ENGINEERING TRANSACTIONS • *Engng. Trans.* • **63**, 1, 77–92, 2015
Polish Academy of Sciences • Institute of Fundamental Technological Research (IPPT PAN)
National Engineering School of Metz (ENIM) • Poznan University of Technology

The Fourier Series Implementation Issues in the Tolerance Modeling of Thermal Conductivity of Periodic Composites

Dorota KULA, Ewaryst WIERZBICKI

Warsaw University of Life Sciences – SGGW
Faculty of Civil and Environmental Engineering
Nowoursynowska 159, 02-776 Warszawa, Poland
e-mail: kuladorota@wp.pl; ewaryst_wierzbicki@sggw.pl

The aim of this study is to propose a partially-averaged model of heat conduction in simple micro-periodic composite conductors. In this model, as in many known models of this type, the type of microstructure is represented by the single scalar parameter, which is referred to as microstructure parameter. Unlike other known averaged models of this type, the resulting model allows for the formulation exact solutions to initial-boundary value problems formulated for the parabolic heat conduction equation. If tolerance approximations will be applied to averaged temperature filed this model becomes asymptotically exact model. The term "asymptotically exact model" refers to models in the framework of which solutions coincident with exact solutions to the mentioned problems for the parabolic heat transfer equation when the microstructure parameter tends to zero.

Key words: tolerance averaging, heat conduction, periodic conductors, boundary effect, Fourier series.

1. Introduction

The aim of this study is to propose a partially-averaged model of heat conduction in simple micro-periodic composite conductors. In this model, as in many known models of this type, the type of microstructure is represented by the single scalar parameter, which is referred to as microstructure parameter. Unlike other known averaged models of this type, the resulting model allows for the formulation exact solutions to initial-boundary value problems formulated for the parabolic heat conduction equation. If tolerance approximations will be applied to averaged temperature filed this model becomes asymptotically exact model. The term "asymptotically exact model" refers to models in the framework of which solutions coincident with exact solutions to the mentioned problems for the parabolic heat transfer equation when the microstructure parameter tends to zero.

Aforementioned property of the proposed model allows us to situations in which we are able to compare the behavior of the composite conductors made of the same finite number of components but differ in their geometric arrangement. This comparison will be made, while the reactions of these composites to the same external thermal conditions (represented by the imposed initial-boundary conditions) will be examined. Other known averaged models of this type does not allow to carry out such comparisons, because the solutions obtained in the framework of these models are only approximate and the accuracy of these solutions can be verified in the present state of knowledge only empirically.

The starting point for the implementation of the proposed method of modeling is the modeling method known as "the tolerance averaging technique", as proposed by Professor Czesław Woźniak. The reader is here referred to the six basic monographs on this subject [1–6] for theoretical foundations and [8–15] for various applications of tolerance modeling approach. In this work, it is proposed to instead apply "micro-macrohypothesis" (used in tolerance modeling), a more general hypothesis which postulates the possibility of developing a Fourier series for residuals between the "exact temperature" and its micro-macro approximation used in tolerance modeling. Hence, in the proposed course of modeling the temperature field is represented as a sum of two parts. The first part which will be referred to as *the long-wave part of the temperature field* coincides with the temperature approximation introducedin the tolerance micro-macro hypothesis. The second part is the mentioned above Fourier expansion and is referred to as *short-wave part of the temperature field*. To allow us to the use of Fourier expansion method we are to impose on the long-part of the temperature field an additional assumption, whereby the heat flux vector component normal to the surface of the bonding components of the composite is continuous on these surface. In fact, this is an additional assumption imposed on tolerance shape functions used in the long parts of the temperature field.

Considerations of work are illustrated with two examples of the application of the proposed model for the study of problems of thermal conductivity.

2. Fundamental concepts

The starting point of consideration is the well-known parabolic heat transfer equation

$$\nabla^T(K\nabla\theta) - c\dot{\theta} = b \tag{2.1}$$

in which $\theta = \theta(z,t)$, $z = (z^1, z^2, z^3) \in R^3$, $t \geq 0$, denotes the temperature field, c is a specific heat field and $K = (k_{ij})$ is the heat conductivity matrix. Here $\nabla \equiv [\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3]^T$ and indices i, j run over 1, 2, 3. Fields $c = c(\cdot)$

and $K = K(\cdot)$ take S values denoted by c^I, ..., c^S and K^I, ..., K^S, respectively, do not depend on the temperature field θ and are restrictions of certain fields defined in R^3 to the region

$$\Omega = (0, L_1) \times (0, L_2) \times (0, L_3) \backslash (\delta, L_1 - \delta) \times (\delta, L_2 - \delta) \times (\delta, L_3 \backslash \delta)] \subset R^3 \tag{2.2}$$

in which L_1, L_2, $L_3 > 0$ and $0 < 2\delta < L_2$, L_3. The fragment of the considered dividing wall and introduced Cartesian coordinate system is illustrated in Fig. 1. An open problem is how to arrange the components in the interior of dividing wall to obtain optimal insulation properties of the wall with respect to a certain given *a priori* criterion.

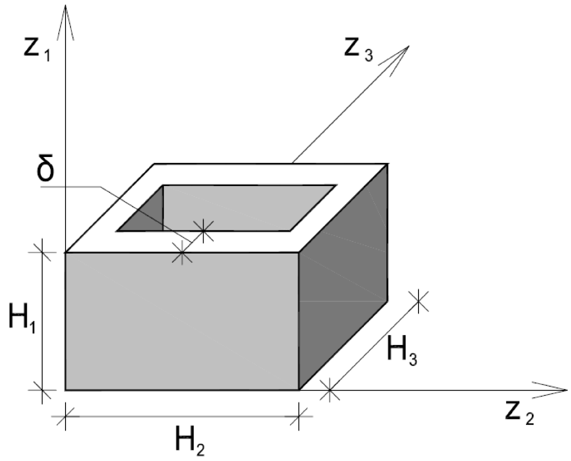

FIG. 1. The fragment of the considered dividing wall.

