}+2\beta_7 s_t\frac{v}{r}.$$

In the radial motion, the distance R of a typical particle at $t=0$ from the centre of the sphere is taken for the Lagrangean variable. Afterwards, according to the need in integration, we shall use the Lagrangean variable (R, t) or the Eulerian variable (r, t). In the formulae (2.2)–(2.5) $\partial/\partial t$ is the time differentiation under the Eulerian variable, sometimes denoted more explicitly by $(\partial/\partial t)_r$, while the differentiation under the Lagrangean variable is denoted by d/dt or $(\partial/\partial t)_R$.

At $t=0$ the sphere with the radius A is assumed to be at rest and stress-free:

$$\rho|_{t=0}=\rho_0, \quad v|_{t=0}=0, \quad s_r|_{t=0}=s_t|_{t=0}=0. \tag{2.6}$$

Under the exterior action the sphere is forced into motion:

$$r=r(R,t) \quad \text{for} \quad t\geqslant 0. \tag{2.7}$$

Only the motion satisfying the continuity axiom will be considered, i.e. the function (2.7) assumed to be continuously differentiable and single-valued, possesses an inverse

$$R=R(r,t) \tag{2.8}$$

with the same property. Thus

$$\frac{\partial r}{\partial R}=\frac{1}{\dfrac{\partial R}{\partial r}}\neq 0. \tag{2.9}$$

Evidently, the following relations must be satisfied:

$$r(R,0)=R, \quad r(0,t)=0, \tag{2.10}$$

$$\left.\frac{\partial r}{\partial R}\right|_{t=0}=\left.\frac{\partial R}{\partial r}\right|_{t=0}=1. \tag{2.11}$$

The in time varying forcing factor can be either the displacement boundary condition

$$r(A,t)=f(t) \tag{2.12}$$

or the stress boundary condition

$$s_r(A,t)=g(t), \tag{2.13}$$

$f(t)$ and $g(t)$ being given functions satisfying

$$f(0)=A, \quad f'(0)=0; \tag{2.14}$$

$$g(0)=0, \quad g'(0)=0. \tag{2.15}$$

Passing from the Eulerian variable to the Lagrangean one, we have

$$\frac{\partial v}{\partial r}=\frac{\partial v}{\partial R}\frac{\partial R}{\partial r}=\frac{1}{\dfrac{\partial r}{\partial R}}\frac{\partial}{\partial R}\left(\frac{\partial r}{\partial t}\right)_R=\frac{1}{\dfrac{\partial r}{\partial R}}\left[\frac{\partial}{\partial t}\left(\frac{\partial r}{\partial R}\right)\right]_R=\frac{d}{dt}\left(\ln\frac{\partial r}{\partial R}\right), \tag{2.16}$$

$$\frac{v}{r}=\frac{1}{r}\left(\frac{\partial r}{\partial t}\right)=\frac{d}{dt}\left(\ln\frac{r}{R}\right).$$

Inserting Eq. (2.16) into Eq. (2.2) and taking into account

$$\frac{d}{dt}\left(\frac{\partial}{\partial t}\right)_R=\left(\frac{\partial}{\partial t}\right)_r+\left(\frac{\partial r}{\partial t}\right)_R\frac{\partial}{\partial r}=\frac{\partial}{\partial t}+v\frac{\partial}{\partial r}, \tag{2.17}$$

we obtain

$$\frac{d\rho}{dt}+\rho\frac{d}{dt}\left[\ln\left(\frac{r^2}{R^2}\frac{\partial r}{\partial R}\right)\right]=0. \tag{2.18}$$

Under the Lagrangean variable we integrate the above equation from 0 to t along the trajectory R=const. Taking into account the initial condition $(2.1)_1$ and (2.10) and (2.11), we obtain

$$\frac{\rho r^2}{\rho_0 R^2}\frac{\partial r}{\partial R}=1. \tag{2.19}$$

This is, in fact, the continuity equation of the Lagrangean type, Now we consider in sequence the hypo-elastic material of grade zero and grade one.

