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THE PRINCIPLE OF VIRTUAL WORK AND THE CONSTITUTIVE EQUATIONS IN GENERALIZED THEORIES OF ELASTICITY

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1. INTRODUCTION

A. C. ERINGEN and E. S. SUHUBI [1] have introduced the model of a continuum with microstructure, giving its physical meaning and its mathematical formulation. In order to include the microstructure in the deformation, they considered the deformation of a macro-element of a body (as a limit volume element in which, from the viewpoint of molecular theory, it is not possible to consider mass as continuous) independent of the motion of its center of gravity. From the viewpoint of continuum theory, when the macro-element is identified with a material point, the deformation is determined not only by the field of displacements of material points but also by the field of micro-displacements, i.e. by deformation gradients and micro-deformation gradients.

Postulating balance equations in the classical form (with a symmetric stress tensor) for points of the macro-element, ERINGEN and SUHUBI have derived the corresponding equations for the model considered. Finally, they obtained non-linear constitutive equations for elastic materials. Constitutive equations were also considered in the second part of their work [2]. In addition, they have shown in the same paper that, remaining in the limits of the linear theory, one could obtain a theory of couple-stresses as a special case, in which, differing from the so-called polar theory considered by R. TOUPIN [3] and R. D. MINDLIN and H. F. TIERSTEN [4], the stress tensor and couple-stress tensor were completely determined. ERINGEN has also considered this theory in particular papers (e.g. [5, 6, 7]), remaining in the domain of linearity, and has given it the name of *micropolar theory*. A similar theory, based on independent rotations of material points, was given by E. L. AERO and E. V. KUVSHINSKII [8, 9]. In all these works, complete determination of the stress tensor and couple-stress tensor is the consequence of the existence of independent rotations of material points in the continuum. Later, C. B. KAFADAR and A. C. ERINGEN [10] gave the non-linear theory of micropolar elastic materials.

In the continuum theory of materials with microstructure, or micromorphic materials, every material point is *phenomenologically equivalent to a deformable body*. In the continuum theory of micropolar materials, however, every material point is *phenomenologically equivalent to a rigid body*. In both cases there exists an influence of microstructure on deformation. In the theory of multipolar materials, given by A. E. GREEN and R. S. RIVLIN [11], as in its special case — dipolar theory, this is not the case. In this theory, the deformation is completely determined by the field of displacements of material points, i.e. by first and higher-order deformation gradients. The same holds in polar theory. For this reason, it seems that for dipolar and polar materials the term "special case of simple materials with microstructure" is not fully correct.

In this paper, we shall consider micro-elastic materials as a generalized elastic Cosserat continuum, without taking into account non-mechanical effects (thermal effects etc.). Special cases: micropolar, dipolar and polar elastic materials will be considered as special cases of generalized elastic Cosserat continua. We shall make use of the terminology introduced by ERINGEN and SUHUBI, and the usual expressions from the theory of Cosserat continuum (see, for example, [12]). We shall also make use of the usual notations from the theory of two-point tensor fields.

2. THE MODEL

Let body $\mathcal{B}$ of volume V, with a boundary given by the closed surface S, be at t_0 in its initial (undeformed) configuration K_0. In the deformed configuration K, which corresponds to $t>t_0$, the body will have volume v enclosed by surface s. A macro-element dV of body $\mathcal{B}$ in the undeformed configuration will become dv after deformation. We suppose that sources of matter do not exist in the body, so that the mass of the macro-element considered in the body remains unchanged during deformation, i.e.

$$\rho_0\, dV = \rho dv = dm = \text{const}, \tag{2.1}$$

where ρ_0 and ρ are average densities of macro-elements dV and dv, respectively, i.e.

$$\rho_0 = \frac{dm}{dV}, \qquad \rho = \frac{dm}{dv}. \tag{2.2}$$

Equation (2.1) represents the equation of conservation of mass and can be written in the form

$$\dot{\overline{dm}} = \frac{\partial \rho}{\partial t} + (\rho v^k)_{,k}, \tag{2.3}$$

where v^k is the velocity of the center of gravity of the macro-element.

If we distinguish in the macro-element dV the point $C(X^K)$ which is the center of gravity, referred to a curvilinear system of material coordinates, then the position of an arbitrary point $A\,(X'^K)$ of dV, relative to the center of gravity, can be determined by a vector D^K, so that we have

$$X'^K = X^K + D^K \tag{2.4}$$

referred to the same coordinate system, since we suppose that the distance between points $C\,(X^K)$ and $A\,(X'^K)$ is infinitesimal (Fig. 1).

Let ρ_0' be the mass density at a point $A\,(X'^K)$; then we have

(2.5) $$\int_{dV} \rho_0'\, dV' = \rho_0\, dV = dm\,, \qquad \int_{dV} \rho_0' D^K\, dV' = 0\,,$$

since $C\,(X^K)$ is the center of gravity of the undeformed macro-element.

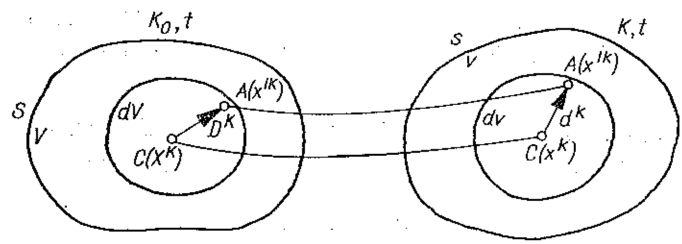

Fig. 1

Through the motion the material point $C\,(X^K)$ is carried into a spatial point $C\,(x^k)$, the vector D^K into d^k (Fig. 1.), so that

(2.6) $$x'^k = x^k + d^k\,,$$

referred to a curvilinear system of spatial coordinates x^k, where

(2.7) $$d^k = d^k\,(X^K, D^K; t)\,.$$

We assume that (2.7) is an analytic function, so that it can be expanded as a power series,

(2.8) $$d^k = \chi^k_{.K} D^K + \chi^k_{.KL} D^K D^L + \ldots\,.$$

Since $|D^K|$ is sufficiently small, we need only retain the linear terms in D^K in the above expansion. Hence, for *simple materials* we have

(2.9) $$d^k = \chi^k_{.K} D^K\,.$$

Now, (2.6) may be written as

(2.10) $$x'^k = x^k + \chi^k_{.K} D^K\,.$$

The micro-deformation gradients

(2.11) $$\chi^k_{.K}(X^L, t) = \left(\frac{\partial d^k}{\partial D^K}\right)_{D^L=0} = \left(\frac{\partial x'^k}{\partial X'^K}\right)_{X'^L=0}$$

are independent of the motion of the point X^K. Hence, the motion is determined by the equations

(2.12) $$x^k = x^k(X^K, t)\,, \qquad \chi^k_{.K} = \chi^k_{.K}(X^L, t)\,.$$

Let ρ' be the mass density at a point $A\,(x'^k)$; then we have

(2.13) $$\int_{dv} \rho'\, dv' = \rho dv = dm\,,$$

and, using $(2.5)_2$,

$$\int_{dv} \rho' d^k dv' = \int_{dv} \rho' \chi^k_{.K} D^K dv' = \chi^k_{.K} \int_{dv} \rho' D^K dv' = \chi^k_{.K} \int_{dV} \rho'_0 D^K dV' = 0, \tag{2.14}$$

where we assume that the mass of the micro-element is conserved, i.e. $dm' = \rho'_0\, dV' = \rho' dv' = \text{const}$. From (2.14) we see that the point $C(x^k)$ is the center of gravity of the macro-element dv. Hence, through the motion (2.12), the center of gravity of the undeformed macro-element is carried into the center of gravity of the deformed element.

Let $D^K_{.(\alpha)}$, $|D^K_{.(\alpha)}| \neq 0$, $\alpha = 1, 2, 3$, be a triad of non-coplanar vectors at the point $C(X^K)$, which is attached to macro-element dV. Then, taking into account (2.9),

$$d^k_{.(\alpha)} = \chi^k_{.K} D^K_{.(\alpha)}, \tag{2.15}$$

wherefrom

$$\chi^k_{.K} = d^k_{.(\alpha)} D^{(\alpha)}_{.K}, \tag{2.16}$$

where $D^K_{.(\alpha)}$ and $D^{(\alpha)}_{.K}$ are mutually reciprocal triads, i.e.

$$D^K_{.(\alpha)} D^{(\alpha)}_{.L} = \delta^K_L, \quad D^K_{.(\alpha)} D^{(\beta)}_{.K} = \delta^\beta_\alpha. \tag{2.17}$$

The micro-deformation gradients $\chi^k_{.K}$ are then completely determined by deformation of the triad vectors $D^K_{.(\alpha)}$. Using (2.16), equation (2.9) can be written in the form

$$d^k = d^k_{.(\alpha)} D^{(\alpha)}_{.K} D^K. \tag{2.18}$$

Vectors $D^K_{.(\alpha)}$ and $d^k_{.(\alpha)}$ can be considered as *directors*, so that the motion is determined by the equations

$$x^k = x^k(X^K, t), \quad d^k_{.(\alpha)} = d^k_{.(\alpha)}\big(D^K_{.(\alpha)}(X^L), t\big). \tag{2.19}$$

It is clear that directors, introduced in that way, are *material vectors* attached to the macro-element.

Let $d^k_{.(\alpha)}$ and $d^{(\alpha)}_{.k}$ be mutually reciprocal triads, i.e.

$$d^k_{.(\alpha)} d^{(\alpha)}_{.l} = \delta^k_l, \quad d^k_{.(\alpha)} d^{(\beta)}_{.k} = \delta^\beta_\alpha. \tag{2.20}$$

Then, from (2.18), we get

$$D^K = D^K_{.(\alpha)} d^{(\alpha)}_{.k} d^k, \tag{2.21}$$

or

$$D^K = \chi^K_{.k} d^k, \tag{2.22}$$

where

$$\chi^k_{.K} \chi^K_{.l} = \delta^k_l, \quad \chi^k_{.K} \chi^L_{.k} = \delta^L_K. \tag{2.23}$$

Making use of (2.18), (2.6) can be written in the form

$$x'^k = x^k + d^k_{.(\alpha)} D^{(\alpha)}_{.K} D^K. \tag{2.24}$$

From (2.24) by differentiation we obtain

$$v'^k = v^k + \dot{d}^k_{.(\alpha)} D^{(\alpha)}_{.\,K} D^K, \tag{2.25}$$

or, taking into account (2.21),

$$v'^k = v^k + \dot{d}^k_{.(\alpha)} d^{(\alpha)}_{.\,l} d^l. \tag{2.26}$$

This expression can be written in the form

$$v'^k = v^k + b^k_{.\,l} d^l, \tag{2.27}$$

where

$$b^k_{.\,l} = \dot{d}^k_{.(\alpha)} d^{(\alpha)}_{.\,l} = \dot{\chi}^k_{.K} \chi^K_{.\,l}. \tag{2.28}$$

The expression (2.27) represents the velocity of any point of the macroelement dv, where b_{kl} is defined at the center of gravity of the macro-element.