The study of the mentioned problem will be restricted here to the microstructural Δ-periodic composites which diameter $\lambda = \operatorname{diam}(\Delta)$ is small where compared to the characteristic length dimension of region Ω. It means that there exists m-tuple $(\mathbf{v}^1, ..., \mathbf{v}^m)$ of independent vectors $\mathbf{v}^1$, ..., $\mathbf{v}^m$ from R^m, determining m directions of periodicity and referred to as periodicity vectors, such that

$$\bigcup \left\{ \Delta_{k_1,...k_m, r_{m+1},...,r_3} \colon k_1, ...k_m = ...-1, 0, 1, ..., \quad r_{m+1}, ..., r_3 \in R \right\} = R^3 \tag{2.3}$$

for $\Delta_{k_1,...k_m, r_{m+1},...,r_3} \equiv k_1 \mathbf{v}^1 + ... + k_m \mathbf{v}^m + r_{m+1} \mathbf{e}^{m+1} + ... + r_3 \mathbf{e}^3 + \Delta$ and such that both fields $c = c(\cdot)$ and $K = K(\cdot)$ are Δ-periodic. Here $\mathbf{e}^i$ denotes the i-th unit vector from R^3, $i = 1, 2, 3$. In the subsequent investigations the averaging $\langle f \rangle(x)$ of an arbitrary integrable field f defined on R^m plays an important role, and is defined as

$$\langle f \rangle(x) = \frac{1}{|\Delta|} \int_{x+\Delta} f(\xi)\, d\xi \tag{2.4}$$

and which is a constant field provided that f is a Δ-periodic field. Investigations of the mentioned above physical problem will be focused on the analysis of initial-boundary condition

$$\begin{aligned} \theta(z,t)|_{z\in\partial\Omega} &= \theta_{\partial\Omega}(z,t), \\ \theta(z,t)|_{t=0} &= \theta_0(z) \end{aligned} \tag{2.5}$$

attached, for given boundary and initial temperature values $\theta_{\partial\Omega}(z,t)$ and $\theta_0(z)$, to heat transfer equation (2.1). The formulated initial-boundary problem is too complicated to be a proper mathematical tool to compare thermal properties of dividing composite walls illustrated in Fig. 1 depending on the geometry of interfaces and material properties of components. Hence, we are to adopt the above problem to the form that allows for the realization of such a comparison. This adaptation will be realized as follows.

Let the layer $V^h = z_0 + V$, $V = (0, H_1) \times (0, H_2) \times (0, \delta)$, determining by a certain $z_0 \in R^3$, and H_1, H_2, $\delta > 0$, be a fragment of the front part of this wall located far from the wall edge. Denote $\tilde{z} \equiv (x, y) = z - z_0$, where $x \equiv (x^1, ..., x^m) \in R^m$ is related to the periodicity directions of the composite and $y \equiv (y^{3-m}, ..., y^3) \in R^{3-m}$. In fact, together with $\tilde{z} \equiv (x, y)$ we have just introduced new coordinate system such that coordinates $\tilde{z}$ and z of the same point are interrelated $\tilde{z} = z - z_0$. For $m = 1$ we deal with one-directional periodicity. In this case $V = \Xi \times \Phi$ for $\Xi = (0, H_1)$. In this case z_0 can be identified with $[\eta l/2,\ \delta,\ 0]$ where $l^I = \eta l$, $l^{II} = (1-\eta)l$ for $0 < \eta < 1$, are the thickness of the first and the second laminas and microstructure parameter λ coincide with the thickness of the repetitive layer, $\lambda = l \equiv l^I + l^{II}$. The interval $(0, l)$ is identified as a basic cell Δ. Case $m = 2$ deals the periodicity in two directions in which $V = \Xi \times \Phi$ for $\Xi = (0, H_1) \times (0, H_2)$ and $\Phi = (0, \delta)$. Both two-constituent conductors together with related basic cells are illustrated in Fig. 2. In the subsequent considerations the denotations $\nabla_\Xi \equiv [\partial/\partial x^1,\ ...,\ \partial/\partial x^m,\ 0,\ ...,0]^T$, $\nabla_\Phi \equiv [0,\ ...,\ 0,\ \partial/\partial x^{m+1},\ ...,\ \partial/\partial x^3]^T$ are

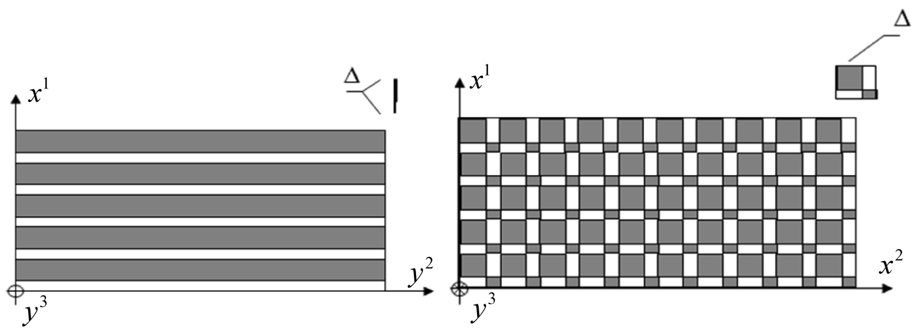

FIG. 2. Chessboard-type periodic rigid conductor. Case m=1 and $m = 2$.

used for partial derivatives in the periodicity and non-periodicity directions respectively. Hence $\nabla_{\Xi} + \nabla_{\Phi} = \nabla \equiv [\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3]^T$.

In the framework of this study we are to investigate the intensity attenuation of temperature fluctuations with respect to the averaged (on cells) temperature field and hence we are to investigate temperature field θ in the form of decomposition

$$\theta = u + \theta_{\text{res}}, \tag{2.6}$$

where *long wave* u and *short-wave* θ_{res} parts of the temperature field should satisfy conditions

$$\langle \theta \rangle = \langle u \rangle, \qquad \langle \theta_{\text{res}} \rangle = 0. \tag{2.7}$$

Hence the boundary and initial values $\theta_{\partial V}$, θ_0 should be represented in the similar form

$$\begin{aligned} \theta_{\partial V} &= u|_{\partial V} + \theta_{\text{res}}|_{\partial V}, \\ \theta_0 &= u_0 + \theta_0^{\text{res}} \end{aligned} \tag{2.8}$$

for a given boundary (on ∂V), $u|_{\partial V}(x,y,t)$, $\theta_{\text{res}}|_{\partial V}(x,y,t)$ and initial $u_0(x,y)$, $\theta_0^{\text{res}}(x,y)$ representing the averaged and the fluctuation parts of the temperature field on the boundary ∂V and at the time instant $t = 0$, respectively. The values $u|_{\partial V}(x,y)$ and $\theta_{\text{res}}|_{\partial V}(x,y)$ should satisfy the conditions

$$\begin{aligned} \langle \theta_{\partial V} \rangle &= \langle u|_{\partial V} \rangle, & \langle \theta_{\text{res}}|_{\partial V} \rangle &= 0, \\ \langle \theta_0 \rangle &= \langle u_0 \rangle, & \langle \theta_0^{\text{res}} \rangle &= 0. \end{aligned} \tag{2.9}$$

Decomposition of the initial and boundary conditions given by (2.8), is justified by the fact that in the outside and inside of the rooms which are surrounded by walls illustrated in Fig. 1, the boundary layer located close to the walls has an abnormal temperature field due to the need to adapt it to the microperiodic structure of the wall.