2.1. *Hypo-elastic body of grade zero*

Inserting Eqs. (2.16) and (2.17) into the constitutive equation (2.4), integrating it from 0 to t along the trajectory R=const and taking into account Eq. $(2.6)_3$ and the conditions (2.10)–(2.11), we obtain

$$\begin{aligned} s_r &= \left(\frac{\lambda}{2\mu}+1\right)\ln\frac{\partial r}{\partial R}+\frac{\lambda}{\mu}\ln\frac{r}{R}, \\ s_t &= \frac{\lambda}{2\mu}\ln\frac{\partial r}{\partial R}+\left(\frac{\lambda}{\mu}+1\right)\ln\frac{r}{R}. \end{aligned} \tag{2.20}$$

Since $dt=0$ is the characteristics of the momentum equation (2.3) substitution of Eqs. (2.9) and (2.20) into it leads to the characteristic relations on the line t=const:

$$\frac{d}{dr}\left(\ln\frac{dR}{dr}\right)+\frac{\kappa}{r}\ln\frac{dR}{dr}+(2-\kappa)\frac{d}{dr}\left(\ln\frac{R}{r}\right)-\frac{\kappa}{r}\ln\frac{R}{r}=0 \tag{2.21}$$

with the notation

$$\kappa=\frac{2}{\dfrac{\lambda}{2\mu}+1}. \tag{2.22}$$

It is easy to verify that

$$R(r,t)=rF(t) \tag{2.23}$$

is the solution of Eq. (2.21), $F(t)$ being the arbitrary function of t. Making use of the boundary condition (2.12) to determine $F(t)$, we obtain the corresponding form of Eq. (2.7)

$$r(R,t)=\frac{R}{A}f(t). \tag{2.24}$$

Further, from Eqs. (2.24), (2.19) and (2.20) we have

$$\rho=\rho_0\frac{A^3}{f^3(t)},$$

$$v=\left(\frac{\partial r}{\partial t}\right)_R=\frac{R}{A}f'(t)=r\frac{d\ln f(t)}{dt}, \tag{2.25}$$

$$s_r=s_t=2\left(\frac{3}{\kappa}-1\right)\ln\frac{f(t)}{A}.$$

The satisfaction of the initial condition $(2.6)_2$ is guaranteed by the condition $(2.14)_2$. The expressions (2.25) are the solution of the problem with the displacement forcing factor.

If the forcing factor is the boundary force (2.13), then the solution of the problem can be obtained from Eq. (2.25).

$$\rho=\rho_0 \exp\left[\frac{3\kappa g(t)}{2(\kappa-3)}\right],$$

$$(2.26)\qquad v=\frac{\kappa r g'(t)}{2(3-\kappa)},$$

$$s_r=s_t=g(t).$$

2.2. Hypo-elastic body of grade one

Subtracting Eq. $(2.5)_2$ from Eq. $(2.5)_1$ and taking into account Eq. (1.6), we obtain

$$(2.27)\qquad \frac{\partial(s_r-s_t)}{\partial t}+v\frac{\partial(s_r-s_t)}{\partial r}=(s_r-s_t)\left[\beta_4\left(\frac{\partial v}{\partial r}+\frac{2v}{r}\right)-\beta_6\left(2\frac{\partial v}{\partial r}+\frac{v}{r}\right)\right]+\beta_5\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right).$$

The substitution of Eqs. (2.16) and (2.17) into Eq. (2.27) leads to

$$(2.28)\qquad \frac{d(s_r-s_t)}{dt}=(s_r-s_t)\frac{d}{dt}\left[(\beta_4-2\beta_6)\ln\frac{\partial r}{\partial R}+(2\beta_4-\beta_6)\ln\frac{r}{R}\right]+ +\frac{d}{dt}\left[\beta_5\left(\ln\frac{\partial r}{\partial R}-\ln\frac{r}{R}\right)\right].$$

Integrating from 0 to t along $R=$const and taking into account Eq. $(2.6)_3$, we obtain

$$(2.29)\qquad s_r-s_t=\exp\left[(\beta_4-2\beta_6)\ln\frac{\partial r}{\partial R}+(2\beta_4-\beta_6)\ln\frac{r}{R}\right]\times \times\int_0^t\left\{\exp\left[(2\beta_6-\beta_4)\ln\frac{\partial r}{\partial R}+(\beta_6-2\beta_4)\ln\frac{r}{R}\right]\frac{d}{dt}\left[\beta_5\left(\ln\frac{\partial r}{\partial R}-\ln\frac{r}{R}\right)\right]\right\}_{t=\tau}d\tau\equiv\Theta(R,t).$$