Making use of (2.25) and (2.14), the kinetic energy of a portion v of a body is

$$2T = \int_v \int_{dv} \rho' v'^k v'_k \, dv' = \int_v \rho v^k v_k \, dv + \int_v \dot{d}^k_{.(\alpha)} \dot{d}_{k(\beta)} D^{(\alpha)}_{.\,K} D^{(\beta)}_{.\,L} \int_{dv} \rho' D^K D^L \, dv',$$

and can be written in the form

$$2T = \int_v \rho (v^k v_k + I^{KL} \dot{d}^k_{.(\alpha)} \dot{d}_{k(\beta)} D^{(\alpha)}_{.\,K} D^{(\beta)}_{.\,L}) \, dv, \tag{2.29}$$

where

$$\rho dv \, I^{KL} = \rho dv I^{LK} = \int_{dv} \rho' D^K D^L \, dv' = \int_{dV} \rho'_0 D^K D^L \, dv'. \tag{2.30}$$

The quantities I^{KL} are the coefficients of inertia of the macro-element with respect to its center of mass.

If we introduce the "director coefficients of inertia" $I^{\alpha\beta}$ as

$$I^{\alpha\beta} = I^{\beta\alpha} = I^{KL} D^{(\alpha)}_{.\,K} D^{(\beta)}_{.\,L}, \tag{2.31}$$

then the kinetic energy can be written in the form

$$2T = \int_v \rho (v^k v_k + I^{\alpha\beta} \dot{d}^k_{.(\alpha)} \dot{d}_{k(\beta)}) \, dv. \tag{2.32}$$

The rate of the kinetic energy is

$$\dot{T} = \int_v \rho (\dot{v}^k v_k + I^{\alpha\beta} \ddot{d}^k_{.(\alpha)} \dot{d}_{k(\beta)}) \, dv, \tag{2.33}$$

and, using (2.28), can be written in the form

$$\dot{T} = \int_v \rho (\dot{v}^k v_k + \Gamma^{ij} b_{ij}) \, dv, \tag{2.34}$$

where

$$\Gamma^{ij} = I^{\alpha\beta} \ddot{d}^i_{.(\alpha)} d^j_{.(\beta)} = I^{KL} \ddot{\chi}^i_{.K} \chi^j_{.L} \tag{2.35}$$

is the inertial spin.

In continuum theory, we assume that a body $\mathcal{B}$ is composed of material points which are distributed continuously. We attribute to these points the features of macro-elements. In other words, we identify now every macro-element with a material point. Hence, every material point is *phenomenologically equivalent to a deformable body*. The micro-deformation gradients are now defined at every point of the body, so that they represent a continuous field. The motion is then determined by equations (2.12) which are mutually independent. Instead of the micro-deformation gradients $\chi^k_{.K}$ we may define at every point a triad of directors $D^K_{.(\alpha)}$, which are carried by the deformation into $d^k_{.(\alpha)}$, so that the motion is determined by the equations (2.19). In undeformed configuration, we choose arbitrarily triads of directors and their deformations characterize the deformations of previously considered macro-elements.

Taking into account the fact that the deformation of the directors is independent of displacements of the points, it is clear that, after deformation, the directors $d^k_{.(\alpha)}$ *are not material vectors*. By introduction of deformable directors in the points of the body, and by describing its motion by the equations (2.19), we interprete a continuum with microstructure as an oriented continuous medium with three deformable directors; in other words, as a *generalized Cosserat continuum*.

It is clear that elements dV or dv have a new meaning with respect to the previously considered macro-element. However, taking into account the fact that quantities, previously defined in the center of mass of the macro-element, are now *continuous functions of the position*, the expression (2.32) for kinetic energy, and the expressions (2.33) or (2.34) for the derivative of kinetic energy, remain unchanged, with a dv which is now interpreted as an element of the body.

3. The principle of virtual work

Let us assume that the surface forces T^i and H^{ij}, as well as the body forces f^i and l^{ij}, act on the body $\mathcal{B}$, so that their time rate of working is

$$\dot{A}=\oint_s (T^i v_i+H^{ij} b_{ij})\, ds+\int_v \rho(f^i v_i+l^{ij} b_{ij})\, dv. \tag{3.1}$$

According to (2.28), (3.1) can be written in the form

$$\dot{A}=\oint_s (T^i v_i+H^{i(\alpha)} \dot{d}_{i(\alpha)})\, ds+\int_v \rho(f^i v_i+l^{i(\alpha)} \dot{d}_{i(\alpha)})\, dv, \tag{3.2}$$

where

$$H^{i(\alpha)}=H^{ij} d^{(\alpha)}_{.j}, \qquad l^{i(\alpha)}=l^{ij} d^{(\alpha)}_{.j}. \tag{3.3}$$

The virtual work of the surface and body forces is

$$\delta A=\oint_s (T^i \delta x_i+H^{i(\alpha)} \delta d_{i(\alpha)})\, ds+\int_v \rho(f^i \delta x_i+l^{i(\alpha)} \delta d_{i(\alpha)})\, dv, \tag{3.4}$$

and, by an identical transformation, can be written in the form

$$\delta A=\oint_s [(T^i ds-t^{ik} ds_k)\, \delta x_i+(H^{i(\alpha)} ds-h^{i(\alpha)k} ds_k)\, \delta d_{i(\alpha)}]+ \\ +\int_v \rho(f^i \delta x_i+l^{i(\alpha)} \delta d_{i(\alpha)})\, dv+\oint_s (t^{ik} ds_k\, \delta x_i+h^{i(\alpha)k} ds_k\, \delta d_{i(\alpha)}). \tag{3.5}$$

By applying the Green-Gauss theorem to the last surface integral, we get

$$\delta A = \oint_s [(T^i ds - t^{ik} ds_k)\,\delta x_i + (H^{i(\alpha)} ds - h^{i(\alpha)k} ds_k)\,\delta d_{i(\alpha)}] + \int_v \rho(f^i \delta x_i + l^{i(\alpha)} \delta d_{i(\alpha)})\,dv + \int_v (t^{ik}{}_{,k}\,\delta x_i + t^{ik}\,\delta x_{i,k} + h^{i(\alpha)k}{}_{,k}\,\delta d_{i(\alpha)} + h^{i(\alpha)k}\,\delta d_{i(\alpha),k})\,dv. \tag{3.6}$$

The time rate of the kinetic energy (2.34) we can write in the form

$$\dot{T} = \int_v \rho(\dot{v}^i v_i + \Gamma^{i(\alpha)} \dot{d}_{i(\alpha)})\,dv, \qquad (\Gamma^{i(\alpha)} = \Gamma^{ij} d^{(\alpha)}{}_{.j}), \tag{3.7}$$

so that the virtual work of the inertial forces is

$$\delta T = \int_v \rho(\dot{v}^i \delta x_i + \Gamma^{i(\alpha)} \delta d_{i(\alpha)})\,dv. \tag{3.8}$$

We assume the principle of virtual work in the form

$$\delta T + \delta W = \delta A, \tag{3.9}$$

where δW is a variation of strain energy, i.e.

$$\delta W = \int_v \rho \delta w\,dv, \tag{3.10}$$

and where w is the specific strain energy.

According to (3.6) and (3.8), we write (3.9) in the form

$$\int_v \rho(\dot{v}^i \delta x_i + \Gamma^{i(\alpha)} \delta d_{i(\alpha)})\,dv + \int_v \rho\delta w\,dv = \oint_s [(T^i ds - t^{ik} ds_k)\,\delta x_i + (H^{i(\alpha)} ds - h^{i(\alpha)k} ds_k)\,\delta d_{i(\alpha)}] + \int_v \rho(f^i \delta x_i + l^{i(\alpha)} \delta d_{i(\alpha)})\,dv + \int_v (t^{ik}{}_{,k}\,\delta x_i + t^{ik}\,\delta x_{i,k} + h^{i(\alpha)k}{}_{,k}\,\delta d_{i(\alpha)} + h^{i(\alpha)k}\,\delta d_{i(\alpha),k})\,dv. \tag{3.11}$$

We now take the volume v in (3.11) to be a tetrahedral element bounded by a plane with an arbitrary unit normal n_k and by planes through the point of the body parallel to the coordinate planes. (By coordinate planes we understand planes which are tangential to coordinate surfaces at a point of the body). If ds is the area of the plane of the tetrahedron normal to n_k, and ds_k are other oriented faces of the tetrahedron, then

$$ds_k = ds\, n_k. \tag{3.12}$$

If we apply (3.11) to the tetrahedron and let the tetrahedron shrink to zero while preserving the orientation of its faces, we obtain the equation

$$(T^i ds - t^{ik} ds_k)\,\delta x_i + (H^{i(\alpha)} ds - h^{i(\alpha)k} ds_k)\,\delta d_{i(\alpha)} = 0. \tag{3.13}$$

This equation is valid for all variations δx_i and $\delta d_{i(\alpha)}$, so that we obtain the boundary conditions

$$T^i = t^{ik} n_k, \tag{3.14}$$

and

$$H^{i(\alpha)} = h^{i(\alpha)k} n_k, \quad \text{or} \quad H^{ij} = h^{ijk} n_k, \tag{3.15}$$

where

$$h^{ijk} = h^{i(\alpha)k} d^j_{.(\alpha)}. \tag{3.16}$$

The vector T^i is *the stress vector*, f^i *is the body force, and* t^{ij} *is the non-symmetric stress tensor. Vectors* $H^{i(\alpha)}$ *are director stresses,* $l^{i(\alpha)}$ *are body director forces, and tensors* $h^{i(\alpha)k}$ *are director stress tensors. The tensor* H^{ij} *is the first surface moment*, l^{ij} *is the first body moment, and* h^{ijk} *is the first surface stress moment.*