The aim of this paper is to formulate the averaged mathematical model of heat transfer in composite periodic media, in a framework for which exact solutions to the boundary problem given by (2.6)–(2.9) can be investigated.

3. Modeling procedure

The course of modeling is based on the two fundamental assumptions.

The first modeling assumption is a certain extension of *themicro-macro hypothesis*, introduced framework of *the tolerance averaging technique*, cf. [1–6].

In accordance with that hypothesis, the temperature field θ can be approximated with an acceptableaccuracy with

$$\theta_M(z) \equiv \vartheta(z) + h^A(x)\psi_A(z), \tag{3.1}$$

where the slowly varying fields $\vartheta(\cdot, z)$ and $\psi_A(\cdot, z, t)$ are the *tolerance averaging of temperature field* and *amplitude fluctuations fields*, respectively. Here and in the sequel the summation convention holds with respect to indices A, B, ... = 1, ..., N. Symbols $h^A(x)$, $A = 1, ..., N$, denote in (3.1) tolerance shape functions which should be periodic and satisfy the conditions

$$h^A \in o(\lambda), \qquad \lambda\nabla_{\Xi} h^A \in o(\lambda), \qquad \langle ch^A \rangle = 0.$$

For particulars the reader is referred to [1–6]. The *tolerance-micro macro hypothesis* can be formulated in the following form:

Tolerance micro-macro hypothesis. The residual part of the temperature field θ_{res} being the difference between the temperature field θ and its tolerance parts θ_M given by Eq. (3.1) can be treated as zero,

$$\theta_{\text{res}} \equiv \theta - \theta_M \approx 0, \tag{3.2}$$

i.e. it vanishes with an acceptable "tolerance approximation".

In contrast to the tolerance modeling in this paper instead of quoted above micro-macro hypothesis *the extended micro-macro hypothesis* will be applied. According to this new hypothesis the right side of Eq. (3.2) is not equal to zero, but is a special infinite analytic expansion. This is a certain attempt to adapt of an idea implemented in signal theory, where we have to deal with the "overlapping" of many signals determining by different parameters. In order to formulate this hypothesis denote by Δ_1, Δ_2, ..., Δ_n the homogeneity subregions of the basic cell Δ and let $\mathit{\Gamma}_1 = \partial\Delta_1$, $\mathit{\Gamma}_2 = \partial\Delta_2$, ..., $\mathit{\Gamma}_n = \partial\Delta_n$, $\mathit{\Gamma}_\Delta \equiv \mathit{\Gamma}_1 \cup \mathit{\Gamma}_2 \cup ... \cup \mathit{\Gamma}_n$. Now, *the extended micro-macro hypothesis* can be formulated as follow:

ASSUMPTION 1. (*Extended micro-macro hypothesis*)
The residual part $\theta_{\text{res}} \equiv \theta - \theta_M$ of the temperature field $\theta \equiv \theta_M - \theta_{\text{res}}$ being the difference between the temperature field θ and its tolerance parts θ_M given by Eq. (3.1) can be represented by the Fourier series $\theta_{\text{res}}(x, y, t) = \lambda a_0(y, t) + \sum_{p=1}^{\infty} a_p(y, t)\phi^p(x)$ formed by an orthogonal Δ-periodic basis $\phi^p(x)$, $p = 1, 2, ...$ which should be constant with respect to periodic variable x. Hence $\psi_p = \psi_p(y, t)$.

The orthogonality condition is here related to the scalar product $f_1 \circ f_2 = \langle f_1 f_2 \rangle$, determined by the averaged value $\langle f_1 f_2 \rangle$ which is constant for any Δ-periodic functions f_1 and f_2 defined on R^m.

In the subsequent considerations it will be assumed that summation convention holds also with respect to $p = 1, 2, ...$. That is why, under the second modeling assumption, formula (2.6) can be rewritten in the form

$$\theta_{\text{res}}(x, y, t) = \lambda a_0(y, t) + a_p(y, t)\phi^p(x). \tag{3.3}$$

In according to *the second modeling assumption* shape functions as well as the orthogonal system $\phi^p(x)$ are independent on the thermal and geometrical properties of the conductor.

Assumption 2. (*Continuity condition of the tolerance heat flux vector*) The component of heat flux vector $q_{\text{res}} \equiv K\nabla\theta_{\text{res}}$ generated by residual part θ_{res} of the temperature field, normal to the surface Γ separating constituents, vanish in almost everywhere on Γ.

Assumption 2 should be satisfied independently of the material structure of the considered composite and hence for any $A = 1, ..., N$ and any positive integer $p > 0$ we have $\nabla_{\Xi} h^A = 0$ and $\nabla_{\Xi}\phi^p = 0$ almost everywhere on Γ and

$$\begin{aligned} \langle c\phi^p \rangle &= 0, & \langle K\phi^p \rangle &= 0, & p &= 1, 2, \dots, \\ \langle ch^A \rangle &= 0, & \langle Kh^A \rangle &= 0, & A &= 1, ..., N. \end{aligned} \tag{3.4}$$

Since fluctuation amplitudes are slowly-varying functions (2.8) yields

$$\begin{aligned} \langle \theta \rangle(x, y) &= \langle \vartheta \rangle(x, y) + \langle h^A\psi_A \rangle(x, y) + \lambda a_0(y), \\ \langle \theta_{\text{res}} \rangle(x, y) &= \lambda a_0(y). \end{aligned} \tag{3.5}$$

Conditions (2.7) and (3.5) imply

$$a_0(y) = 0 \qquad \text{for} \;\; y \in \Phi \tag{3.6}$$

and we conclude from Assumptions 1 and 2 that the temperature field can be written as a sum (2.6) for

$$u(x, z) = \vartheta(x, z) + h^A\psi_A(x, z) \tag{3.7}$$

and θ_{res} represented by Eqs. (3.3) and (3.6). At the same time conditions (2.7) are satisfied. Moreover, bearing in mind Eqs. (3.5), we conclude that $\langle u - \vartheta \rangle = \langle h^A\psi_A \rangle = \langle h^A \rangle\psi_A = 0$ and hence u can be treated as an averaged temperature field.