Applying Eq. (2.29) to eliminate s_t from Eq. $(2.5)_1$ and substituting Eqs. (2.16) and (2.17) into it, we obtain

$$(2.30)\qquad \frac{ds_r}{dt}=\xi s_r\frac{d}{dt}\left(\ln\frac{\partial r}{\partial R}+\ln\frac{r^2}{R^2}\right)+\left[(\beta_1+\beta_5)-2(\beta_2+\beta_6)\Theta\right]\frac{d}{dt}\left(\ln\frac{\partial r}{\partial R}\right)+ +[\beta_1-(2\beta_2+\beta_3)\Theta]\frac{d}{dt}\left(\ln\frac{r^2}{R^2}\right).$$

with the notation

$$\xi = 3\beta_2 + \beta_3 + \beta_4 . \tag{2.31}$$

Integrating Eq. (2.30) from 0 to t along R=const and taking into account Eq. $(2.6)_3$, we obtain

$$s_r = \left(\frac{r^2}{R^2}\frac{\partial r}{\partial R}\right)^{\xi} \times \tag{2.32}$$

$$\times \int_0^t \left\{ \frac{[(\beta_1+\beta_5)-2(\beta_2+\beta_6)\Theta]\frac{d}{dt}\left(\ln\frac{\partial r}{\partial R}\right) + [\beta_1-(2\beta_2+\beta_3)\Theta]\frac{d}{dt}\left(\ln\frac{r^2}{R^2}\right)}{\left(\frac{r^2}{R^2}\frac{\partial r}{\partial R}\right)^{\xi}} \right\}_{t=\tau} d\tau .$$

Substitution of Eqs. (2.29) and (2.32) to Eq. (2.3) gives the characteristic relation of the momentum equation at lines t=const. This relation has a somewhat complicated form. Similarly to the body of grade zero, it is easily verified that

$$R(r,t) = rF(t) . \tag{2.33}$$

satisfies the mentioned characteristic relation. Hence

$$r(R,t) = \frac{R}{A} f(t), \qquad \rho = \rho_0 \frac{A^3}{f^3(t)}, \qquad v = \frac{R}{A} f'(t) = r\frac{d\ln f(t)}{dt} . \tag{2.34}$$

Substituting Eq. (2.34) into Eqs. (2.29) and (2.32), we obtain

$$s_r = s_t = \begin{cases} \dfrac{3\beta_1+\beta_5}{3\xi}\left\{\left[\dfrac{f(t)}{A}\right]^{3\xi} - 1\right\}, & \xi \neq 0, \quad (2.35) \\[2ex] (3\beta_1+\beta_5)\ln\dfrac{f(t)}{A}, & \xi = 0. \quad (2.36) \end{cases}$$

The expressions (2.34)–(2.36) are the solution of the problem with the displacement forcing factor (2.12). It should be noted that for the physical reason $3\beta_2+\beta_5$ can not equal zero.

If the forcing factor is the boundary force (2.13), then the solution of the problem is as follows:

$$r = \begin{cases} R\left[\dfrac{3\xi g(t)}{3\beta_1+\beta_5} + 1\right]^{\frac{1}{3\xi}}, & \xi \neq 0, \\[2ex] R\exp\left[\dfrac{g(t)}{3\beta_1+\beta_5}\right], & \xi = 0, \end{cases} \tag{2.37}$$

$$\rho = \begin{cases} \rho_0\left[\dfrac{3\beta_1+\beta_5}{3\beta_1+\beta_5+3\xi g(t)}\right]^{\frac{1}{\xi}}, & \xi \neq 0, \\[2ex] \rho_0\exp\left[\dfrac{-3g(t)}{3\beta_1+\beta_5}\right], & \xi = 0, \end{cases}$$

$$v=\frac{rg'(t)}{3\beta_1+\beta_5+3\xi g(t)},$$

$$s_r=s_t=g(t).$$

If we put $\beta_1=\lambda/2\mu$, $\beta_5=1$, $\beta_2=\beta_3=\beta_4=\beta_6=0$ in the solution for the body of grade one, then as a special case we obtain the solution for the body of grade zero.