Upon substituting (3.13) into (3.11), we obtain

$$\int_v \rho(\dot{v}^i \delta x_i + \Gamma^{i(\alpha)} \delta d_{i(\alpha)})\, dv + \int_v \rho \delta w\, dv = \int_v \rho(f^i \delta x_i + l^{i(\alpha)} \delta d_{i(\alpha)})\, dv + \int_v (t^{ik}{}_{,k} \delta x_i + t^{ik} \delta x_{i,k} + h^{i(\alpha)k}{}_{,k} \delta d_{i(\alpha)} + h^{i(\alpha)k} \delta d_{i(\alpha)k})\, dv. \tag{3.17}$$

This equation is valid for all variations δx_i, $\delta x_{i,k}$, $\delta d_{i(\alpha)}$ and $\delta d_{i(\alpha),k}$. Furthermore, we suppose that $\delta W = 0$ for any virtual rigid displacement, i.e. the virtual work of the surface and body forces is equal to the virtual work of the inertial forces for any rigid displacement,

$$\int_v \rho(\dot{v}^i \delta x_i + \Gamma^{i(\alpha)} \delta d_{i(\alpha)})\, dv = \int_v \rho(f^i \delta x_i + l^{i(\alpha)} \delta d_{i(\alpha)})\, dv + \int_v (t^{ik}{}_{,k} \delta x_i + t^{ik} \delta x_{i,k} + h^{i(\alpha)k}{}_{,k} \delta d_{i(\alpha)} + h^{i(\alpha)k} \delta d_{i(\alpha),k})\, dv. \tag{3.18}$$

To derive the equations of motion, we shall apply the theorem of PIOLA in the form presented by TRUESDELL and TOUPIN [13]: *the equation* (3.18) *for virtual translations is equivalent to Cauchy's first law of motion, and for rigid displacements, to Cauchy's second law.*

For virtual translations we have

$$\delta x_i = \text{const}, \quad \delta x_{i,k} = 0, \quad \delta d_{i(\alpha)} = 0, \quad \delta d_{i(\alpha),k} = 0, \tag{3.19}$$

and equation (3.18) becomes

$$\int_v \rho \dot{v}^i \delta x_i\, dv = \int_v (t^{ik}{}_{,k} + \rho f^i)\, dv, \tag{3.20}$$

wherefrom

$$\rho \dot{v}^i = t^{ik}{}_{,k} + \rho f^i. \tag{3.21}$$

This is Cauchy's first law of motion, i.e. a necessary and sufficient condition for the balance of momentum, which represents three classical differential equations of motion.

Now, if we assume that Cauchy's first law is valid, equation (3.17) becomes

$$\int_v \rho \Gamma^{i(\alpha)} \delta d_{i(\alpha)} dv + \int_v \rho \delta w \, dv = \\ = \int_v [t^{ik} \delta x_{i,k} + (h^{i(\alpha)k}{}_{,k} + \rho l^{i(\alpha)}) \delta d_{i(\alpha)} + h^{i(\alpha)k} \delta d_{i(\alpha),k}] \, dv, \tag{3.22}$$

and (3.18)

$$\int_v \rho \Gamma^{i(\alpha)} \delta d_{i(\alpha)} dv = \int_v [t^{ik} \delta x_{i,k} + (h^{i(\alpha)k}{}_{,k} + \rho l^{i(\alpha)}) \delta d_{i(\alpha)} + h^{i(\alpha)k} \delta d_{i(\alpha),k}] \, dv. \tag{3.23}$$

For virtual rigid displacements, we have

$$\begin{aligned} &\delta x_{(i,k)} = 0, \quad \delta x_{[i,k]} = \text{const}, \\ &\delta d_{i(\alpha)} = \delta x_{[i,l]} d^l{}_{.(\alpha)}, \quad \delta d_{i(\alpha),k} = \delta x_{[i,l]} d^l{}_{.(\alpha),k}, \end{aligned} \tag{3.24}$$

and the equation (3.23) becomes

$$\int_v [t^{ij} + h^{ijk}{}_{,k} + \rho (l^{ij} - \Gamma^{ij})] \, \delta x_{[i,j]} dv = 0. \tag{3.25}$$

If we write

$$t^{ij} + h^{ijk}{}_{,k} + (l^{ij} - \Gamma^{ij}) = \tau^{ij}, \tag{3.26}$$

then the equation (3.25) can be written in the form

$$\int_v \tau^{ij} \delta x_{[i,j]} dv = 0. \tag{3.27}$$

Since (3.27) is valid for any $\delta x_{[i,j]}$, we obtain

$$\tau^{[ij]} = 0, \quad \text{or} \quad t^{[ij]} + h^{[ij]k}{}_{,k} + \rho (l^{[ij]} - \Gamma^{[ij]}) = 0. \tag{3.28}$$

This is Cauchy's second law of motion, i.e. a necessary and sufficient condition for the balance of the moment of momentum.

From (3.28) it follows that τ^{ij} is a symmetric tensor. Then, (3.26) represents the system of nine differential equations of motion. They include the three differential equations in (3.28). In (3.26), the first body moment l^{ij} is prescribed, while t^{ij}, τ^{ij} and h^{ijk} have to be determined from the constitutive equations.

Equation (3.26) can be written in the form

$$\tau^{ij} = t^{ij} + h^{i(\alpha)k}{}_{,k} d^j{}_{.(\alpha)} + h^{i(\alpha)k} d^j{}_{.(\alpha),k} + \rho (l^{i(\alpha)} - \Gamma^{i(\alpha)}) d^j{}_{.(\alpha)}, \tag{3.29}$$

from which follows

$$h^{i(\alpha)k}{}_{,k} + \rho l^{i(\alpha)} = \tau^{ij} d^{(\alpha)}{}_{.j} - t^{ij} d^{(\alpha)}{}_{.j} + h^{ijk} d^{(\alpha)}{}_{.j,k} + \rho \Gamma^{i(\alpha)}. \tag{3.30}$$

Upon substituting this into (3.22), we get

$$\int_v \rho \delta w \, dv = \int_v [t^{ik} \delta x_{i,k} + (\tau^{ij} d^{(\alpha)}{}_{.j} - t^{ij} d^{(\alpha)}{}_{.j} + h^{ijk} d^{(\alpha)}{}_{.j,k}) \delta d_{i(\alpha)} + h^{i(\alpha)k} \delta d_{i(\alpha),k}] \, dv, \tag{3.31}$$

wherefrom

$$\delta w = t^{ik} \delta x_{i,k} + (\tau^{ij} d^{(\alpha)}{}_{.j} - t^{ij} d^{(\alpha)}{}_{.j} + h^{ijk} d^{(\alpha)}{}_{.j,k}) \delta d_{i(\alpha)} + h^{i(\alpha)k} \delta d_{i(\alpha),k}. \tag{3.32}$$

This is the expression for the variation of the specific strain energy, which is form-invariant with respect to the superposed rigid motion.

According to (3.32) and using (2.25), we obtain

$$\rho\dot{w}=t^{ij}v_{i,j}+(\tau^{ij}-t^{ij})\,b_{ij}+h^{ijk}b_{ij,k}. \tag{3.33}$$

This is the expression of local conservation of energy, which is in accordance with the expression obtained by Eringen and Suhubi [1], where the tensor τ^{ij} was named by them "micro-stress average".

4. Constitutive equations

Equation (3.32) can be written in the form

$$\rho\delta w=t^{ij}X^K_{;j}\,\delta x_{i;K}+(\tau^{ij}d^{(\alpha)}_{\cdot j}-t^{ij}d^{(\alpha)}_{\cdot j}+h^{ijk}X^K_{;k}d^{(\alpha)}_{\cdot j;K})\,\delta d_{i(\alpha)}+ +h^{ijk}d^{(\alpha)}_{\cdot j}X^K_{;k}\,\delta d_{i(\alpha);K}. \tag{4.1}$$

We assume that the specific strain energy is a function of the form

$$w=w(x^k_{;K},\,d^k_{\cdot(\alpha)},\,d^k_{\cdot(\alpha);K}). \tag{4.2}$$

Then, we have

$$\delta w=g^{il}\frac{\partial w}{\partial x^l_{;K}}\delta x_{i;K}+g^{il}\frac{\partial w}{\partial d^l_{\cdot(\alpha)}}\delta d_{i(\alpha)}+g^{il}\frac{\partial w}{\partial d^l_{\cdot(\alpha);K}}\delta d_{i(\alpha);K}. \tag{4.3}$$

With regard to (4.1) and (4.3), we obtain

$$\begin{aligned} t^{ij}&=\rho g^{il}\frac{\partial w}{\partial x^l_{;K}}x^j_{;K},\\ \tau^{ij}&=\rho g^{il}\frac{\partial w}{\partial x^l_{;K}}x^j_{;K}+\rho g^{il}\frac{\partial w}{\partial d^l_{\cdot(\alpha)}}d^j_{\cdot(\alpha)}+\rho g^{il}\frac{\partial w}{\partial d^l_{\cdot(\alpha);K}}d^j_{\cdot(\alpha);K},\\ h^{ijk}&=\rho g^{il}\frac{\partial w}{\partial d^l_{\cdot(\alpha);K}}d^j_{\cdot(\alpha)}x^k_{;K}. \end{aligned} \tag{4.4}$$

These are the constitutive equations for micro-elastic materials. However, the right side of $(4.4)_2$ must satisfy the condition

$$\left(g^{il}\frac{\partial w}{\partial x^l_{;K}}x^j_{;K}+g^{il}\frac{\partial w}{\partial d^l_{\cdot(\alpha)}}d^j_{\cdot(\alpha)}+g^{il}\frac{\partial w}{\partial d^l_{\cdot(\alpha);K}}d^j_{\cdot(\alpha);K}\right)_{[ij]}=0, \tag{4.5}$$

to the end that the Cauchy's second law of motion be satisfied. This represents the *objectivity condition* of the specific strain energy (4.2) and of the constitutive equations (4.4).

