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Research Paper

Slow Flow of Couple Stress Fluid Past a Cylinder Embedded in a Porous Medium: Slip Effect

Priya SARKAR, Krishna Prasad MADASU*

Department of Mathematics, National Institute of Technology
Raipur, 492010, Chhattisgarh, India

*Corresponding Author e-mails: madaspra.maths@nitrr.ac.in, kpm973@gmail.com

An analytical study for the creeping flow of a couple stress fluid past a cylinder embedded in a porous medium is presented using the slip condition. The uniform flow is considered far away from a cylinder. The boundary conditions used are zero couple stress and tangential slip conditions. The modified Bessel functions represent the stream function (the velocity). The drag exerted on a solid cylinder immersed in a porous medium is derived. The impacts of the couple stress, permeability, and slip parameters on the normalized drag force are presented graphically. The drag forces of well-known exceptional cases are reduced. The drag force is a decreasing function of the permeability and couple stress parameters and an increasing function of the slip parameter.

Keywords: couple stress fluid; cylinder; Brinkman's equation; saturated porous medium; slip coefficient; drag force.

Notations

a – radius of solid cylinder [m],
d_{ij} – deformation rate tensor,
D_N – normalized drag force,
e_{ijk} – alternating tensor,
F_D – drag force [N],
k – permeability [m^2],
m_{ij} – couple stress tensor,
p – fluid pressure [$\mathrm{N/m}^2$],
$\mathbf{q}$ – fluid velocity [m/s],
q_r, q_θ – velocity components of fluid [m/s],
r – radial coordinate measured from cylinder axis [m],
R_e – Reynolds number $\left(\frac{2aU\rho}{\mu}\right)$,
t_{ij} – stress tensor [$\mathrm{N/m}^2$],
U – uniform velocity of fluid [m/s].

Greek symbols

α – dimensionless permeability parameter,
β – slip coefficient,
δ_{ij} – Kronecker delta,
η, η' – couple stress viscosities [N · s],
θ – angular coordinate [deg],
λ – length parameter of couple stress fluid [m],
μ – viscosity coefficient of fluid [(N · s)/m^2],
ρ – fluid density [kg/m^3],
τ – couple stress viscosity ratio parameter,
ψ – stream function,
ω – vorticity vector [s^{-1}],
ω_{ij} – spin tensor,
(r, θ, z) – cylindrical polar coordinates system.

Operators

, – differentiation with respect to variable,
∇^2 – Laplacian operator [m^{-2}],
∇ – gradient operator [m^{-1}].

1. Introduction

The study of creeping flow through cylindrical particles in a porous medium has always been a topic of interest. It is an exciting area for research in biomedical engineering, chemical engineering, biophysics of membranes, industrial applications, hydrology, and geothermal studies because it is used for purification processes, filtration, oil recovery techniques, thermal insulation, heat storage systems, and emergency cooling of nuclear reactors [1–4]. Either Darcy's law [5] or Brinkman's equation [6] are commonly adopted to model the flow in the porous medium. In particular, Brinkman's model validation has shown substantial exposition in the literature [7–9].

Spielman and Goren [10] studied Brinkman's model for the flow past porous media over a circular cylinder and presented their findings in the form of modified Bessel functions. Pop and Cheng [11] examined the steady flow through a circular cylinder embedded in a medium with constant porosity using Brinkman's model. They found that there is no flow separation when a circular cylinder is implanted in a medium with constant porosity. Wang [12] investigated Newtonian flow in a Darcy-Brinkman's porous medium with impermeable inclusions. It was found that when permeability tends to zero, the solution approaches the Stokes flow for a sphere, confirming the Stokes paradox (the absence of Stokes flow over a cylinder) for a cylinder.

Leontev [13] examined the viscous fluid flow past a cylinder and a sphere within a porous medium with the slip effect. The author found that the slip condition influences the flow behavior at the boundary. MADASU and SRINIVASACHARYA [14] investigated a micropolar fluid flowing through a cylinder and a sphere placed in a Brinkman's porous medium. They concluded that the drag force is a decreasing function of the permeability of a porous medium. MARTIN [15] discussed two-dimensional Brinkman flows and their relation with Stokes flows. The author observed that Brinkman's model provides a regularization of the flow problem as it does not exhibit a Stokes-like paradox.

Non-Newtonian fluid flow holds significant importance in modern technology and industries. In the field of non-Newtonian fluids, couple stress fluids explain all the essential features and effects of couple stresses. The main effect of couple stresses is introducing a size-dependent effect determined by material constants and dynamic viscosity, which is not present in the classical viscous theory. STOKES [16] introduced the couple stress theory. Examples of couple stress fluid are blood, electro-rheological fluid, synthetic fluids, and lubricants.