3. Plane-radial motion of a circular cylinder

Let the z-axis of the cylindrical system $\{r, \vartheta, z\}$ coincide with the axis of the cylinder, of which the quasi-static motion is to be studied. For the plane-radial motion, only 5 physical components are not identically equal to zero:

$$\begin{aligned}&\rho(r,t)\equiv\rho, \quad v_r(r,t)\equiv v,\\ &s_{rr}(r,t)\equiv s_r, \quad s_{\vartheta\vartheta}(r,t)\equiv s_\vartheta, \quad s_{zz}(r,t)\equiv {}_z.\end{aligned}\tag{3.1}$$

Now the continuity equation (1.7) and momentum equation (1.8) reduce to

$$\frac{\partial\rho}{\partial t}+v\frac{\partial\rho}{\partial r}+\rho\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)=0,\tag{3.2}$$

$$\frac{\partial s_r}{\partial r}+\frac{1}{r}(s_r-s_\vartheta)=0.\tag{3.3}$$

The constitutive equations for bodies of grade zero and grade one, are, respectively,

$$\begin{aligned}&\frac{\partial s_r}{\partial t}+v\frac{\partial s_r}{\partial r}=\frac{\lambda}{2\mu}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+\frac{\partial v}{\partial r},\\ &\frac{\partial s_\vartheta}{\partial t}+v\frac{\partial s_\vartheta}{\partial r}=\frac{\lambda}{2\mu}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+\frac{v}{r},\\ &\frac{\partial s_z}{\partial t}+v\frac{\partial s_z}{\partial r}=\frac{\lambda}{2\mu}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right);\end{aligned}\tag{3.4}$$

$$\begin{aligned}\frac{\partial s_r}{\partial t}+v\frac{\partial s_r}{\partial r}&=[\beta_1+\beta_2(s_r+s_\vartheta+s_z)]\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+\beta_3\left(s_r\frac{\partial v}{\partial r}+s_\vartheta\frac{v}{r}\right)+\\ &\quad+\beta_4 s_r\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+[\beta_5+\beta_6(s_r+ +s_\vartheta+s_z)]\frac{\partial v}{\partial r}+2\beta_7 s_r\frac{\partial v}{\partial r},\\ \frac{\partial s_\vartheta}{\partial t}+v\frac{\partial s_\vartheta}{\partial r}&=[\beta_1+\beta_2(s_r+s_\vartheta+s_z)]\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+\beta_3\left(s_r\frac{\partial v}{\partial r}+s_\vartheta\frac{v}{r}\right)+\\ &\quad+\beta_4 s_\vartheta\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+[\beta_5+\beta_6(s_r+s_\vartheta+s_z)]\frac{v}{r}+2\beta_7 s_\vartheta\frac{v}{r},\\ \frac{\partial s_z}{\partial t}+v\frac{\partial s_z}{\partial r}&=[\beta_1+\beta_2(s_r+s_\vartheta+s_z)]\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)+\\ &\quad+\beta_3\left(s_r\frac{\partial v}{\partial r}+s_\vartheta\frac{v}{r}\right)+\beta_4 s_z\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right).\end{aligned}\tag{3.5}$$

Take the distance R of the typical particle at $t=0$ from the axis of the cylinder as the Lagrangean coordinate of that particle, then the meaning of $(\partial/\partial t)_r$ and $(\partial/\partial t)_R$ remains the same as in the case of the sphere. We consider the motion

$$r=r(R,t), \quad t\geqslant 0 \tag{3.6}$$

of the cylinder with a radius A for the initial conditions

$$\rho|_{t=0}=\rho_0, \quad v|_{t=0}=0, \quad s_r|_{t=0}=s_\vartheta|_{t=0}=s_z|_{t=0}=0 \tag{3.7}$$

and the forcing factor

$$r(A,t)=f(t) \quad \text{or} \quad s_r(A,t)=g(t), \tag{3.8}$$

satisfying Eq. (2.4) or Eq. (2.5) respectively. Because of the validity of Eqs. (2.16) and (2.17), it is just as easy as in the case of the sphere to translate the Eulerian continuity equation (3.2) into the Lagrangean form:

$$\frac{\rho r}{\rho_0 R}\frac{\partial r}{\partial R}=1. \tag{3.9}$$

Now we consider in sequence the hypo-elastic material of grade zero and grade one.