Now introduce the new representations $h^A(x) \equiv \lambda g^A(\lambda^{-1}x)$ and $\phi^p(x) \equiv \lambda\phi^p(\lambda^{-1}x)$ for tolerance shape functions $h^A(x)$ and orthogonal basis $\phi^p(x)$. Assumptions 1 and 2 yields the conclusion that if the temperature field is represented by (2.7) and u and θ_{res} fulfil Eqs. (3.3), (3.6), (3.7) then the following integral-differential equations hold:

$$
\begin{gathered}
\langle c\dot{u}\rangle - \nabla^T\left[\langle K\nabla u\rangle - \langle K\nabla_{\Xi} g\rangle\psi_A - \langle K\nabla_{\Xi}\phi^p\rangle a_p\right] = -\langle b\rangle,\\
\lambda^2\left\{\begin{bmatrix}\langle g_\varepsilon^A c g^B\rangle & \langle g^A c\phi^p\rangle\\ \langle \phi^p c g^B\rangle & \langle g^A c\phi^p\rangle\end{bmatrix}\begin{bmatrix}\dot{\psi}_B\\ \dot{a}_p\end{bmatrix} - \nabla_{\Phi}^T\left(\begin{bmatrix}\langle g_\varepsilon^A c g^B\rangle & \langle g^A c\phi^p\rangle\\ \langle \phi^p K g^B\rangle & \langle \phi^p c\phi^q\rangle\end{bmatrix}\begin{bmatrix}\nabla_{\Phi}\psi_B\\ \nabla_{\Phi}a_p\end{bmatrix}\right)\right\}\\
+\lambda\left(\begin{bmatrix}\langle\nabla_{\Xi}^T g^A K g^B\rangle & \langle\nabla_{\Xi}^T g^A K\phi^p\rangle\\ \langle\nabla_{\Xi}^T\phi^p K g^B\rangle & \langle\nabla_{\Xi}^T\phi^p k\phi^q\rangle\end{bmatrix}\right.\\
\left.-\begin{bmatrix}\langle\nabla_{\Xi}^T g^B K g^B\nabla_{\Phi}\psi_B\rangle & \langle\nabla_{\Xi}^T\phi^q K g^A\rangle\\ \langle\nabla_{\Xi}^T\phi^q K g^B\nabla_{\Phi}\psi_B\rangle & \langle\nabla_{\Xi}^T\phi^q K\phi^p\rangle\end{bmatrix}\right)\begin{bmatrix}\nabla_{\Phi}\psi_B\\ \nabla_{\Phi}a_q\end{bmatrix}\\
+\begin{bmatrix}\langle\nabla_{\Xi}^T g^A K\nabla_{\Xi} g^B\rangle & \langle\nabla_{\Xi}^T g^A K\nabla_{\Xi}\phi^q\rangle\\ \langle\nabla_{\Xi}^T\phi^p K\nabla_{\Xi} g^B\rangle & \langle\nabla_{\Xi}^T\phi^p k\nabla_{\Xi}\phi^q\rangle\end{bmatrix}\begin{bmatrix}\psi_B\\ a_q\end{bmatrix} + \begin{bmatrix}\langle\nabla_{\Xi}^T g^A K\nabla u\rangle\\ \langle\nabla_{\Xi}^T\phi^p K\nabla u\rangle\end{bmatrix} = \lambda\begin{bmatrix}\langle g^A b\rangle\\ \langle\phi^p b\rangle\end{bmatrix}.
\end{gathered}
\tag{3.8}
$$

The above equations can be treated as a certain exact model of heat transfer in periodic rigid composite conductors provided that tolerance shape functions mentioned in Assumption 2 can be found. It is mean that solutions to Eqs. (3.8) via (2.6) and Assumption 1 and Assumption 2 can be treated as a certain representation of solutions to heat transfer equation (2.1) and *vice versa*.

To make this paper self consistent we should formulate, as a certain remark, a *third modeling assumption*. This is and additional assumption which provides the ability to perform tolerance modeling procedure with respect to the fields $u(\cdot)$ as the slowly varying tolerance averaged temperature field, and to the fields $\psi_A(\cdot)$ and $a_p(\cdot)$ as fluctuation amplitudes, respectively.

ASSUMPTION 3. (*The validity of tolerance modeling procedure*)
Shape functions $h^A(x) \equiv \lambda g^A(\lambda^{-1}x)$ together with orthogonal basis $\phi^p(x) \equiv \lambda\varphi^p(\lambda^{-1}x)$ can be treated as an infinite sequence of tolerance shape functions and with respect to them the tolerance modeling procedure can be realized with slowly-varying (with respect to the periodic directions variables $x \in R^m$) averaged temperature field $u(x, y, t)$ which is, together with fluctuation amplitudes $\psi_A(x, y, t)$, $A = 1, ..., N$, and Fourier coefficients $a_p = a_p(y)$, $p = 1, 2, ...,$ as basic unknowns of the *extended tolerance model.*

Under Assumption 3 we can apply to (3.8) typical tolerance approximations with respect to and obtain

$$
\begin{gathered}
\langle c\rangle\dot{u} - \nabla^T\left[\langle K\rangle\nabla u - \langle K\nabla_{\Xi} g\rangle\psi_A - \langle K\nabla_{\Xi}\varphi^p\rangle a_p\right] = -\langle b\rangle,\\
\lambda^2\left\{\begin{bmatrix}\langle g_\varepsilon^A c g^B\rangle & \langle g^A c\varphi^p\rangle\\ \langle \varphi^p c g^B\rangle & \langle g^A c\varphi^p\rangle\end{bmatrix}\begin{bmatrix}\dot{\psi}_B\\ \dot{a}_p\end{bmatrix} - \nabla_{\Phi}^T\left(\begin{bmatrix}\langle g_\varepsilon^A c g^B\rangle & \langle g^A c\varphi^p\rangle\\ \langle \varphi^p K g^B\rangle & \langle \varphi^p c\varphi^q\rangle\end{bmatrix}\begin{bmatrix}\nabla_{\Phi}\psi_B\\ \nabla_{\Phi}a_p\end{bmatrix}\right)\right\}
\end{gathered}
\tag{3.9}
$$

$$
\begin{aligned}
(3.9)[\text{cont}] \qquad &+\lambda\left(\begin{bmatrix}\langle\nabla_{\Xi}^{T}g^{A}Kg^{B}\rangle & \langle\nabla_{\Xi}^{T}g^{A}K\varphi^{p}\rangle\\ \langle\nabla_{\Xi}^{T}\varphi^{p}Kg^{B}\rangle & \langle\nabla_{\Xi}^{T}\varphi^{p}k\varphi^{q}\rangle\end{bmatrix}\right.\\
&\left.-\begin{bmatrix}\langle\nabla_{\Xi}^{T}g^{B}Kg^{B}\rangle & \langle\nabla_{\Xi}^{T}\varphi^{q}Kg^{A}\rangle\\ \langle\nabla_{\Xi}^{T}\varphi^{q}Kg^{B}\rangle & \langle\nabla_{\Xi}^{T}\varphi^{q}K\varphi^{p}\rangle\end{bmatrix}\right)\begin{bmatrix}\nabla_{\Phi}\psi_{B}\\ \nabla_{\Phi}a_{q}\end{bmatrix}\\
&+\begin{bmatrix}\langle\nabla_{\Xi}^{T}g^{A}K\nabla_{\Xi}g^{B}\rangle & \langle\nabla_{\Xi}^{T}g^{A}K\nabla_{\Xi}\varphi^{q}\rangle\\ \langle\nabla_{\Xi}^{T}\varphi^{p}K\nabla_{\Xi}g^{B}\rangle & \langle\nabla_{\Xi}^{T}\varphi^{p}k\nabla_{\Xi}\varphi^{q}\rangle\end{bmatrix}\begin{bmatrix}\psi_{B}\\ a_{q}\end{bmatrix}+\begin{bmatrix}\langle\nabla_{\Xi}^{T}g^{A}K\rangle\\ \langle\nabla_{\Xi}^{T}\varphi^{p}K\rangle\end{bmatrix}\nabla u=\lambda\begin{bmatrix}\langle g^{A}b\rangle\\ \langle\varphi^{p}b\rangle\end{bmatrix}.
\end{aligned}
$$