MURTHY and NAGARAJU [17] studied a couple stress fluid flow past a cylinder subjected to longitudinal and torsional oscillations. They determined that the amplitude of the oscillations for the drag in the viscous fluid case is smaller than that of the couple stress fluid case. KHAN *et al.* [18] developed a solution for a couple stress fluid within a porous rectangular channel. DEVAKAR *et al.* [19] analytically solved the couple stress fluid flow between the concentric cylinders with slip conditions. They observed that the presence of couple stress decreases the fluid velocity. The flow of a couple stress fluid past two parallel porous plates was studied by SRINIVASACHARYA *et al.* [20]. NAGARAJU *et al.* [21] investigated heat transfer in a couple stress fluid within two rotating cylinders in a porous lining with a magnetic effect. The entropy generation rate for the couple stress fluid flow past a porous medium was examined by ADESANYA *et al.* [22]. HASSAN [23] discussed a couple stress hydro-magnetic fluid flow past a porous channel. YADAV *et al.* [24] examined internal heat generation and variable viscosity effects in Darcy-Brinkman convection motion in a porous layer saturated by a couple stress fluid. PALAIAH *et al.* [25] studied the effects of thermal radiation and viscous dissipation on magnetized couple stress fluid flow through a cylinder. MADASU and SARKAR [26, 27] separately studied the governing equations for the flow of couple stress fluid through an isotropic porous medium and the MHD effect on the flow of couple stress fluid past a sphere, respectively.

Fluid flow problems related to various geometries depend on the boundary conditions applied at the solid-fluid or solid-porous interfaces. Traditionally, the no-slip condition has been employed at the interfaces. However, recent studies have shown that the no-slip condition might not always hold, and slippage of fluid particles could occur on the solid boundary surface [28, 29]. NAVIER [30] proposed a general boundary condition that includes the potential occurrence of fluid slip at a rigid boundary. Sherief *et al.* [31] examined the motion of a slipping sphere in a micropolar fluid along the axis of a circular cylindrical pore. They observed that the slip coefficient value increase leads to an augmentation of the drag force. Ashmawy [32] studied the impact of slip on a solid sphere immersed in an unbounded couple stress fluid. The study concluded that the drag force acting on a sphere is a decreasing function of the couple stress viscosity ratio parameter and an increasing function of the slip parameter. Madasu *et al.* [33] analyzed the slip effect on a spheroid embedded in a Brinkman medium. Madasu and Sarkar [34] studied the couple stress fluid flow through a sphere embedded in a porous medium with a slip effect. They found that the drag force is an increasing function of the slip parameter. Madasu and Sarkar [35] examined the influence of the MHD and slip on a rigid sphere in a cell model. Texier *et al.* [36] studied the impact of the physical parameters on the propulsion of a rotating helical filament in a granular medium. Chen *et al.* [37] examined helical locomotion in Brinkman's porous medium. Nganguia *et al.* [38] studied squirming in a viscous fluid enclosed by Brinkman's model.

To the authors' best knowledge, the flow behavior of a couple stress fluid through a cylinder implanted in Brinkman's porous medium has not been investigated yet. The article aims to examine the slip coefficient influence on the couple stress fluid flow around a cylinder placed within Brinkman's porous medium. Specifically, the study focuses on the cylinder surface, where both a zero couple stress and a slip condition are applied. The drag force exerted on an impermeable cylinder is obtained, and some well-known cases are discussed. The effects of couple stress, permeability, and slip parameters on the drag force are represented graphically.

2. Mathematical modelling

The flow of a couple stress fluid through a rigid cylinder with a radius $r = a$ situated in a porous medium is shown in Fig. 1. It is assumed that a uniform

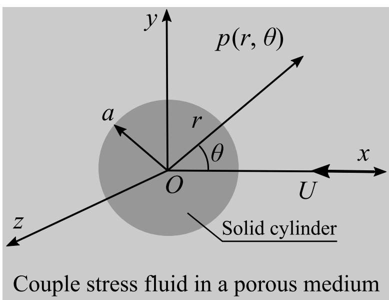

Fig. 1. A schematic representation of the problem.

velocity U is far away from the cylinder. The analysis considers a small Reynolds number, i.e., only viscous terms are present in the fluid momentum.

The governing equations of the motion of a couple stress fluid through a solid cylinder implanted in a porous medium in the absence of body couple and body force are given in [16, 22, 23, 26, 34]

$$\nabla \cdot \mathbf{q} = 0, \tag{2.1}$$

$$\eta \nabla \times \nabla \times \nabla \times \nabla \times \mathbf{q} + \nabla p + \frac{\mu}{k}\mathbf{q} + \mu \nabla \times \nabla \times \mathbf{q} = 0, \tag{2.2}$$

where $\mathbf{q}$, p, μ, k are the velocity vector, fluid pressure term, classical viscosity coefficient, and permeability, respectively, and η is the first viscosity coefficient of a couple stress fluid. When the viscosity coefficient η approaches zero, Eq. (2.2) reduces to Brinkman's equation.