Making use of

$$d^l_{\cdot(\alpha)}=\chi^l_{\cdot K}D^K_{\cdot(\alpha)},\qquad d^l_{\cdot(\alpha);K}=\chi^l_{\cdot L;K}D^L_{\cdot(\alpha)},$$

we can take the specific strain energy as a function of the form

$$w = w(x^k_{;K}, \chi^k_{.K}, \chi^k_{.L;K}). \tag{4.6}$$

Now, the constitutive equations (4.4) can be written in the form

$$\begin{aligned} t^{ij} &= \rho g^{il} \frac{\partial w}{\partial x^l_{;K}} x^j_{;K}, \\ \tau^{ij} &= \rho g^{il} \frac{\partial w}{\partial x^l_{;K}} x^j_{;K} + \rho g^{il} \frac{\partial w}{\partial \chi^l_{.K}} \chi^j_{.K} + \rho g^{il} \frac{\partial w}{\partial \chi^l_{.L;K}} \chi^j_{.L;K}, \\ h^{ijk} &= \rho g^{il} \frac{\partial w}{\partial \chi^l_{.L;K}} \chi^j_{.L} x^k_{;K}, \end{aligned} \tag{4.7}$$

and the objectivity condition (4.5) in the form

$$\left(g^{il} \frac{\partial w}{\partial x^l_{;K}} x^j_{;K} + g^{il} \frac{\partial w}{\partial \chi^l_{.K}} \chi^j_{.K} + g^{il} \frac{\partial w}{\partial \chi^l_{.L;K}} \chi^j_{.L;K}\right)_{[ij]} = 0. \tag{4.8}$$

The specific strain energy (4.6) is a function of 45 independent variables $x^k_{;K}$, $\chi^k_{.K}$ and $\chi^k_{.L;K}$. Conditions (4.8) represent a set of three linear partial differential equations and w is an arbitrary function of the integrals of (4.8). The system admits $45-3=42$ independent integrals. There are many possibilities for the choise of the basic integrals of (4.8). We shall take the following (1)

$$\begin{aligned} \Psi_{KL} &= g_{kl} \chi^k_{.K} \chi^l_{.L}, \\ \Sigma_{KL} &= \chi_{Kk} x^k_{;L}, \\ D_{KLM} &= \chi_{Kk} \chi^k_{.L;M}, \end{aligned} \tag{4.9}$$

so that the system (4.8) has *general solution*

$$w = w(\Psi_{KL}, \Sigma_{KL}, D_{KLM}). \tag{4.10}$$

Substituting (4.10) into (4.7), and using (4.9), we obtain

$$\begin{aligned} t^{ij} &= \rho \frac{\partial w}{\partial \Sigma_{KL}} \chi_K^{\cdot i} x^j_{;L}, \\ \tau^{ij} &= 2\rho \frac{\partial w}{\partial \Psi_{KL}} \chi^i_{.K} \chi^j_{.L}, \\ h^{ijk} &= \rho \frac{\partial w}{\partial D_{KLM}} \chi_K^{\cdot i} \chi^j_{.L} x^k_{;M}. \end{aligned} \tag{4.11}$$

These are the non-linear constitutive equations of non-isotropic micro-elastic materials, which are form-invariant with respect to the superposed rigid motion.

If, instead of Ψ_{KL} and Σ_{KL}, we introduce the following material measure of deformations

$$2F_{KL} = \Psi_{KL} - G_{KL}, \qquad \varepsilon_{KL} = \Sigma_{KL} - G_{KL}, \tag{4.12}$$

(1) The reason for choosing this minimal integrity basis is discussed in the appendix.

the constitutive equations (4.11) have the form

$$t^{ij}=\rho\frac{\partial w}{\partial \varepsilon_{KL}}\chi_K^{\;;i}x^j_{;L},$$

(4.13)
$$\tau^{ij}=\rho\frac{\partial w}{\partial F_{KL}}\chi^i_{.K}\chi^j_{.L},$$

$$h^{ijk}=\rho\frac{\partial w}{\partial D_{KLM}}\chi_K^{\;;i}\chi^j_{.L}x^k_{;M}.$$

If we introduce *the director displacement vectors* $\varphi^k_{.(\alpha)}$, i.e.

(4.14)
$$d^k_{.(\alpha)}=D^k_{.(\alpha)}+\varphi^k_{.(\alpha)}=g^k_K D^K_{.(\alpha)}+\varphi^k_{.(\alpha)},$$

then, multiplying this by $D^{(\alpha)}_{.K}$, we obtain

(4.15)
$$\chi^k_{.K}=g^k_K+\varphi^k_{.K}.$$

From (4.14) it follows

(4.16)
$$D^K_{.(\alpha)}=d^K_{.(\alpha)}-\varphi^K_{.(\alpha)}=g^K_k(d^k_{.(\alpha)}-\varphi^k_{.(\alpha)}),$$

and from this, multiplying by $d^{(\alpha)}_{.k}$,

(4.17)
$$\chi^K_{.k}=g^K_k-\varphi^K_{.k}.$$

From (4.15) and (4.17) we see that the micro-deformation gradients $\chi^k_{.K}$ and $\chi^K_{.k}$ are related to the micro-displacement gradients $\varphi^k_{.K}$ and $\varphi^K_{.k}$ in the same way that the deformation gradients are related to the gradients of displacement vectors, i.e.

(4.18)
$$x^k_{;K}=g^k_K+u^k_{;K},\qquad X^K_{;k}=g^K_k-u^K_{;k}.$$

With regard to

(4.19)
$$d^k_{.(\alpha)}d^{(\alpha)}_{.l}=\chi^k_{.K}\chi^K_{.l}=\delta^k_l,\qquad D^K_{.(\alpha)}D^{(\alpha)}_{.L}=\chi^K_{.k}\chi^k_{.L}=\delta^K_L,$$

the micro-displacement gradients $\varphi^k_{.L}$ and $\varphi^K_{.l}$, i.e. $\varphi^k_{.l}$ and $\varphi^K_{.L}$, are mutually related

(4.20)
$$\varphi^K_{.L}=\varphi^K_{.l}g^l_L+\varphi^K_{.l}\varphi^l_{.L}=\varphi^K_{.l}g^l_L+\varphi^K_{.M}\varphi^M_{.l}g^l_L=\varphi^k_{.L}g^K_k=\varphi^K_{.l}\chi^l_{.L},$$

(4.21)
$$\varphi^k_{.l}=\varphi^k_{.L}g^L_l-\varphi^k_{.L}\varphi^L_{.l}=\varphi^k_{.L}g^L_l-\varphi^k_{.m}\varphi^m_{.L}g^L_l=\varphi^K_{.l}g^k_K=\varphi^k_{.L}\varphi^L_{.l}.$$

The deformation tensors F_{KL}, ε_{KL} and D_{KLM}, by the use of (4.9), (4.12), (4.15), (4.17), (4.18) and (4.20), can be expressed in the form

$$2F_{KL}=\varphi_{KL}+\varphi_{LK}+\varphi_{MK}\varphi^M_{.L},$$

(4.22)
$$\varepsilon_{KL}=u_{K,L}-\varphi_{KL}-(u_{M,L}-\varphi_{ML})\varphi_{Kk}g^{Mk},$$

$$D_{KLM}=\varphi_{KL,M}-\varphi_{SL,M}\varphi_{Kk}g^{Sk}.$$

For an infinitesimal deformation, when the displacement and micro-displacement gradients are very small, the above deformation tensors are in the linear theory of the form

(4.23)
$$F_{KL}=\frac{1}{2}(\varphi_{KL}+\varphi_{LK}),\qquad \varepsilon_{KL}=u_{K,L}-\varphi_{KL},\qquad D_{KLM}=\varphi_{KL,M}.$$

Constitutive equations (4.11) and (4.13) satisfy the principle of objectivity. However, they can be further reduced but that depends on the material symmetries.

For the isotropic materials we can introduce the following spatial deformation tensors

$$\psi_{kl}=G_{KL}\,\chi^K_{,k}\,\chi^L_{,l},$$
$$\sigma_{kl}=\chi_{kK}\,X^K_{;l}, \tag{4.24}$$
$$d_{klm}=\chi_{kK;m}\,\chi^K_{.l}=-\chi_{kK}\,\chi^K_{.l;m},$$

so that

$$w=w(\psi_{kl},\sigma_{kl},d_{klm}). \tag{4.25}$$

Substituting (4.25) into (4.7), and using (4.24), we obtain non-linear constitutive equations for the isotropic materials

$$t^{ij}=-\rho\frac{\partial w}{\partial\sigma_{kj}}\sigma_k^{;i}-\rho\frac{\partial w}{\partial d_{klj}}d_{kl}^{;;i},$$

$$\tau^{ij}=-2\rho\frac{\partial w}{\partial\psi_{jk}}\psi^i_k+\rho\frac{\partial w}{\partial\sigma_{ik}}\sigma^j_{.k}-\rho\frac{\partial w}{\partial\sigma_{kj}}\sigma_k^{;i}+\rho\frac{\partial w}{\partial d_{ilm}}d^j_{.lm}- \tag{4.26}$$
$$-\rho\frac{\partial w}{\partial d_{kjm}}d_{k.m}^{;i}-\rho\frac{\partial w}{\partial d_{klj}}d_{kl}^{;;i},$$

$$h^{ijk}=\rho\frac{\partial w}{\partial d_{ijk}}.$$

These equations, however, do not satisfy the principle of objectivity. In order that the principle of objectivity be satisfied, i.e. that Cauchy's second law of motion (3.28) be satisfied, it must be

$$\left(-2\frac{\partial w}{\partial\psi_{jk}}\psi^i_k+\frac{\partial w}{\partial\sigma_{ik}}\sigma^j_{.k}-\frac{\partial w}{\partial\sigma_{kj}}\sigma_k^{;i}+\frac{\partial w}{\partial d_{ilm}}d^j_{.lm}-\right. \tag{4.27}$$
$$\left.-\frac{\partial w}{\partial d_{kjm}}d_{k.m}^{;i}-\frac{\partial w}{\partial d_{klj}}d_{kl}^{;;i}\right)_{[ij]}=0.$$