System of partial differential equations (3.9) represents a certain averaged model of the heat transfer which will be referred to as *the extended tolerance heat transfer model.* To equations (3.8) as well as to the model equations (3.9) should be added related boundary and initial conditions for basic unknowns which are long-wave term, fluctuation amplitudes ψ_B and Fourier coefficients a_p. The basic unknowns of models (3.8) and (3.9) can be treated as slowly-varying, provided that solutions to both above initial-boundary problems a closed. Since, after introducing limit passage $\lambda \to 0$, model equations (3.8) and (3.9) coincide we conclude that solutions to these initial-boundary problems for (3.8) and for (3.9) asymptotically coincide.

The extended tolerance heat transfer model is an approximated model of heat transfer in periodic rigid composite conductors and has been examined in [7]. In this paper our attention is focused on the analysis of the exact model of heat conduction in periodic composites represented by model equations (3.8). In this paper consideration will be illustrated in an attempt to obtain the exact solution of heat transfer equation (2.1) for a special case of composite conductor.

The infinite number of equations in (3.2) is an important inconvenience of both limit model and the extended tolerance model. Although, there are many possibilities to avoid this inconvenience. One of these possibilities is rescaling of the tolerance shape function system $h^A = \lambda g^A$. The procedure of rescaling is similar to that applied to the extended tolerance heat transfer model in [7]. For the extended limit model it will be described in the next section.

4. Rescaling of the tolerance shape function system

Suppose that we have properly chosen a finite N-tuple $(h^1(x), ..., h^N(x))$ of Δ-periodic tolerance shape functions and infinite orthogonal and Δ-periodic Fourier basis $\phi^p(x)$, $p = 1, 2, ...$ We are to construct the infinite sequence of N-tuples $S_{(n)} = (h^1_{(n)}(x), ..., h^N_{(n)}(x))$ of differentiable (almost everywhere in R^m) Δ-periodic functions. To this end, let $S_{(1)} = (h^1_{(1)}(x), ..., h^N_{(1)}(x)) =$

$(h^1(x), ..., h^N(x))$ and N-tuple $S_{(n+1)} = (h^1_{(n+1)}(x), ..., h^N_{(n+1)}(x))$ is defined for $x \in R^m$ as follow:

$$h^A_{(n+1)}(x) = \begin{cases} h^A_{(n)}(x) & \text{for } |h^A_{(n)}(x)| \le \frac{1}{2}\mu^A_{(n)}, \\ \mu^A_{(n)} - h^A_{(n)}(x) & \text{for } h^A_{(n)}(x) > \frac{1}{2}\mu^A_{(n)}, \\ -\mu^A_{(n)} - h^A_{(n)}(x) & \text{for } h^A_{(n)}(x) < -\frac{1}{2}\mu^A_{(n)}, \end{cases} \tag{4.1}$$

where $\mu^A_{(n)} \equiv \max\{|h^A_{(n)}(x)|\colon x \in R^m\}$ satisfy inequality $\mu^A_{(n)}(x) < \dfrac{\lambda}{n}$ and hence $\lim\limits_{n\to+\infty} h^A_{(n)}(x) = \lim\limits_{n\to+\infty} \left[\dfrac{\lambda}{n} g^A_{(n)}(\lambda^{-1}x)\right] = 0$ for $h^A_{(n)}(x) = \dfrac{\lambda}{n} g^A_{(n)}(\lambda^{-1}x)$. Shape functions $h^A_{(n)}(x)$ are differentiable almost everywhere in R^m. Now, we can replace in (3.8) and (3.9) tolerance shape functions g^A onto $g^A_{(n)}$ and next apply the limit passage $n \to +\infty$ to equations obtained on this way. As the result of this procedure instead of Eq. (3.8) we obtain

$$\langle c\dot{u}\rangle - \nabla^T \left[\langle K\nabla u\rangle - \langle K\nabla_\Xi g\psi_A\rangle - \langle K\nabla_\Xi \phi^p\rangle a_p\right] = -\langle b\rangle,$$

$$\begin{aligned} &\lambda^2 \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \langle \phi^p c \phi^q\rangle \end{bmatrix} \begin{bmatrix} \dot{\psi}_B \\ \dot{a}_q \end{bmatrix} - \nabla^T_\Phi \left(\begin{bmatrix} 0 & 0 \\ 0 & \langle \phi^p c \phi^q\rangle \end{bmatrix} \begin{bmatrix} \nabla_\Phi \psi_B \\ \nabla_\Phi a_q \end{bmatrix} \right) \right\} \\ &\quad + \lambda \left(\begin{bmatrix} 0 & \langle \nabla^T_\Xi g^A K \phi^q\rangle \\ 0 & \langle \nabla^T_\Xi \phi^p K \phi^q\rangle \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \langle \nabla^T_\Xi g^B K \phi^p\rangle & \langle \nabla^T_\Xi \phi^q K \phi^p\rangle \end{bmatrix} \right) \begin{bmatrix} \nabla_\Phi \psi_B \\ \nabla_\Phi a_q \end{bmatrix} \\ &\quad + \begin{bmatrix} \langle \nabla^T_\Xi g^A K \nabla_\Xi g^B \psi_B\rangle & \langle \nabla^T_\Xi g^A K \nabla_\Xi \phi^q\rangle \\ \langle \nabla^T_\Xi \phi^p K \nabla_\Xi g^B \psi_B\rangle & \langle \nabla^T_\Xi \phi^p K \nabla_\Xi \phi^q\rangle \end{bmatrix} \begin{bmatrix} 1 \\ a_q \end{bmatrix} \\ &\quad + \begin{bmatrix} \langle \nabla^T_\Xi g^A K \nabla u\rangle \\ \langle \nabla^T_\Xi \phi^p K \nabla u\rangle \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ \langle \phi^p b\rangle \end{bmatrix} \end{aligned} \tag{4.2}$$

which will be referred to as *the reduced limit model*.