The stress tensor is

$$t_{ij} = -p\delta_{ij} + 2\mu d_{ij} - \frac{1}{2} e_{ijk} m_{sk,s,}. \tag{2.3}$$

The couple stress tensor is

$$m_{ij} = m\delta_{ij} + 4(\eta' \omega_{i,j} + \eta \omega_{j,i}), \tag{2.4}$$

where m, $\omega_{i,j}$, e_{ijk}, and δ_{ij} represent the trace of the couple stress tensor, the spin tensor, the alternating tensor, and the Kronecker delta, respectively. Additionally, η' is the second viscosity coefficient of a couple stress fluid and satisfies the inequalities $\eta \geq 0$ and $\eta \geq \eta'$.

The d_{ij} deformation rate tensor is written as

$$d_{i,j} = \frac{1}{2}(q_{i,j} + q_{j,i}). \tag{2.5}$$

The vorticity vector ω_i is

$$\omega_i = -\frac{1}{2} e_{ijk} q_{k,j}. \tag{2.6}$$

To convert the governing equations into a non-dimensional form, we must use the following non-dimensional variables:

$$r = a\widetilde{r}, \qquad \nabla = \frac{\widetilde{\nabla}}{a}, \qquad p = \frac{\mu U}{a}\widetilde{p}, \qquad \mathbf{q} = U\widetilde{\mathbf{q}}. \tag{2.7}$$

Inserting Eq. (2.7) into the governing equations and removing the tildes, we obtain

$$\nabla \cdot \mathbf{q} = 0, \tag{2.8}$$

$$\frac{1}{\lambda^2} \nabla \times \nabla \times \nabla \times \nabla \times \mathbf{q} + \nabla p + \alpha^2 \mathbf{q} + \nabla \times \nabla \times \mathbf{q} = 0, \tag{2.9}$$

where $\alpha^2 = \frac{a^2}{k}$ and $\lambda = \sqrt{\frac{\mu a^2}{\eta}}$ are the permeability and length-dependent parameters, respectively.

Consider a cylindrical coordinate system $(r,\ \theta,\ z)$. As the flow is axisymmetric, the velocity vector is independent of z.

The velocity vector $\mathbf{q}$ is written as

$$\mathbf{q} = q_r(r,\theta)\mathbf{e}_r + q_\theta(r,\theta)\mathbf{e}_\theta, \tag{2.10}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors in the r and θ direction, respectively.

The q_r and q_θ are defined as

$$q_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad q_\theta = -\frac{\partial\psi}{\partial r}. \tag{2.11}$$

By removing the pressure term from Eq. (2.9) using Eq. (2.11), we obtain

$$\nabla^2(\nabla^2 - \xi_1^2)(\nabla^2 - \xi_2^2)\psi = 0, \tag{2.12}$$

where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{2.13}$$

$$\xi_1^2 = \frac{\lambda^2 + \lambda\sqrt{\lambda^2 - 4\alpha^2}}{2}, \tag{2.14}$$

$$\xi_2^2 = \frac{\lambda^2 - \lambda\sqrt{\lambda^2 - 4\alpha^2}}{2}. \tag{2.15}$$

3. Solution of the problem

The sixth-order partial differential Eq. (2.12) is solved with the help of the method of separation variables, giving the following solution:

$$\psi = \left[r + \frac{A}{r} + BK_1(\xi_1 r) + CK_1(\xi_2 r)\right]\sin\theta, \tag{3.1}$$

where A, B, and C are unknowns to be determined, and $K_1(*)$ is the modified Bessel function of the second kind of order one.

The non-zero vorticity vector is

$$\omega_z = -\frac{1}{2}\left[B\xi_1^2 K_1(\xi_1 r) + C\xi_2^2 K_1(\xi_2 r)\right]\sin\theta. \tag{3.2}$$

Additionally, the pressure term is

$$p = -\alpha^2\left(r - \frac{A}{r}\right)\cos\theta + \text{const}. \tag{3.3}$$

4. Boundary condition

The appropriate boundary conditions for the slip flow problem [13, 32–35, 39] are

$$q_r = 0, \tag{4.1}$$

$$\beta_1 q_\theta = t_{r\theta}, \tag{4.2}$$

$$m_{rz} = 0, \tag{4.3}$$

where $\beta_1 = \frac{a\beta}{\mu}$ is the slip parameter, and β is the coefficient of sliding friction. The slip parameter depends on the solid surface and fluid properties. When $\beta_1 \to \infty$ in Eq. (4.2), it is reduced to the no-slip condition.