If, instead of ψ_{kl} and σ_{kl}, we introduce the following spatial measure of deformations

$$2f_{kl}=g_{kl}-\psi_{kl}=g_{kl}-G_{KL}\,\chi^K_{.k}\,\chi^L_{.l}, \tag{4.28}$$
$$\varepsilon_{kl}=g_{kl}-\sigma_{kl}=g_{kl}-\chi_{kK}\,X^K_{;l},$$

the constitutive equations (4.26) take the form

$$t^{ij}=\rho\left(\frac{\partial w}{\partial\varepsilon_{ij}}-\frac{\partial w}{\partial\varepsilon_{kj}}\varepsilon_k^{;i}-\frac{\partial w}{\partial d_{klj}}d_{kl}^{;;i}\right),$$

$$\tau^{ij}=\rho\left(\frac{\partial w}{\partial f_{ij}}-2\frac{\partial w}{\partial f_{jk}}f^i_k+\frac{\partial w}{\partial\varepsilon_{ik}}\varepsilon^j_{.k}-\frac{\partial w}{\partial\varepsilon_{kj}}\varepsilon_k^{;i}+\frac{\partial w}{\partial d_{ilm}}d^j_{.lm}-\right. \tag{4.29}$$
$$\left.-\frac{\partial w}{\partial d_{kjm}}d_{k.m}^{;i}-\frac{\partial w}{\partial d_{klj}}d_{kl}^{;;i}\right),$$

$$h^{ijk}=\rho\frac{\partial w}{\partial d_{ijk}},$$

and the condition of objectivity (4.27),

$$(4.30)\quad \left(-2\frac{\partial w}{\partial f_{jk}}f^{i}_{\cdot k}+\frac{\partial w}{\partial \varepsilon_{ik}}\varepsilon^{j}_{\cdot k}-\frac{\partial w}{\partial \varepsilon_{kj}}\varepsilon_{k}^{\cdot i}+\frac{\partial w}{\partial d_{ilm}}d^{j}_{\cdot lm}-\frac{\partial w}{\partial d_{kjm}}d_{k\cdot m}^{\cdot i}-\frac{\partial w}{\partial d_{klj}}d_{kl}^{\cdot\cdot i}\right)_{[ij]}=0.$$

The deformation tensors f_{kl}, ε_{kl} and d_{klm}, making use of (4.15), (4.17), (4.18), (4.21), (4.24) and (4.28), may be expressed in the form

$$(4.31)\quad \begin{aligned} 2f_{kl}&=\varphi_{kl}+\varphi_{lk}-\varphi_{mk}\,\varphi^{m}_{\cdot l},\\ \varepsilon_{kl}&=u_{k,l}-\varphi_{kl}+(u_{m,l}-\varphi_{ml})\,\varphi_{kK}\,g^{mK},\\ d_{klm}&=\varphi_{kl,m}+\varphi_{rl,m}\,\varphi_{kK}\,g^{rK}. \end{aligned}$$

For an infinitesimal deformation, in the linear theory, these tensors are of the form

$$(4.32)\quad f_{kl}=\frac{1}{2}(\varphi_{kl}+\varphi_{lk}),\qquad \varepsilon_{kl}=u_{k,l}-\varphi_{kl},\qquad d_{klm}=\varphi_{kl,m}.$$

The material deformation tensors (4.22) and the spatial deformation tensors (4.31) are related by the following expressions

$$(4.33)\quad \begin{aligned} F_{KL}&=f_{kl}\,\chi_{K}^{\cdot k}\,\chi_{L}^{\cdot l}, & f_{kl}&=F_{KL}\,\chi^{K}_{\cdot k}\,\chi^{L}_{\cdot l},\\ \varepsilon_{KL}&=\varepsilon_{kl}\,\chi_{K}^{\cdot k}\,x^{l}_{;L}, & \varepsilon_{kl}&=\varepsilon_{KL}\,\chi_{k}^{\cdot K}\,X^{L}_{;l},\\ D_{KLM}&=d_{klm}\,\chi_{K}^{\cdot k}\,\chi^{l}_{\cdot L}\,x^{m}_{;M}, & d_{klm}&=D_{KLM}\,\chi_{k}^{\cdot K}\,\chi^{L}_{\cdot l}\,X^{M}_{;m}. \end{aligned}$$

In the linear theory, we omit the non-linear terms in (4.29), so that the constitutive equations become

$$(4.34)\quad t^{ij}=\rho\frac{\partial w}{\partial \varepsilon_{ij}},\qquad \tau^{ij}=\rho\frac{\partial w}{\partial f_{ij}},\qquad h^{ijk}=\rho\frac{\partial w}{\partial d_{ijk}}.$$

In the constitutive equations (4.29) and (4.34), the specific strain energy is a function of the form

$$(4.35)\quad w=w(f_{kl},\varepsilon_{kl},d_{klm}).$$

That is an isotropic function of its arguments and can be expanded in the polynomial form. If we suppose that the initial stresses do not exist, in the linear theory the specific strain energy is a quadratic polynomial of the form

$$(4.36)\quad \rho w=\frac{1}{2}A^{ijkl}f_{ij}f_{kl}+B^{ijkl}f_{ij}\varepsilon_{kl}+\frac{1}{2}C^{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\frac{1}{2}D^{ijklmn}d_{ijk}d_{lmn},$$

where

$$(4.37)\quad \begin{aligned} A^{ijkl}&=\lambda g^{ij}g^{kl}+\mu(g^{ik}g^{jl}+g^{il}g^{jk}),\\ B^{ijkl}&=\lambda_1 g^{ij}g^{kl}+\mu_1(g^{ik}g^{jl}+g^{il}g^{jk}),\\ C^{ijkl}&=\nu_1 g^{ij}g^{kl}+\nu_2 g^{ik}g^{jl}+\nu_3 g^{il}g^{jk},\\ D^{ijklmn}&=\gamma_1(g^{ij}g^{kl}g^{mn}+g^{jk}g^{in}g^{lm})+\gamma_2(g^{ij}g^{km}g^{nl}+g^{kl}g^{jn}g^{im})+\gamma_3 g^{ij}g^{kn}g^{lm}+\\ &+\gamma_4 g^{jk}g^{il}g^{mn}+\gamma_5(g^{jk}g^{im}g^{nl}+g^{kl}g^{jl}g^{mn})+\gamma_6 g^{kl}g^{jm}g^{nl}+\gamma_7 g^{il}g^{jm}g^{kn}+\\ &+\gamma_8(g^{jl}g^{km}g^{in}+g^{kl}g^{im}g^{jn})+\gamma_9 g^{il}g^{jn}g^{km}+\gamma_{10}g^{jl}g^{kn}g^{im}+\gamma_{11}g^{kl}g^{in}g^{jm}, \end{aligned}$$

are isotropic tensors, and $\lambda, \mu, \lambda_1, \mu_1, \nu_1, \nu_2, \nu_3$ and $\gamma_1, \gamma_2, \ldots, \gamma_{11}$ are material constants.

Making use of (4.36), (4.37) and (4.34), the linear constitutive equations take the form

$$t^{ij} = \lambda_1 f_I g^{ij} + 2\mu_1 f^{ij} + \nu_1 \varepsilon_I g^{ij} + \nu_2 \varepsilon^{ij} + \nu_3 \varepsilon^{ji}, \tag{4.38}$$

$$\tau^{ij} = \lambda f_I g^{ij} + 2\mu f^{ij} + \lambda_1 \varepsilon_I g^{ij} + 2\mu_1 \varepsilon^{(ij)}, \tag{4.39}$$

$$\begin{aligned} h^{ijk} = {} & \gamma_1 (d^{kl}_{..l} g^{ij} + d^{l.i}_{.l} g^{jk}) + \gamma_2 (d^{lk}_{..l} g^{ij} + d^{l.j}_{.l} g^{ki}) + \gamma_3 d^{l.k}_{.l} g^{ij} + \gamma_4 d^{il}_{..l} g^{jk} + \\ & + \gamma_5 (d^{li}_{..l} g^{jk} + d^{jl}_{..l} g^{ki}) + \gamma_6 d^{lj}_{..l} g^{ki} + \gamma_7 d^{ijk} + \gamma_8 (d^{jki} + d^{kij}) + \\ & + \gamma_9 d^{ikj} + \gamma_{10} d^{jik} + \gamma_{11} d^{kji}, \end{aligned} \tag{4.40}$$

where the deformation tensors are of the form (4.32).

The linear constitutive equations (4.38), (4.39) and (4.40) differ in the form from the linear constitutive equations obtained by SUHUBI and ERINGEN [2], because we have introduced other measures of deformations. However, it is easy to show that these constitutive equations are equivalent, if we establish the connection between the corresponding deformation tensors.

5. MICROPOLAR ELASTIC MATERIALS

We shall consider now a special case of the materials with microstructure in which the macro-elements have a rigid motion. The motion of every macro-element of the body is then determined by the displacement of its center of gravity and by its rotation. The triad of directors $D^K_{.(\alpha)}$, which is attached to the macro-element, is now carried into $d^k_{.(\alpha)}$ by a translation and rotation.

From the viewpoint of continuum theory, when the macro-element is identified with a material point, every material point is *phenomenologically equivalent to a rigid body*. Since the motion of the directors is independent of displacements of the points, it is clear that, after deformation, the directors $d^k_{.(\alpha)}$ *are not material vectors*. By introduction of rigid directors in the points of the body, we interprete a micropolar elastic continuum as *an elastic Cosserat continuum*.

Since the motion of directors is a rigid motion, then

$$g_{kl}\, d^k_{.(\alpha)} d^l_{.(\beta)} = G_{KL} D^K_{.(\alpha)} D^L_{.(\beta)}, \tag{5.1}$$

wherefrom, using (2.15),

$$g_{kl}\, \chi^k_{.K} \chi^l_{.L} = G_{KL}, \qquad G^{KL} \chi_{kK} \chi_{lL} = g_{kl}. \tag{5.2}$$

This is the condition that $\chi^k_{.K}$ is an orthogonal tensor. However, it must be

$$|\chi^k_{.K}| = \sqrt{\frac{G}{g}}, \tag{5.3}$$

to represent the rotation.

The motion of micropolar continua is determined by the equations

$$x^i = x^i(X^K, t), \quad \chi^k_{.K} = \chi^k_{.K}(X^L, t). \tag{5.4}$$

The first equation determines the displacements of points of the body, i.e. the translations of directors. The second determines independent rotations of the directors, since $\chi^k_{.K}$ is an orthogonal tensor.

The condition of orthogonality of the tensor $\chi^k_{.K}$ can be written in the form

$$\chi_{kK} = \chi_{Kk} \quad (\chi^T = \chi^{-1}). \tag{5.5}$$

Since $\chi^k_{.K}$ is an orthogonal tensor, i.e. during the motion the components $\chi^k_{.K}$ have to satisfy six equations (5.2) (the condition (5.3) is included in (5.2)), we deduce that $\chi^k_{.K}$ have only three mutually independent coordinates.