It must be emphasized that the means of basic unknowns ψ_A, a_p before and after completing the procedure of rescaling (4.1) and of limit passage $n \to +\infty$ are different. Particularly, terms $h^A(x)\psi_A(x,y)$ in (3.7) vanish and hence representation of temperature field given by extended micro-macro hypothesis changes to the form

$$\theta(x,z) = u(x,z) + \theta_{\text{res}}(x,z). \tag{4.3}$$

for θ_{res} given by Eq. (3.3). Similarly, formula (3.5), under $a_0 = 0$, takes the form

$$\langle\theta\rangle(x,z) = \langle u\rangle(x,z) = \langle\vartheta\rangle(x,z). \tag{4.4}$$

At the same time from expansion

$$\theta(x,z) = u(x,z) + \lambda a_p(z)\varphi_p(x,z) \tag{4.5}$$

cannot be conclude that temperature gradient $\nabla\theta(x,z)$ is equal to $\nabla u(x,z) + \lambda\nabla[a_p(z)\varphi_p(x,z)]$ since from Eq. (3.1) one can obtain

$$\begin{aligned} \nabla_{\Xi}\theta(x,z) &= \lim_{n\to+\infty}\Big\{\nabla_{\Xi}u(x,z) + \nabla_{\Xi}[h^A_{(n)}(x)\psi_A(x,z)] + \nabla_{\Xi}\theta_{\rm res}(x,z)\Big\}, \\ \nabla_{\Phi}\theta(x,z) &= \lim_{n\to+\infty}\Big\{\nabla_{\Phi}u(x,z) + h^A_{(n)}(x)\nabla_{\Phi}\psi_A(x,z)] + \nabla_{\Phi}\theta_{\rm res}(x,z)\Big\}. \end{aligned} \tag{4.6}$$

It is easy to verify that all coefficients non-vanishing under passage from (3.9) to (4.1) and (4.2) remain unchanged.

5. Two-phased laminated rigid conductor

In the case of two-phased one-directionally periodic composite the orthogonal basis mentioned in Assumption 1 will be assumed in regions occupied by constituents in the form of the well-known infinite trigonometric system, cf. Fig. 3, i.e.,

$$\varphi^p\left(\frac{x}{l}\right) = \begin{cases} \dfrac{1}{2}\cos\dfrac{p\pi}{l'}x, & x\in\langle 0,l'\rangle, \\ \dfrac{1}{2}\cos\dfrac{p\pi}{l''}(x-l), & x\in\langle l',l\rangle. \end{cases} \tag{5.1}$$

Now, Assumptions 1 and 2 are fulfilled provided that the saw-like function is used as the single tolerance shape function and the tolerance heat flux component normal to the planes separating components is continuous, cf. [3].

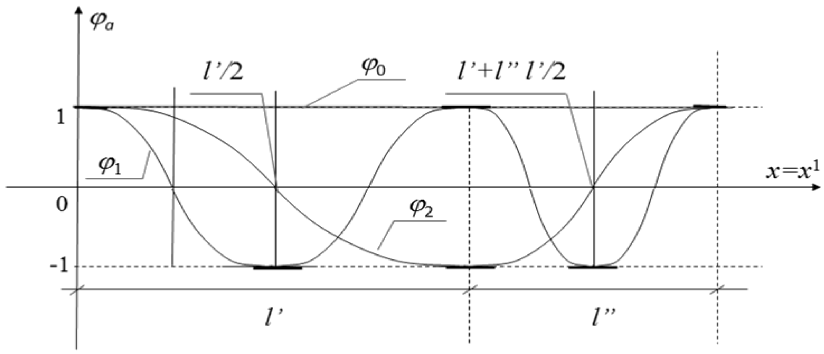

Fig. 3. The initial three elements of trigonometric orthogonal basis for two-phased uniperiodic layer. Assumptions 1–3 are satisfied.

The characteristic property of the *reduced limit model equations* (4.2) that we can distinguish from this model single equations for averaged temperature u independent on the other basic unknown of this model. To this end introduce denotations $\langle\sigma\rangle_+ = \dfrac{\sigma_{II}}{\eta_{II}} + \dfrac{\sigma_I}{\eta_I}$ and $[\sigma] = \sigma_{II} - \sigma_I$ for any σ taking two constant values σ_I and σ_{II} for both constituents. The first two equations from (4.2) can be rewritten in the form

$$\begin{aligned} \langle c\dot{u}\rangle - \nabla^T[\langle K\nabla u\rangle - [0,\,[k_{12}],\,[k_{13}]]^T\psi &= -\langle b\rangle, \\ \langle k_{11}\rangle_+\psi + [[k_{11}],\,[k_{12}],\,[k_{13}]]\nabla u &= \lambda\langle gb\rangle. \end{aligned} \tag{5.2}$$

The rest of equations can be rewritten as infinite system of partial differential equation system for a_p

$$\begin{aligned} &\frac{\lambda^2}{8}\left\{\langle c\rangle\begin{bmatrix}\dot{a}_{2p}\\ \dot{a}_{2q-1}\end{bmatrix} - \nabla_\Phi^T\left(\langle K\rangle\nabla_\Phi^T\begin{bmatrix}a_{2p}\\ a_{2q-1}\end{bmatrix}\right)\right\} + \begin{bmatrix}0\\ \langle K_{11}^I\rangle_+\psi\end{bmatrix} \\ &\quad + \frac{\lambda}{2}\begin{bmatrix}0 & \dfrac{s^2+p^2}{\left(p-\frac{1}{2}\right)^2 - s^2}\\ \dfrac{\left(s-\frac{1}{2}\right)^2+\left(s-\frac{1}{2}\right)^2}{q^2-\left(s-\frac{1}{2}\right)^2} & 0\end{bmatrix}\begin{bmatrix}([k_{12}]\partial_2+[k_{13}]\partial_3)a_{2r}\\ ([k_{12}]\partial_2+[k_{13}]\partial_3)a_{2s-1}\end{bmatrix} \\ &\quad + \frac{1}{2}\pi^2\langle k_{11}\rangle_+\begin{bmatrix}p^2 & 0\\ 0 & \left(q-\frac{1}{2}\right)^2\end{bmatrix}\begin{bmatrix}a_{2p}\\ a_{2q-1}\end{bmatrix} \\ &\quad + \langle[[k_{11},[k_{12}],[k_{13}]]\nabla u\rangle\begin{bmatrix}0\\ 1\end{bmatrix} = \lambda\begin{bmatrix}\langle\phi^{2p}b\rangle\\ \langle\phi^{2q-1}b\rangle\end{bmatrix}. \end{aligned} \tag{5.3}$$