The condition at a far distance from the cylinder is $\psi = r \sin\theta$.

The stress components t_{rr} and $t_{r\theta}$ are

$$t_{rr} = \Big[\alpha^2(r - Ar^{-1}) - 4Ar^{-3} - 2Br^{-2}\big(2K_1(\xi_1 r) + \xi_1 r K_0(\xi_1 r)\big) - 2Cr^{-2}(2K_1\big(\xi_2 r) + \xi_2 r K_0(\xi_2 r)\big)\Big]\cos\theta, \tag{4.4}$$

$$t_{r\theta} = -\Big[4Ar^{-3} + Br^{-2}\big((4 + r^2\alpha^2)K_1(\xi_1 r) + 2\xi_1 r K_0(\xi_1 r)\big) + Cr^{-2}\big((4 + r^2\alpha^2)K_1(\xi_2 r) + 2\xi_2 r K_0(\xi_2 r)\big)\Big]\sin\theta. \tag{4.5}$$

The couple stress component m_{rz} is

$$m_{rz} = 2\eta\Big[B\xi_1^3\big(K_0(\xi_1 r) + r^{-1}\xi_1^{-1}K_1(\xi_1 r)\big) + C\xi_2^3\big(K_0(\xi_2 r) + r^{-1}\xi_2^{-1}K_1(\xi_2 r)\big)\Big]\sin\theta. \tag{4.6}$$

Using the boundary condition Eqs. (4.1)–(4.3), we obtain

$$a + Aa^{-1} + BK_1(\xi_1 a) + CK_1(\xi_2 a) = 0, \tag{4.7}$$

$$A(4a^{-3} + \beta_1 a^{-2}) + Ba^{-2}\Big[(4 + a^2\alpha^2 + a\beta_1)K_1(\xi_1 a) + \xi_1 a(2 + a\beta_1)K_0(\xi_1 a)\Big] + Ca^{-2}\Big[(4 + a^2\alpha^2 + a\beta_1)K_1(\xi_2 a) + \xi_2 a(2 + a\beta_1)K_0(\xi_2 a)\Big] = \beta_1, \tag{4.8}$$

$$2\Big[B\xi_1^3\big(K_0(\xi_1 a)+a^{-1}\xi_1^{-1}K_1(\xi_1 a)\big) + C\xi_2^3\big(K_0(\xi_2 a)+a^{-1}\xi_2^{-1}K_1(\xi_2 a)\big)\Big]=0. \tag{4.9}$$

By solving the system of linear Eqs. (4.7)–(4.9), we have

$$\begin{aligned} A &= -\frac{a\big(K_0(\xi_1 a)a\xi_1 S_8 - K_1(\xi_1 a)S_9\big)}{\Delta}, \\ B &= -\frac{2S_4\xi_2^2\big(K_0(\xi_2 a)a\xi_2 + K_1(\xi_2 a)\big)}{\Delta}, \\ C &= \frac{2S_4\xi_1^2\big(K_0(\xi_1 a)a\xi_1 + K_1(\xi_1 a)\big)}{\Delta}, \end{aligned} \tag{4.10}$$

where $S_1 - S_9$ and Δ are defined in the Appendix.

As the couple stress parameter $\lambda \to \infty$ ($\xi_1 \to \infty$, $\xi_2 \to \alpha$, $a = 1$), the unknowns: A, B, and C are reduced to the case of a viscous fluid past a slip cylinder in a porous medium. Here, the flow is governed by Brinkman's equation. The values of A, B, and C are

$$\begin{aligned} A &= -\frac{\alpha K_0(\alpha)(\beta_1+2) + K_1(\alpha)\big(2(\beta_1+2)+\alpha^2\big)}{\alpha K_0(\alpha)(\beta_1+2)+\alpha^2 K_1(\alpha)}, \\ B &= 0, \\ C &= \frac{2(\beta_1+2)}{\alpha K_0(\alpha)(\beta_1+2)+\alpha^2 K_1(\alpha)}. \end{aligned} \tag{4.11}$$

These results agree with the findings of Leontev [13].

As the couple stress parameter $\lambda \to \infty$ ($\xi_1 \to \infty$, $\xi_2 \to \alpha$, $a = 1$) and slip parameter $\beta_1 \to \infty$, the values of A, B, and C in the case of the viscous flow past a no-slip cylinder embedded in a porous medium are as follows:

$$\begin{aligned} A &= -\Big[1 + 2K_1(\alpha)\big(\alpha K_0(\alpha)\big)^{-1}\Big], \\ B &= 0, \\ C &= 2\big(\alpha K_0(\alpha)\big)^{-1}. \end{aligned} \tag{4.12}$$

These values match with the results presented by Pop and Cheng [11] and Wang [12].