By differentiation with respect to the time, from $(5.2)_2$ we get

$$G^{KL} \dot{\chi}_{kK} \chi_{lL} = -G^{KL} \chi_{kK} \dot{\chi}_{lL}, \tag{5.6}$$

wherefrom, using (5.5) and (2.27),

$$b_{kl} = \dot{\chi}_{kK} \chi^K_{.l} = -\dot{\chi}_{lK} \chi^K_{.k} = -b_{lk}. \tag{5.7}$$

We see that b_{kl} is a skew-symmetric tensor, so that it has three mutually independent coordinates. According to this, from (2.34) we see that only skew-symmetric part of the inertial spin makes contribution to the rate of the kinetic energy. For this reason, we can take, without loss of generality, that the inertial spin is skew-symmetric, i.e. $\Gamma^{ij} = -\Gamma^{ji}$. In the same way, we see that only skew-symmetric parts of the first surface and body moments make contribution to the time rate of working of the surface and body forces. Consequently, without loss of generality, we can take that $H^{ij} = -H^{ji}$ and $l^{ij} = -l^{ji}$, i.e. that H^{ij} represents *the surface couple*, and l^{ij} *the body couple*. From this it follows that $h^{ijk} = -h^{jik} = m^{ijk}$, so that (3.15) becomes $H^{ij} = m^{ijk} n_k$, where m^{ijk} is *the couple stress tensor* which has nine mutually independent coordinates.

Since b_{kl} is a skew-symmetric tensor, then, taking into account (2.27), we have

$$(\dot{d}_{i(\alpha)} d^{(\alpha)}_{.j})_{(ij)} = 0, \tag{5.8}$$

and, consequently,

$$(\delta d_{i(\alpha)} d^{(\alpha)}_{.j})_{(ij)} = 0. \tag{5.9}$$

The variations $\delta d_{i(\alpha)}$ are not mutually independent, but have to satisfy six equations (5.9), so that only three of them are mutually independent. According to this, we can write

$$\delta d_{i(\alpha)} = \delta\varphi_{ij} d^j_{.(\alpha)}, \quad \delta d_{i(\alpha),k} = \delta\varphi_{ij,k} d^j_{.(\alpha)} + \delta\varphi_{ij} d^j_{.(\alpha),k}, \tag{5.10}$$

where $\delta\varphi_{ij} = -\delta\varphi_{ji}$ are three independent variations.

Making use of (5.10), the equation (3.17) becomes

$$\int_v \rho(\dot{v}^i \delta x_i + \Gamma^{ij} \delta\varphi_{ij})\, dv + \int_v \rho \delta w\, dv = \int_v \rho (f^i \delta x_i + l^{ij} \delta\varphi_{ij})\, dv + \tag{5.11}$$
$$+ \int_v (t^{ij}_{,j} \delta x_i + t^{ij} \delta x_{i,j} + m^{ijk}_{,k} \delta\varphi_{ij} + m^{ijk} \delta\varphi_{ij,k})\, dv,$$

and it is valid for all variations δx_i, $\delta x_{i,j}$, $\delta\varphi_{ij}$ and $\delta\varphi_{ij,k}$. However, we suppose that $\delta w=0$ for a rigid motion, so that the above equation becomes

$$(5.12)\quad \int_v \rho(\dot{v}^i\,\delta x_i+\Gamma^{ij}\,\delta\varphi_{ij})\,dv=\int_v \rho(f^i\,\delta x_i+l^{ij}\,\delta\varphi_{ij})\,dv+ \\ +\int_v (t^{ij}{}_{,j}\,\delta x_i+t^{ij}\,\delta x_{i,j}+m^{ijk}{}_{,k}\,\delta\varphi_{ij}+m^{ijk}\,\delta\varphi_{ij,k})\,dv.$$

This equation for virtual translations is equivalent to Cauchy's first law of motion, and for rigid virtual displacements — to Cauchy's second law.

For virtual translations, we have

$$(5.13)\quad \delta x_i=\text{const},\quad \delta x_{i,j}=0,\quad \delta\varphi_{ij}=0,\quad \delta\varphi_{ij,k}=0,$$

and the equation (5.12) gives

$$(5.14)\quad \rho\dot{v}^i=t^{ij}{}_{,j}+\rho f^i.$$

This is Cauchy's first law of motion, i.e. a necessary and sufficient condition for the balance of momentum, which represents three classical differential equations of motion.

If we assume that Cauchy's first law is valid, the equation (5.12) becomes

$$(5.15)\quad \int_v \rho\Gamma^{ij}\,\delta\varphi_{ij}\,dv=\int_v [\rho l^{ij}\,\delta\varphi_{ij}+t^{ij}\,\delta x_{i,j}+m^{ijk}{}_{,k}\,\delta\varphi_{ij}+m^{ijk}\,\delta\varphi_{ij,k}]\,dv,$$

and, for virtual rigid displacements, it is equivalent to Cauchy's second law of motion.

For virtual rigid displacements, we have

$$(5.16)\quad \delta x_{(i,j)}=0,\quad \delta\varphi_{ij}=\delta x_{[i,j]}=\text{const},\quad \delta\varphi_{ij,k}=0,$$

and the equation (5.15) gives

$$(5.17)\quad t^{[ij]}+m^{ijk}{}_{,k}+\rho(l^{ij}-\Gamma^{ij})=0.$$

This is Cauchy's second law of motion, i.e. a necessary and sufficient condition for the balance of the moment of momentum, which represents three differential equations of motion.

Substituting (5.14) and (5.17) into (5.11), we obtain

$$(5.18)\quad \int_v \rho\delta w\,dv=\int_v (t^{ij}\,\delta x_{i,j}-t^{[ij]}\,\delta\varphi_{ij}+m^{ijk}\,\delta\varphi_{ij,k})\,dv,$$

whence it follows

$$(5.19)\quad \rho\delta w=t^{ij}\,\delta x_{i,j}-t^{[ij]}\,\delta\varphi_{ij}+m^{ijk}\,\delta\varphi_{ij,k}.$$

This is the expression for the variation of the specific strain energy, which is form-invariant with respect to the superposed rigid motion.

According to (5.19), we obtain the expression of local conservation of energy in the form

$$(5.20)\quad \rho\dot{w}=t^{ij}v_{i,j}-t^{[ij]}b_{ij}+m^{ijk}b_{ij,k}.$$

Comparing this expression with (3.33), we see that (5.20) follows directly from (3.33) if we take into account that b_{ij} is a skew-symmetric tensor.

From (5.19), as well as (5.20), we see that t^{ij}, $t^{[ij]}$ and m^{ijk} can be separately determined through the constitutive equations. If we assume, as well as in the case of microelastic materials, that the specific strain energy is a function of the form

$$w=w(x^k_{;K}, d^k_{.(\alpha)}, d^k_{.(\alpha);K}), \tag{5.21}$$

then, using (5.10), we obtain

$$\delta w=g^{il}\frac{\partial w}{\partial x^l_{;K}}x^j_{;K}\,\delta x_{i,j}+\left(\frac{\partial w}{\partial d^l_{.(\alpha)}}g^{l[i}d^{j]}_{.(\alpha)}+\frac{\partial w}{\partial d^l_{.(\alpha);K}}g^{l[i}d^{j]}_{.(\alpha);K}\right)\delta\varphi_{ij}+ \\ +\frac{\partial w}{\partial d^l_{.(\alpha);K}}g^{l[i}d^{j]}_{.(\alpha)}x^k_{;K}\,\delta\varphi_{ij,k}. \tag{5.22}$$

Comparing this expression with (5.19), we obtain the constitutive equations

$$\begin{aligned} t^{ij}&=\rho g^{il}\frac{\partial w}{\partial x^l_{;K}}x^j_{;K}, \\ t^{[ij]}&=-\rho\frac{\partial w}{\partial d^l_{.(\alpha)}}g^{l[i}d^{j]}_{.(\alpha)}-\rho\frac{\partial w}{\partial d^l_{.(\alpha);K}}g^{l[i}d^{j]}_{.(\alpha);K}, \\ m^{ijk}&=\rho\frac{\partial w}{\partial d^l_{.(\alpha);K}}g^{l[i}d^{j]}_{.(\alpha)}x^k_{;K}. \end{aligned} \tag{5.23}$$

However, the first two equations must be compatible, i.e.

$$\left(g^{il}\frac{\partial w}{\partial x^l_{;K}}x^j_{;K}+g^{il}\frac{\partial w}{\partial d^l_{.(\alpha)}}d^j_{.(\alpha)}+g^{il}\frac{\partial w}{\partial d^l_{.(\alpha);K}}d^j_{.(\alpha);K}\right)_{[ij]}=0. \tag{5.24}$$

This represents *the condition of objectivity* of the specific strain energy (5.21) and of the constitutive equations (5.23). If the equation (5.24) is satisfied, then the equation $(5.23)_2$ is superfluous, because it is included in $(5.23)_1$ as its skew-symmetric part.

If we assume that the specific strain energy is a function of the form

$$w=w(x^k_{;K}, \chi^k_{.K}, \chi^k_{.L;K}), \tag{5.25}$$

then the constitutive equations (5.23) become

$$\begin{aligned} t^{ij}&=\rho g^{il}\frac{\partial w}{\partial x^l_{;K}}x^j_{;K}, \\ m^{ijk}&=\rho\frac{\partial w}{\partial \chi^l_{.L;K}}g^{l[i}\chi^{j]}_{.L}x^k_{;K}, \end{aligned} \tag{5.26}$$

and the condition of objectivity (5.24),

$$\left(g^{il}\frac{\partial w}{\partial x^l_{;K}}x^j_{;K}+g^{il}\frac{\partial w}{\partial \chi^l_{.K}}\chi^j_{.K}+g^{il}\frac{\partial w}{\partial \chi^l_{.L;K}}\chi^j_{.L;K}\right)_{[ij]}=0. \tag{5.27}$$

The specific strain energy (5.25) is a function of 21 independent variables $x^k_{;K}$, $\chi^k_{.K}$ and $\chi^k_{.L;K}$. Since (5.27) represents the system of three linear partial differential equations, then it admits $21-3=18$ independent integrals. There are many possibilities for the choice of the independent integrals of (5.27). We shall take the following

(5.28) $$\Sigma_{KL}=\chi_{Kk}x^k_{;L}, \qquad K_{KLM}=\chi_{Kk}\chi^k_{.L;M},$$

which, using (5.5), we can write in the form

(5.29) $$\Sigma_{KL}=\chi_{kK}x^k_{;L}, \qquad K_{KLM}=\chi_{kK}\chi^k_{.L;M}=-\chi_{kL}\chi^k_{.K;M}.$$

It follows from $(5.2)_1$, by differentiation with respect to X^M, that the tensor K_{KLM} is skew-symmetric in the indices K and L. Hence, it has nine mutually independent components.