3.1. *Hypo-elastic body of grade zero*

Substituting Eqs. (2.16) and (2.17) into Eq. (3.4), integrating it and making use of the initial conditions (3.1) we obtain

$$\begin{aligned} s_r&=\left(\frac{\lambda}{2\mu}+1\right)\ln\frac{\partial r}{\partial R}+\frac{\lambda}{2\mu}\ln\frac{r}{R},\\ s_\vartheta&=\frac{\lambda}{2\mu}\ln\frac{\partial r}{\partial R}+\left(\frac{\lambda}{2\mu}+1\right)\ln\frac{r}{R},\\ s_z&=\frac{\lambda}{2\mu}\ln\frac{\partial r}{\partial R}+\frac{\lambda}{2\mu}\ln\frac{r}{R}. \end{aligned} \tag{3.10}$$

By inserting Eq. (3.10) into the momentum equation (3.3), we have at the characteristic lines $dt=0$

$$\frac{d}{dr}\left(\ln\frac{dR}{dr}\right)+\frac{\eta}{r}\ln\frac{dR}{dr}+(1-\eta)\frac{d}{dr}\left(\ln\frac{R}{r}\right)-\frac{\eta}{r}\ln\frac{R}{r}=0, \tag{3.11}$$

with the notation

$$\eta=\frac{1}{\dfrac{\lambda}{2\mu}+1}. \tag{3.12}$$

It is easily verified that the solution of Eq. (3.11) has also the form (2.23):

$$R(r,t)=rF(t). \tag{3.13}$$

Making use of the boundary condition $(3.8)_1$ to determine $F(t)$, we finally obtain the solution of the problem with the displacement forcing factor:

$$r=\frac{R}{A}f(t), \quad v=\frac{R}{A}f'(t)=r\frac{d\ln f(t)}{dt},$$

$$\rho=\rho_0\frac{A^2}{f^2(t)}, \tag{3.14}$$

$$s_r=s_\vartheta=\left(\frac{\lambda}{\mu}+1\right)\ln\frac{f(t)}{A}, \quad s_z=\frac{\lambda}{\mu}\ln\frac{f(t)}{A}.$$

For the boundary stress forcing factor, the solution is

$$r=R\exp\left[\frac{g(t)}{\frac{\lambda}{\mu}+1}\right], \quad v=\frac{rg'(t)}{\frac{\lambda}{\mu}+1},$$

$$\rho=\rho_0\exp\left[\frac{-2g(t)}{\frac{\lambda}{\mu}+1}\right], \tag{3.15}$$

$$s_r=s_\vartheta=g(t), \quad s_z=\frac{g(t)}{\frac{\mu}{\lambda}+1}.$$

3.2. Hypo-elastic body of grade one

It can be proved that the relation

$$r=\frac{R}{A}f(t) \tag{3.16}$$

holds true also in this case. Because of complexity, the explicit form of further formulae will not be given here.

4. Conclusion

The above studies lead to a common conclusion: In all cases treated here the mass density and stress state remain at any moment homogeneous, while the velocity of the particle is in proportion to its distance from the centre of the sphere or from the axis of the cylinder, respectively.

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STRESZCZENIE

QUASI-STATYCZNY RUCH ŚCIŚLIWYCH, HYPO-SPRĘŻYSTYCH KUL I CYLINDRÓW

Praca zajmuje się quasi-statycznym ruchem ściśliwych ciał hypo-sprężystych stopnia zerowego i pierwszego. Zostało szczegółowo zbadane zagadnienie początkowo-brzegowe ruchu radialnego kuli i ruchu płaskiego radialnego walca. Otrzymano dokładne rozwiązania. Wykazano, że dla dowolnej ustalonej chwili gęstość masy i pole naprężeń ciała są jednorodne, prędkość cząstki zaś jest proporcjonalna do odległości od środka kuli lub od osi walca.

Резюме

КВАЗИСТАТИЧЕСКОЕ ДВИЖЕНИЕ СЖИМАЕМЫХ ГИПОУПРУГИХ ШАРОВ И ЦИЛИНДРОВ

Работа занимается квазистатическим движением сжимаемых, гипоупругих, нулевой и первой степени, тел. Подробно исследованы начально-краевые задачи радиального движения шара и плоского радиального движения цилиндра. Получены точные решения. Показано, что для произвольного, установленного момента времени плотность массы и поле напряжений тела однородны, скорость частицы же пропорциональна расстоянию от центра шара или от оси цилиндра.

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