5. Drag force

The drag force exerted by a couple stress fluid on a slip cylinder in a Brinkman porous medium is obtained by the integral formula:

$$F_D = \int_0^{2\pi} r(t_{rr}\cos\theta - t_{r\theta}\sin\theta)|_{r=a}\,\mathrm{d}\theta. \tag{5.1}$$

By inserting the values of Eqs. (4.4) and (4.5) into the above integral formula, we obtain

$$F_D = \mu\pi U F, \tag{5.2}$$

where

$$F = \alpha^2\big(a^2 - A + BaK_1(\xi_1 a) + CaK_1(\xi_2 a)\big). \tag{5.3}$$

Putting A, B, and C values in Eq. (5.2), we have

$$F_D = \frac{2\mu\pi U a\alpha^2\big(K_0(\xi_1 a)a\xi_1 S_8 + K_1(\xi_1 a)S_{15}\big)}{\Delta}, \tag{5.4}$$

where $S_8 - S_{15}$ and Δ are given in the Appendix.

As the permeability parameter $\alpha \to \infty$ ($\xi_1 \to \infty$, $\xi_2 \to \infty$, $a = 1$), the slip parameter $\beta_1 \to \infty$, and the couple stress parameter $\lambda \to \infty$ in Eq. (5.4), the drag force becomes undefined. Therefore, this shows the Stokes paradox, i.e., no solution for a viscous fluid flow across a solid cylinder [10, 12]. However, Martin [15] recently has demonstrated that the Brinkman flow does not exhibit the Stokes paradox.

As the couple stress parameter $\lambda \to \infty$ ($\xi_1 \to \infty$, $\xi_2 \to \alpha$, $a = 1$) and the slip parameter $\beta_1 \to \infty$ in Eq. (5.4), the resulting drag exerted on a no-slip cylinder embedded in a porous medium is

$$F_D = 2\pi\mu U\alpha^2\left[1 + 2K_1(\alpha)\big(\alpha K_0(\alpha)\big)^{-1}\right]. \tag{5.5}$$

This result coincides with the findings of Spielman and Goren [10] and Wang [12].

6. Results and discussion

The calculations of the normalized drag force $D_N = \frac{F_D}{\frac{1}{2}\rho U^2 2a} = \frac{2\pi a F}{R_e}$ exerted by a couple stress fluid on a cylindrical surface are presented graphically in Figs. 2–4 for various values of the following parameters:

- slip parameter: β_1 $(0 \leq \beta_1 < \infty)$,
- couple stress parameter: λ $(3 \leq \lambda < \infty)$,
- permeability parameter: k_1 $\left(= \frac{1}{\alpha^2} = \frac{k}{a^2}(k_1 \geq 0)\right)$.

The variation of the drag force $D_N R_e$ against β_1 for different values of the couple stress parameter λ with fixed values of the permeability parameter k_1 and radius of cylinder a is observed in Fig. 2. This indicates that the drag force decreases as λ increases, while it increases as the value of β_1 increases. This result is in agreement with the results presented by Madasu and Sarkar [34]. The curve $\lambda \to \infty$ represents the flow without couple stresses, i.e., a Newtonian fluid. It can be observed that the drag force exerted on a solid cylinder immersed in a porous medium of couple stress fluid is greater than the drag force acting on a solid cylinder embedded in a porous medium of a Newtonian fluid $(\lambda \to \infty)$.

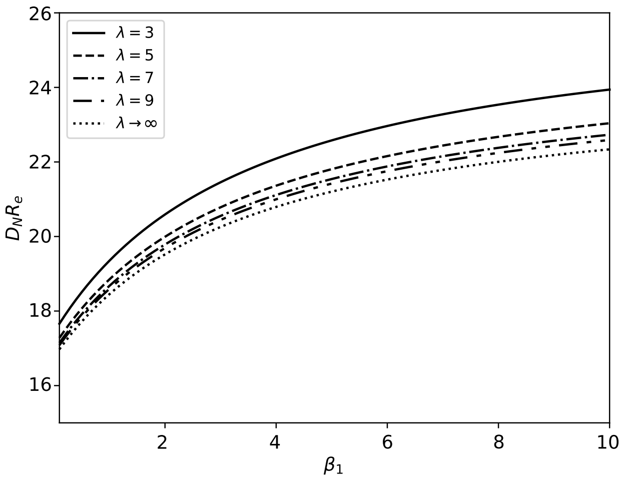

Fig. 2. The variation of $D_N R_e$ with β_1 when $a = 1$ and $k_1 = 1$.

The variation of $D_N R_e$ acting on a cylinder embedded in a porous medium saturated with a couple stress fluid against the slip parameter β_1 with several values of permeability parameter k_1 and fixed values of the couple stress parameter and cylinder radius is presented in Fig. 3. The observed trend indicates that the drag force is a decreasing function of the permeability parameter.