According to (5.29), a general solution of the system (5.27) is

(5.30) $$w=w(\Sigma_{KL}, K_{KLM}).$$

Substituting now (5.30) into (5.26), and using (5.29), we obtain the constitutive equations

(5.31) $$t^{ij}=\rho\frac{\partial w}{\partial \Sigma_{KL}}\chi^i_{.K}x^j_{;L}, \qquad m^{ijk}=\rho\frac{\partial w}{\partial K_{KLM}}\chi^i_{.K}\chi^j_{.L}x^k_{;M},$$

which are form-invariant with respect to the superposed rigid motion.

If, instead of Σ_{KL}, we introduce the following measure of deformation

(5.32) $$\varepsilon_{KL}=\Sigma_{KL}-G_{KL}=\chi_{kK}x^k_{;L}-G_{KL},$$

the constitutive equations (5.31) become

(5.33) $$t^{ij}=\rho\frac{\partial w}{\partial \varepsilon_{KL}}\chi^i_{.K}x^j_{;L}, \qquad m^{ijk}=\rho\frac{\partial w}{\partial K_{KLM}}\chi^i_{.K}\chi^j_{.L}x^k_{;M}.$$

These are the non-linear constitutive equations for anisotropic micropolar elastic materials. We can obtain these equations directly from the constitutive equations (4.13) for micro-elastic materials, if we take into account that $\chi^k_{.K}$ is an orthogonal tensor; then, according to $(5.2)_1$, the tensor $F_{KL}=0$, while ε_{KL} and D_{KLM} in (4.13) become ε_{KL} and K_{KLM} in (5.33). Hence, the constitutive equation $(4.13)_2$ vanishes.

Making use of (4.15), from $(5.2)_1$ we get

(5.34) $$\varphi_{KL}+\varphi_{LK}+\varphi_{MK}\varphi^M_{.L}=0,$$

wherefrom we see that in the linear theory

(5.35) $$\varphi_{KL}+\varphi_{LK}=0.$$

Also, from (5.5), using (4.14) and (4.17), we obtain

$$\varphi_{kK} = -\varphi_{Kk}. \tag{5.36}$$

Hence

$$\varphi_{KL} = -\varphi_{Lk} g^k_K = -\varphi_{lk} g^l_L g^k_K, \quad \varphi_{kl} = -\varphi_{lK} g^K_k = -\varphi_{LK} g^L_l g^K_k. \tag{5.37}$$

The deformation tensors ε_{KL} and K_{KLM}, according to (5.32) and $(5.29)_2$, and using (4.15), $(4.18)_1$ and (5.34), may be expressed in the form

$$\begin{aligned} \varepsilon_{KL} &= u_{K,L} + \varphi_{LK} + \varphi_{MK} u^M_{,L} = u_{K,L} - \varphi_{KL} + (u_{M,L} - \varphi_{ML}) \varphi^M_{.K}, \\ K_{KLM} &= \varphi_{KL,M} + \varphi_{SK} \varphi^S_{.L,M} = -\varphi_{LK,M} - \varphi_{SL} \varphi^S_{.K,M}. \end{aligned} \tag{5.38}$$

In the linear theory, for an infinitesimal deformation, these tensors are

$$\begin{aligned} \varepsilon_{KL} &= u_{K,L} + \varphi_{LK} = u_{K,L} - \varphi_{KL}, \\ K_{KLM} &= \varphi_{KL,M} = -\varphi_{LK,M}. \end{aligned} \tag{5.39}$$

For isotropic materials, we can introduce the following spatial tensors

$$\begin{aligned} \sigma_{kl} &= \chi_{kK} X^K_{;l}, \\ \kappa_{klm} &= G^{KL} \chi_{kK;m} \chi_{lL} = -G^{KL} \chi_{lk;m} \chi_{kL}, \end{aligned} \tag{5.40}$$

so that

$$w = w(\sigma_{kl}, \kappa_{klm}). \tag{5.41}$$

The tensor κ_{klm} is skew-symmetric with respect to k and l, i.e. $\kappa_{klm} = -\kappa_{lkm}$. It follows from $(5.2)_2$ by differentiation with respect to X^m. Hence, κ_{klm} has nine mutually independent components.

Substituting (5.41) into (5.26) and using (5.40), we get the constitutive equations for isotropic materials

$$\begin{aligned} t^{ij} &= -\rho \frac{\partial w}{\partial \sigma_{kj}} \sigma_k^{;i} - \rho \frac{\partial w}{\partial \kappa_{klj}} \kappa_{kl}^{;;i}, \\ m^{ijk} &= \rho \frac{\partial w}{\partial \kappa_{ijk}}. \end{aligned} \tag{5.42}$$

From (5.27), using (5.40) and (5.5), we obtain the condition of objectivity in the form

$$\left(\frac{\partial w}{\partial \sigma_{kj}} \sigma_k^{;i} + \frac{\partial w}{\partial \sigma_{jk}} \sigma^i_{.k} + 2 \frac{\partial w}{\partial \kappa_{jlm}} \kappa^i_{.lm} + \frac{\partial w}{\partial \kappa_{klj}} \kappa_{kl}^{;;i} \right)_{[ij]} = 0. \tag{5.43}$$

If, instead of σ_{kl}, we introduce the following spatial measure of deformation

$$\varepsilon_{kl} = g_{kl} - \sigma_{kl} = g_{kl} - \chi_{kK} X^K_{;l}, \tag{5.44}$$

the constitutive equations (5.42) become

$$\begin{aligned} t^{ij} &= \rho \frac{\partial w}{\partial \varepsilon_{ij}} - \rho \frac{\partial w}{\partial \varepsilon_{kj}} \varepsilon_k^{;i} - \rho \frac{\partial w}{\partial \kappa_{klj}} \kappa_{kl}^{;;i}, \\ m^{ijk} &= \rho \frac{\partial w}{\partial \kappa_{ijk}}, \end{aligned} \tag{5.45}$$

and the condition of objectivity (5.43),

$$\left(\frac{\partial w}{\partial \varepsilon_{kj}}\varepsilon_{k}^{\cdot\,i}+\frac{\partial w}{\partial \varepsilon_{jk}}\varepsilon^{i}_{\cdot k}+2\frac{\partial w}{\partial \kappa_{jlm}}\kappa^{i}_{\cdot lm}+\frac{\partial w}{\partial \kappa_{klj}}\kappa_{kl}^{\cdot\cdot i}\right)_{[ij]}=0. \tag{5.46}$$

From $(5.2)_2$, using (4.14) and $(5.37)_2$, we obtain

$$\varphi_{kl}+\varphi_{lk}-\varphi_{mk}\varphi^{m}_{\cdot l}=0, \tag{5.47}$$

and we see that, in the linear theory,

$$\varphi_{kl}+\varphi_{lk}=0. \tag{5.48}$$

The deformation tensors ε_{kl} and κ_{klm}, making use of (5.40), (5.44), (4.15), (4.18), $(5.37)_2$ and (5.47), may be expressed in the form

$$\begin{aligned}\varepsilon_{kl}&=u_{k,l}+\varphi_{lk}-\varphi_{mk}u^{m}_{,l}=u_{k,l}-\varphi_{kl}-(u_{m,l}-\varphi_{ml})\,\varphi^{m}_{\cdot k},\\ \kappa_{klm}&=\varphi_{kl,m}-\varphi_{rk}\varphi^{r}_{\cdot l,m}=-\varphi_{lk,m}+\varphi_{rl}\varphi^{r}_{\cdot k,m}.\end{aligned} \tag{5.49}$$

In the linear theory, these tensors will be

$$\begin{aligned}\varepsilon_{kl}&=u_{k,l}+\varphi_{lk}=u_{k,l}-\varphi_{kl},\\ \kappa_{klm}&=\varphi_{kl,m}=-\varphi_{lk,m}.\end{aligned} \tag{5.50}$$

The constitutive equations (5.42) and (5.45) are the non-linear constitutive equations for isotropic micropolar elastic materials. We note that these equations can be obtained directly from the constitutive equations (4.26) and (4.29) for micro-elastic isotropic materials, if we take into account the influence of the condition of orthogonality (5.5) of the tensor $\chi^{k}_{\cdot K}$ upon the deformation tensors. Upon this condition, the tensor ψ_{kl} becomes g_{kl}, $f_{kl}=0$, and d_{klm} becomes κ_{klm}.

In the linear theory, disregarding the non-linear terms in (5.45), the constitutive equations read

$$t^{ij}=\rho\frac{\partial w}{\partial \varepsilon_{ij}},\qquad m^{ijk}=\rho\frac{\partial w}{\partial \kappa_{ijk}}, \tag{5.51}$$

where the deformation tensors are of the form (5.50). These equations, also, can be obtained directly from (4.34), taking into account the influence of the condition of orthogonality of the tensor $\chi^{k}_{\cdot K}$ upon the deformation tensors.

According to the skew-symmetry of the tensors κ_{ijk} and m^{ijk} with respect to the first two indices, we can reduce them to the second-order tensors

$$\kappa_{ij}=\frac{1}{2}\varepsilon_{ikl}\kappa^{kl}_{\cdot\cdot j},\qquad m_{ij}=\frac{1}{2}\varepsilon_{ikl}m^{kl}_{\cdot\cdot j}. \tag{5.52}$$

The constitutive equations (5.51) then become

$$t^{ij}=\rho\frac{\partial w}{\partial \varepsilon_{ij}},\qquad m^{ij}=\rho\frac{\partial w}{\partial \kappa_{ij}}, \tag{5.53}$$

where

(5.54)
$$\varepsilon_{ij}=u_{i,j}-\varphi_{ij}=u_{i,j}-\varepsilon_{ijk}\varphi^k,$$
$$\kappa_{ij}=\frac{1}{2}\varepsilon_{ikl}\varphi^{kl}_{..,j}=\varphi_{i,j},$$

and

(5.55)
$$w=w(\varepsilon_{ij},\kappa_{ij}).$$

If there are no initial stresses, the specific strain energy in the linear theory is a quadratic polynomial of the form

(5.56)
$$\rho w=\frac{1}{2}A^{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\frac{1}{2}B^{ijkl}\kappa_{ij}\kappa_{kl},$$

where

(5.57)
$$A^{ijkl}=\nu_1 g^{ij}g^{kl}+\nu_2 g^{ik}g^{jl}+\nu_3 g^{il}g^{jk},$$
$$B^{ijkl}=\tau_1 g^{ij}g^{kl}+\tau_2 g^{ik}g^{jl}+\tau_3 g^{il}g^{jk}$$

are isotropic tensors, and $\nu_1,\nu_2,\nu_3,\tau_1,\tau_2$ and τ_3 are material constants.