It should be emphasized that the convergence of the infinite sum β is not any restriction but is a certain consequence of Assumptions 1–3. Let us observe that from Eq. $(5.2)_2$ we can determine $\overline{\psi}$:

$$\psi + \beta = \frac{1}{\langle k_{11}\rangle_+}(\lambda\langle gb\rangle - \langle[[k_{11}],\,[k_{12}],\,[k_{13}]]\nabla u\rangle) \tag{5.4}$$

and put it into Eq. $(5.2)_1$. As a result of this elimination we obtain equation for u

$$\langle c\dot{u}\rangle - \nabla^T\langle\overline{K}\nabla u\rangle = -\langle b\rangle - \lambda\frac{[k_{11}]^2+[k_{12}]^2+[k_{13}]^2}{\langle k_{11}\rangle_0}\nabla^T\langle gb\rangle, \tag{5.5}$$

where

(5.6) $$\overline{K} = \begin{bmatrix} \overline{k}_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & \langle k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}, \qquad \overline{k}_{11} \equiv k_{11} - \frac{[k_{12}]^2 + [k_{13}]^2}{\langle k_{11} \rangle_+}$$

and $K^{\text{eff}} \equiv \langle \overline{K} \rangle$ is equal to the well-known tolerance effective matrix. It must be emphasized that the above elimination procedure, with a view to, inter alia, to extract from the equations of the so-called *effective modulus matrix* K^{eff}, was so far (in known in the literature approaches) performed in the asymptotic case, i.e. with the required execution of a limit passage $\lambda \to 0$. In the framework of the proposed *limit model* this procedure is realizable without having to use such a limit passage.

Summing up Eqs. (5.3), having regard to the convergence of corresponding series and denote

$$\alpha_1 \equiv \lim_{nn \to +\infty} \frac{1}{m} \sum_{m=1}^{+\infty} \left(p - \frac{1}{2}\right)^2 a_{2p}, \qquad \beta_1 \equiv \lim_{nn \to +\infty} \frac{1}{m} \sum_{m=1}^{+\infty} q^2 a_{2q-1},$$

we conclude that

(5.7) $$\frac{\lambda^2}{8} \left\{ \langle c \rangle \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} - \nabla_{\Phi}^T \left(\langle K \rangle \nabla_{\Phi}^T \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right) \right\} - \frac{\lambda}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} ([k_{12}]\partial_2 + [k_{13}]\partial_3)\alpha \\ ([k_{12}]\partial_2 + [k_{13}]\partial_3)\beta \end{bmatrix} + \langle [[k_{11}], [k_{12}], [k_{13}]] \nabla u \rangle \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{2}\pi^2 \langle k_{11} \rangle_+ \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$$

for $\alpha \equiv a_1 + a_3 + a_5 + \dots$.

Figure 4 illustrates the phenomenon of boundary effect in the two-component layered composite subjected by fluctuations $\theta_{\text{res}}(x, 0) = \theta_{\text{res}}(x, \delta)$ imposed on both sides of the dividing wall by constant different temperatures $u(x, 0) = u_{\text{ext}}$, $u(x, \delta) = u_{\text{int}}$ and arbitrary fluctuation tractions lead to the conditions α, $\beta =$ const. At the same time $\langle k_{11} \rangle / \langle k_{33} \rangle_H = 10$ and $\delta / H_1 = 0.1$. The horizontal axes are related to the dimensionless variables $\xi' = x/H_1$ and $\xi^3 = f^3/H_3$. The vertical axis is the temperature axis represented by $\theta(x, z) = u(x, y) + \lambda a_p(y)\varphi^p(x)$, where $a_p \neq 0$ will be assumed only for an arbitrary odd p. The boundary effect phenomenon is here described by system model equations obtained from (5.3) with the assumption of the independence of basic unknowns on time and the absence of the heat sources, i.e. $b = 0$. Under the aforementioned independence boundary conditions on the y^2 variable we shall also limit model equations sim-

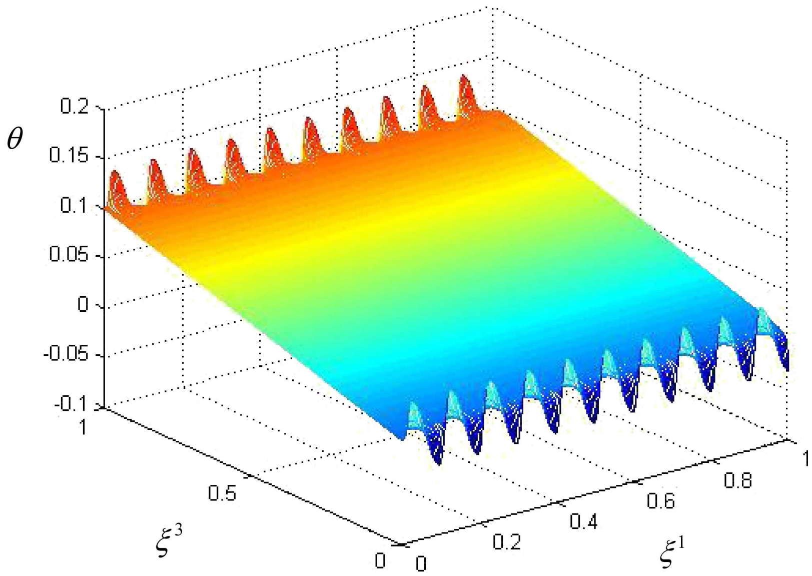

FIG. 4. Illustration of the boundary effect behavior.

plified to the independent equations:

$$
(5.8)\qquad
\begin{aligned}
\nabla^T \langle K^{\text{eff}} \nabla u \rangle &= 0, \\
-\frac{\lambda^2}{8} \langle k_{33} \rangle \frac{\partial^2 a_p}{\partial (y^3)^2} + \frac{1}{4} \pi^2 \langle k_{11} \rangle_0 p^2 a_p &= \frac{1}{\delta} (k_{11}^{II} - k_{11}^{I})(u_{\text{ext}} - u_{\text{int}}).
\end{aligned}
$$

The characteristic feature of the description (5.3) of the expansion on the temperature fluctuation into the interior of the dividing wall is that it gives independent equations for

$$
(5.9)\qquad \omega \equiv \frac{\delta}{\lambda} \sqrt{\frac{\langle k_{11} \rangle_+}{\langle k_{33} \rangle}}.
$$