Figure 4 shows the variation of the drag force against k_1 with different values of β_1 and fixed values of the couple stress parameter and cylinder radius. It is noticed that when β_1 increases, the drag force increases. One can see that the drag force acting on a no-slip cylinder is higher than the drag force acting on a slip cylinder. It is inferred that the impact of the slip on the drag force is significant in fluid flows past solid inclusions in a porous medium.

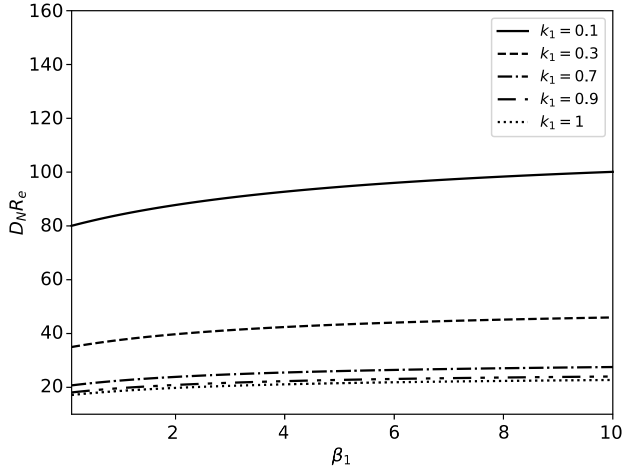

FIG. 3. The variation of $D_N R_e$ with β_1 when $a = 1$ and $\lambda = 7$.

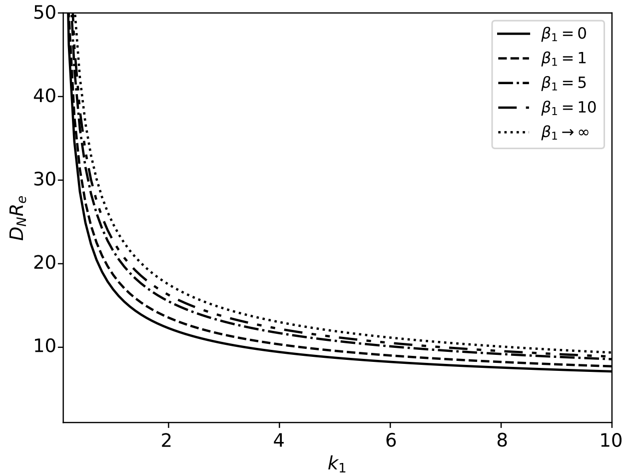

FIG. 4. The variation of $D_N R_e$ with k_1 when $a = 1$ and $\lambda = 7$.

7. CONCLUSIONS

We have investigated the couple stress fluid flow through a solid cylinder implanted in a porous medium by considering non-zero flow velocity and zero couple stress at the boundary. The pressure, stream function, stress components, and couple stress components are determined in this study. The drag exerted by a couple stress fluid on an impermeable cylinder was derived. The specific cases examined agree with the published studies of SPIELMAN and GOREN [10], POP and CHENG [11], WANG [12], and LEONTEV [13]. Furthermore, the influences of couple stress, permeability, and slip parameters on the drag force were presented graphically. We conclude that the drag force exerted on a solid cylinder immersed in a porous medium of a couple stress fluid is greater than that in a Newtonian fluid. Additionally, the drag force acting on a no-slip cylinder is higher than that on a slip cylinder. This investigation finds applications in heat, momentum, mass transfer in porous media [12] and helical locomotion in Brinkman's medium [36–38].

Appendix

The constants that appear in Eqs. (4.10) and (5.4) are as follows:

$$\begin{aligned}
S_1 &= \xi_1^2 - \xi_2^2,\\
S_2 &= \xi_1^2 - 2\xi_2^2,\\
S_3 &= 2\xi_1^2 - \xi_2^2,\\
S_4 &= a\beta_1 + 2,\\
S_5 &= a^2\alpha^2\xi_1^2 + S_3S_4,\\
S_6 &= a^2\alpha^2\xi_2^2 - S_2S_4,\\
S_7 &= a^2\alpha^2 + 2S_4,\\
S_8 &= K_0(\xi_2 a)a\xi_2S_1S_4 + K_1(\xi_2 a)S_5,\\
S_9 &= K_0(\xi_2 a)a\xi_2S_6 - K_1(\xi_2 a)S_1S_7,\\
S_{10} &= a^2\alpha^2\xi_1^2 - \xi_2^2S_4,\\
S_{11} &= a^2\alpha^2\xi_2^2 - \xi_1^2S_4,\\
S_{12} &= K_0(\xi_2 a)a\xi_2S_1S_4 + K_1(\xi_2 a)S_{10},\\
S_{13} &= K_0(\xi_2 a)\xi_2S_{11} - K_1(\xi_2 a)a\alpha^2S_1,\\
S_{14} &= \xi_1^2S_4 - \xi_2^2S_7,\\
S_{15} &= K_0(\xi_2 a)a\xi_2S_{14} + K_1(\xi_2 a)S_1S_7,\\
\Delta &= K_0(\xi_1 a)\xi_1S_{12} - K_1(\xi_1 a)S_{13}.
\end{aligned}$$

References

1. Pop I., Ingham D.B., *Convective heat transfer: mathematical and computational modelling of viscous fluids and porous media*, Elsevier Science, 2001.

2. Ehlers W., Bluhm J. [Eds.], *Porous Media: Theory, Experiments and Numerical Applications*, Springer Science & Business Media, 2002.

3. Bejan A., Dincer I., Lorente S., Miguel A.F., Reis H.A., *Porous and Complex Flow Structures in Modern Technologies*, Springer Science & Business Media, 2004.

4. Nield D.A., Bejan A., *Convection in Porous Media*, Springer, 2006.

5. Bear J., *Dynamics of Fluids in Porous Media*, Courier Corporation, 1988.

6. Brinkman H.C., A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, *Flow, Turbulence and Combustion*, **1**(1): 27–34, 1949, doi: 10.1007/BF02120313.

7. Durlofsky L., Brady J.F., Analysis of the Brinkman equation as a model for flow in porous media, *The Physics of Fluids*, **30**(11): 3329–3341, 1987, doi: 10.1063/1.866465.

8. Phillips R.J, Deen W.M., Brady J.F., Hindered transport in fibrous membranes and gels: effect of solute size and fiber configuration, *Journal of Colloid and Interface Science*, **139**(2): 363–373, 1990, doi: 10.1016/0021-9797(90)90110-A.

9. Auriault J.L., On the domain of validity of Brinkman's equation, *Transport in Porous Media,* **79**(2): 215–223, 2009, doi: 10.1007/s11242-008-9308-7.

10. Spielman L., Goren S.L., Model for predicting pressure drop and filtration efficiency in fibrous media, *Environmental Science & Technology*, **2**(4): 279–287, 1968, doi: 10.1021/es60016a003.

11. Pop I., Cheng P., Flow past a circular cylinder embedded in a porous medium based on the Brinkman model, *International Journal of Engineering Science*, **30**(2): 257–262, 1992, doi: 10.1016/0020-7225(92)90058-O.

12. Wang C.Y., Darcy-Brinkman flow with solid inclusions, *Chemical Engineering Communications,* **197**(3): 261–274, 2009, doi: 10.1080/00986440903088603.

13. Leontev N.E., Flow past a cylinder and a sphere in a porous medium within the framework of the Brinkman equation with the Navier boundary condition, *Fluid Dynamics*, **49**(2): 232–237, 2014, doi: 10.1134/S0015462814020112.

14. Madasu K.P., Srinivasacharya D., Micropolar fluid flow through a cylinder and a sphere embedded in a porous medium, *International Journal of Fluid Mechanics Research*, **44**(3): 229–240, 2017, doi: 10.1615/InterJFluidMechRes.2017015283.

15. Martin P.A., Two-dimensional Brinkman flows and their relation to analogous Stokes flows, *IMA Journal of Applied Mathematics*, **84**(5): 912–929, 2019, doi: 10.1093/imamat/hxz020.

16. Stokes V.K., Couple stresses in fluids, [in:] *Theories of Fluids with Microstructure*, pp. 34–80, Springer, 1984, doi: 10.1007/978-3-642-82351-0_4.

17. Murthy J.V.R., Nagaraju G., Flow of a couple stress fluid generated by a circular cylinder subjected to longitudinal and torsional oscillations, *Contemporary Engineering Sciences*, **2**(10): 451–461, 2009.

18. Khan N.A., Mahmood A., Ara A., Approximate solution of couple stress fluid with expanding or contracting porous channel, *Engineering Computations,* **30**(3): 399–408, 2013, doi: 10.1108/02644401311314358.

19. Devakar M., Sreenivasu D., Shankar B., Analytical solutions of some fully developed flows of couple stress fluid between concentric cylinders with slip boundary conditions, *International Journal of Engineering Mathematics*, **2014**: Article ID 785396, 2014, doi: 10.1155/2014/785396.

20. Srinivasacharya D., Srinivasacharyulu N., Odelu O., Flow of couple stress fluid between two parallel porous plates, *IAENG International Journal of Applied Mathematics*, **41**(2): 5, 2011.

21. Nagaraju G., Matta A., Aparna P., Heat transfer on the MHD flow of couple stress fluid between two concentric rotating cylinders with porous lining, *International Journal of Advances in Applied Mathematics and Mechanics*, **3**(1): 77–86, 2015.