Formulas

$$
(5.10)\qquad
\begin{aligned}
\theta(x, z^3) &= u(x, z^3) + \lambda a_p(z^3) \varphi^p(x)] \quad \text{(no summation)}, \\
u(x, z^3) &= \left(1 - \frac{z^3}{\delta}\right) u_{\text{ext}} + \frac{z^3}{\delta} u_{\text{int}}, \\
a_p(z^3) &= \frac{\sinh\left\{\pi p \omega \left(1 - \frac{z^3}{\delta}\right)\right\}}{\sinh \pi p \omega} \overline{a}_p^{\text{int}}(z^3) + \frac{\sinh\left\{\pi p \omega \frac{z^3}{\delta}\right\}}{\sinh \pi p \omega} \overline{a}_p^{\text{ext}}(z^3),
\end{aligned}
$$

where

$$\begin{aligned}\overline{a}_p^{\text{int}}(z^3) &= a_p^{\text{int}}(z^3) - (\pi p\omega)^{-1}(u_{\text{int}} - u_{\text{ext}}),\\ \overline{a}_p^{\text{ext}}(z^3) &= a_p^{\text{ext}}(z^3) - (\pi p\omega)^{-1}(u_{\text{int}} - u_{\text{ext}})\end{aligned}$$

for $z^3 = y$ represents the solution to the boundary value problem. That is why the boundary effect behavior is described exclusively by residual temperature $\theta_{\text{res}} = \lambda a_p \varphi^p$ but only by coefficients a_p and hence the intensityof this effect is independent on the boundary fluctuations uniquely determined by boundary values $a_p^{\text{int}}(z^3)$, $a_p^{\text{ext}}(z^3)$ of a_p. The attenuation of temperature fluctuations is measured exclusively by the coefficient ω given by Eq. (5.9) located in every exponential expression for a_p in Eqs. (5.10) and hence this coefficient will be referred to as *the intensity of the boundary effect behavior*.

Figure 4 shows a relatively high intensity of the boundary effect behavior, i.e. rapid appearance of temperature fluctuations with a distance from sides of the wall designated by equations $\xi^3 = 0$ and $\xi^3 = 1$, respectively. The dependence of the intensity of the effect behavior with respect to various values of parameters λ/δ and $\langle k_{11}\rangle_H/\langle k_{33}\rangle$ and under $\eta \equiv l^I/l = 0.5$.

6. Concluding remarks

As a first concluding remark we emphasize that the tolerance modeling technique in the case of heat conduction theory of the simple micro-periodic composites proposed in this paper, leads to the averaged models in the framework of which is possible:

1) to achieve reproducible cell periodicity;
2) to achieve a scalar parameter characterizing the degree of compaction of the microstructure and having an impact on the temperature field and temperature gradient field in the composite;
3) to represent the temperature field as a sum of the fields locally averaged temperature and temperature fluctuation fields (fields difference between the temperature field and the field locally averaged temperature) regardless of the type of periodicity of the composite.

Moreover, as a second remark we emphasize that the representation of the temperature field mentioned in Assumption 3, let us refer to the comparison of solutions for the temperature field of the same boundary problem but for the conductors with various geometrical structure of periodicity. The realization of this comparison is possible exclusively after previous investigation of the appropriate tolerance shape functions. and the related orthogonal series. Shape functions and orthogonal series should agree with the character of the investigated problem and hence the realization of the proposed modeling procedure needs previous detailed analysis of the physical background of the investigated problem.

Finally, we emphasize that there are no reasons to guess that effective conductivity matrix $\mathbf{K}^{\text{eff}}$ is identical with the related matrices of effective conductivity modulus obtained on the other ways, particularly in the framework of the well known asymptotic homogenization approach.

References

1. Woźniak Cz., Wierzbicki E., *Averaging techniques In thermomechanics of composite solids, Tolerance averaging versus homogenization*, Wydawnictwo Politechniki Częstochowskiej, Częstochowa, 2000.
2. Nagórko W., *Wybrane metody modelowania płyt niejednorodnych*, Publications of Warsaw University of Life Sciences – SGGW, Warszawa, 2008.
3. Woźniak Cz. [Ed.], *Thermomechanics of microheterogeneous solids and structures. Tolerance averaging approach*, Wydawnictwo Politechniki Łódzkiej, Łódź, 2009.
4. Woźniak Cz. [Ed.], *Developments In Mathematical Modeling and Analysis of Microstructured Media*, Wydawnictwo Politechniki Śląskiej, Gliwice, 2010.
5. Jędrysiak J., *Termomechanika laminatów, płyt i powłok o funkcyjnej gradacji własności*, Wydawnictwo Politechniki Łódzkiej, Łódź, 2010.
6. Michalak B., *Termomechanika ciał z pewną niejednorodną mikrostrukturą: technika tolerancyjnej aproksymacji*, Wydawnictwo Politechniki Łódzkiej, Lódź, 2010.
7. Wierzbicki E., Kula D., Mazewska M., *O fourierowskiej realizacji tolerancyjnego modelowania zagadnień przewodnictwa ciepła prostych kompozytów periodycznych*, [in:] *Modelowanie struktur i konstrukcji inżynierskich*, G. Jemielita, M. Wągrowska [Eds.], pp. 253–265, Wydawnictwo SGGW, Warszawa, 2014.
8. Kula D., *On Fourier implementation of tolerance modeling of linear elastodynamics of periodic composites*, Journal of Applied Mathematics and Computational Mechanics, **13**, 4, 73–80, 2014.
9. Mazewska M., Kula D., Wierzbicki E., *Boundary effect behavior in the hexagonal-type biperiodic structures with rotational symmetry*, Journal of Applied Mathematics and Computational Mechanics, **13**, 2, 73–85, 2014.
10. Kula D., Mazewska M., Wierzbicki E., *Some remarks on the tolerance averaging of heat conduction In chessboard palisade-type periodic composites*, Scientific Review, Engineering and Enviromental Sciences, **21**, 3, 131–140, 2012.
11. Mazewska M., *Tolerance modeling of boundary effect behavior in hexagonal-type composites*, doctoral dissertation, Warsaw University of Life Sciences, Warszawa, 2013.
12. Mazewska M., Kula D., Wierzbicki E., *On the boundary effect behavior in the hexagonal-type biperiodic structures with rotational symmetry*, Journal of Applied Mathematics and Computational Mechanics, **13**, 2, 73–85, 2014.
13. Mazewska M., Wierzbicki E., *Modelowanie tolerancyjne przewodnictwa ciepła w kompozytach o strukturze dwukierunkowo-periodycznej* [in Polish], Acta Scientarum Polonarum, Seria Architectura, **12**, 1, 3–17, 2013.
14. Wierzbicki E., Woźniak Cz., *On the dynamic behaviour of honeycomb based composite solids*, Acta Mechanica, **141**, 161–172, 2000.
15. Matysiak S.J., Jevtushenko A.A., Ivanyk E.G., *Contact temperature and wear of composite elements during braking*, Int. J. Heat and Mass Transfer, **45**, 193–199, 2002.