22. Adesanya S.O., Kareem S.O., Falade J.A., Arekete S.A., Entropy generation analysis for a reactive couple stress fluid flow through a channel saturated with porous material, *Energy*, **93**(Part 1): 1239–1245, 2015, doi: 10.1016/j.energy.2015.09.115.

23. Hassan A.R., The entropy generation analysis of a reactive hydromagnetic couple stress fluid flow through a saturated porous channel, *Applied Mathematics and Computation*, **369**: 124843, 2020, doi: 10.1016/j.amc.2019.124843.

24. Yadav D., Mahabaleshwar U.S., Wakif A., Chand R., Significance of the inconstant viscosity and internal heat generation on the occurrence of Darcy-Brinkman convective motion in a couple-stress fluid saturated porous medium: An analytical solution, *International Communications in Heat and Mass Transfer*, **122**: 105165, 2021, doi: 10.1016/j.icheatmasstransfer.2021.105165.

25. Palaiah S.S., Basha H., Reddy G.J., Magnetized couple stress fluid flow past a vertical cylinder under thermal radiation and viscous dissipation effects, *Nonlinear Engineering*, **10**(1): 343–362, 2021, doi: 10.1515/nleng-2021-0027.

26. Madasu K.P., Sarkar P., A study of couple stress fluid past an isotropic porous medium, *Special Topics & Reviews in Porous Media: An International Journal*, **13**(4): 23–31, 2022, doi: 10.1615/SpecialTopicsRevPorousMedia.2022043960.

27. Madasu K.P., Sarkar P., An analytical study of couple stress fluid through a sphere with an influence of the magnetic field, *Journal of Applied Mathematics and Computational Mechanics*, **21**(3): 99–110, 2022, doi: 10.17512/jamcm.2022.3.08.

28. Tretheway D.C., Meinhart C.D., Apparent fluid slip at hydrophobic microchannel walls, *Physics of Fluids*, **14**(3): L9–L12, 2002, doi: 10.1063/1.1432696.

29. Neto C., Evans D.R., Bonaccurso E., Butt H.J., Craig V.S.J., Boundary slip in Newtonian liquids: a review of experimental studies, *Reports on Progress in Physics*, **68**(12): 2859–2897, 2005, doi: 10.1088/0034-4885/68/12/R05.

30. Navier C.L.M.H., Mémoire sur les lois du Mouvement dea Fluides, *Mémoires de l'Acadé mie Royale de Sciences de l'Institut de France*, 1823.

31. Sherief H.H., Faltas M.S., Ashmawy E.A., Nashwan M.G., Slow motion of a slip spherical particle along the axis of a circular cylindrical pore in a micropolar fluid, *Journal of Molecular Liquids*, **200**(Part B): 273–282, 2014, doi: 10.1016/j.molliq.2014.10.030.

32. Ashmawy E.A., Drag on a slip spherical particle moving in a couple stress fluid, *Alexandria Engineering Journal*, **55**(2): 1159–1164, 2016, doi: 10.1016/j.aej.2016.03.032.

33. Madasu K.P., Kaur M., Bucha T., Slow motion past a spheroid implanted in a Brinkman medium: slip condition, *International Journal of Applied and Computational Mathematics*, **7**(4): 162, 2021, doi: 10.1007/s40819-021-01104-4.

34. Madasu K.P., Sarkar P., Couple stress fluid past a sphere embedded in a porous medium, *Archive of Mechanical Engineering,* **69**(1): 5–19, 2022, doi: 10.24425/ame.2021.139314.

35. Madasu K.P., Sarkar P., Slow flow past a slip sphere in cell model: magnetic effect, [in:] *Recent Trends in Fluid Dynamics Research*, *Lecture Notes in Mechanical Engineering*, Bharti R.P., Gangawane K.M. [Eds.], pp. 25–36, Springer, Singapore, 2022, doi: 10.1007/978-981-16-6928-6_3.

36. Texier B.D., Ibarra A., Melo F., Helical locomotion in a granular medium, *Physical Review Letters*, **119**(6): 068003, 2017, doi: 10.1103/PhysRevLett.119.068003.

37. Chen Y., Lordi N., Taylor M., Pak O.S., Helical locomotion in a porous medium, *Physical Review E*, **102**(4): 043111, 2020, doi: 10.1103/PhysRevE.102.043111.

38. Nganguia H., Zhu L., Palaniappan D., Pak O.S., Squirming in a viscous fluid enclosed by a Brinkman medium, *Physical Review E*, **101**(6): 063105, 2020, doi: 10.1103/Phys RevE.101.063105.

39. Happel J., Brenner H., *Low Reynolds number hydrodynamics with special applications to particulate media*, Springer Science & Business Media, 2012.

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