Using (5.56) and (5.57), from (5.53) we finally obtain the linear constitutive equations

(5.58)
$$t^{ij}=\nu_1\varepsilon_1 g^{ij}+\nu_2\varepsilon^{ij}+\nu_3\varepsilon^{ij},$$
$$m^{ij}=\tau_1\kappa_1 g^{ij}+\tau_2\kappa^{ij}+\tau_3\kappa^{ji},$$

where $\varepsilon_{\rm I}=e_{\rm I}=u^i_{,i}=\operatorname{div}\mathbf{u}$ and $\kappa_{\rm I}=\varphi^i_{,i}=\operatorname{div}\boldsymbol{\varphi}$ are the first invariants of the tensors ε_{ij} and κ_{ij}.

If we write

(5.59)
$$\varepsilon_{ij}=u_{i,j}-\varphi_{ij}=e_{ij}+r_{ij}-\varphi_{ij}=e_{ij}+\varepsilon_{ijk}(r^k-\varphi^k),$$

where

(5.60)
$$r_{ij}=\frac{1}{2}(u_{i,j}-u_{j,i}),\quad e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),$$

the constitutive equations (5.58) may be expressed in the equivalent form

(5.61)
$$t^{ij}=\lambda e_{\rm I} g_{ij}+2\mu e_{ij}+k\varepsilon_{ijk}(r^k-\varphi^k),$$
$$m_{ij}=\tau_1\varphi^k_{,k}g_{ij}+\tau_2\varphi_{i,j}+\tau_3\varphi_{j,i},$$

where λ and μ are classical Lame's constants, and φ^k is the vectorial representation of the angle of rotation of material points.

6. Dipolar elastic materials

In the micromorphic and micropolar theories, directors are not material vectors. We shall now consider a generalized Cosserat continuum in which the directors *are material vectors*. Then, the deformation is completely determined by the deformation gradients, since the deformation of directors is not independent of displacements of the material points. Hence, the equation

$$x^k = x^k(X^K, t) \tag{6.1}$$

is sufficient for description of motion.

The deformation of directors is determined by the deformation gradients,

$$d^k_{\cdot(\alpha)} = x^k_{;K} D^K_{\cdot(\alpha)}. \tag{6.2}$$

It follows from (6.2), that

$$\dot{d}^k_{\cdot(\alpha)} = v^k_{,l} x^l_{;K} D^K_{\cdot(\alpha)} = v^k_{,l} d^l_{\cdot(\alpha)}, \tag{6.3}$$

wherefrom

$$v_{k,l} = \dot{d}_{k(\alpha)} d^{(\alpha)}_{\cdot l}. \tag{6.4}$$

Using (6.3), it follows from (2.32) the expression for the kinetic energy

$$2T = \int_v \rho (v^k v_k + i^{lm} v^k_{,l} v_{k,m})\, dv, \tag{6.5}$$

where

$$i^{lm} = I^{\alpha\beta} d^l_{\cdot(\alpha)} d^m_{\cdot(\beta)} = I^{KL} x^l_{;K} x^m_{;L}. \tag{6.6}$$

The time rate of the kinetic energy is

$$\dot{T} = \int_v \rho (\dot{v}^k v_k + \Gamma^{ij} v_{i,j})\, dv, \tag{6.7}$$

where

$$\Gamma^{ij} = i^{lj} \dot{v}^i_{,l} \qquad (\dot{v}^i_{,l} = (\dot{v}^i)_{,l}) \tag{6.8}$$

is inertial spin.

According to (6.3), we obtain

$$\delta d_{i(\alpha)} = \delta x_{i,j} d^j_{\cdot(\alpha)}, \tag{6.9}$$

and (3.17) gives

$$\int_v \rho (\dot{v}^i \delta x_i + \Gamma^{ij} \delta x_{i,j})\, dv + \int_v \rho \delta w\, dv = \int_v \rho (f^i \delta x_i + l^{ij} \delta x_{i,j})\, dv + \\ + \int_v [t^{ik}_{,k} \delta x_i + (t^{ij} + h^{ijk}_{,k}) \delta x_{i,j} + h^{ijk} \delta x_{i,jk}]\, dv \tag{6.10}$$

for all variations δx_i, $\delta x_{i,j}$ and $\delta x_{i,jk}$. However, for virtual rigid displacements, this equation becomes

$$\int_v \rho (\dot{v}^i \delta x_i + \Gamma^{ij} \delta x_{i,j})\, dv = \int_v \rho (f^i \delta x_i + l^{ij} \delta x_{i,j})\, dv + \\ + \int_v [t^{ik}_{,k} \delta x_i + (t^{ij} + h^{ijk}_{,k}) \delta x_{i,j} + h^{ijk} \delta x_{i,jk}]\, dv, \tag{6.11}$$

since $\delta w = 0$.

From (6.11) for virtual translations, we get

$$\rho\dot{v}^i = t^{ij}{}_{,j} + \rho f^i . \tag{6.12}$$

This is Cauchy's first law of motion.

If we assume that (6.12) is valid, the equation (6.11) takes the form

$$\int_v \rho \Gamma^{ij}\delta x_{i,j}\,dv = \int_v \rho l^{ij}\delta x_{i,j}\,dv + \int_v [(t^{ij}+h^{ijk}{}_{,k})\delta x_i + h^{ijk}\delta x_{i,jk}]\,dv \tag{6.13}$$

and it is equivalent to Cauchy's second law of motion, for virtual rigid displacements, i.e.

$$t^{[ij]} + h^{[ij]k}{}_{,k} + \rho(l^{[ij]} - \Gamma^{[ij]}) = 0 . \tag{6.14}$$

Substituting (6.12) and (6.14) into (6.10), we obtain

$$\int_v \rho\delta w\,dv = \int_v (\tau^{ij}\delta x_{i,j} + h^{ijk}\delta x_{i,jk})\,dv , \tag{6.15}$$

where, according to (6.14),

$$\tau^{ij} = t^{ij} + h^{ijk}{}_{,k} + \rho(l^{ij} - \Gamma^{ij}) = \tau^{ji} . \tag{6.16}$$

It follows from (6.15) that

$$\rho\delta w = \tau^{ij}\delta x_{(i,j)} + h^{i(jk)}\delta x_{i,jk} . \tag{6.17}$$

This is the expression for the variation of the specific strain energy, which is form-invariant with respect to the superposed rigid motion.

The time rate of the specific strain energy is

$$\rho\dot{w} = \tau^{ij}d_{ij} + h^{i(jk)}v_{i,jk} , \quad (d_{ij} = v_{(i,j)}) , \tag{6.18}$$

wherefrom, as well as from (6.17), we conclude that τ^{ij} and $h^{i(jk)}$ can be determined through the constitutive equations. Consequently, the stress tensor t^{ij} is not determined through the constitutive equations, but we determine it from the system of equations (6.16). However, with regard to the fact that through the constitutive equations we cannot determine all components of the tensor h^{ijk}, but only its part $h^{i(jk)}$, from (6.16) it follows that we cannot determine all components of the tensor t^{ij}. Consequently, the undetermined part $h^{i[jk]}$ may be regarded as an arbitrary function, with a corresponding contribution $h^{i[jk]}{}_{,k}$ in the stress t^{ij}. It is evident, however, that $h^{i[jk]}$ makes no contribution to the equations of motion (6.12), since $h^{i[jk]}{}_{,kj} = 0$. Although $h^{i[jk]}$ is undetermined, it plays an important role in determining correct boundary conditions, as it was shown by A. E. Green and R. S. Rivlin [11] and J. L. Bleustein and A. E. Green [14].

In order to obtain the constitutive equations, we assume that the specific strain energy is a function of the form

$$w = w(d^i{}_{.(\alpha)}, d^i{}_{.(\alpha);K}) , \tag{6.19}$$

wherefrom we get

$$\delta w = g^{il} \frac{\partial w}{\partial d^l_{.(\alpha)}} \delta d_{i(\alpha)} + g^{il} \frac{\partial w}{\partial d^l_{.(\alpha);K}} \delta d_{i(\alpha);K}, \tag{6.20}$$

or, using (6.9),

$$\delta w = g^{il} \left(\frac{\partial w}{\partial d^l_{.(\alpha)}} d^j_{.(\alpha)} + \frac{\partial w}{\partial d^l_{.(\alpha),k}} d^j_{.(\alpha),k} \right) \delta x_{i,j} + g^{il} \frac{\partial w}{\partial d^l_{.(\alpha),k}} d^j_{.(\alpha)} \delta x_{i,jk}. \tag{6.21}$$

From this equation and (6.17), we obtain the constitutive equations

$$\begin{aligned} \tau^{ij} &= \rho g^{il} \left(\frac{\partial w}{\partial d^l_{.(\alpha)}} d^j_{.(\alpha)} + \frac{\partial w}{\partial d^l_{.(\alpha),k}} d^j_{.(\alpha),k} \right), \\ h^{i(jk)} &= \rho g^{il} \frac{\partial w}{\partial d^j_{.(\alpha),k}} d^j_{.(\alpha)}, \end{aligned} \tag{6.22}$$

which, according to (6.2), may be expressed in the form

$$\begin{aligned} \tau^{ij} &= \rho g^{il} \left(\frac{\partial w}{\partial x^l_{;K}} x^j_{;K} + \frac{\partial w}{\partial x^l_{;KL}} x^j_{;KL} \right), \\ h^{i(jk)} &= \rho g^{il} \frac{\partial w}{\partial x^l_{;KL}} x^j_{;K} x^k_{;L}. \end{aligned} \tag{6.23}$$

However, to the end that Cauchy's second law of motion be satisfied, i.e. that the stress tensor τ^{ij} be symmetric, it must be

$$\left(g^{il} \frac{\partial w}{\partial x^l_{;K}} x^j_{;K} + g^{il} \frac{\partial w}{\partial x^l_{;KL}} x^j_{;KL} \right)_{[ij]} = 0. \tag{6.24}$$

This is the objectivity condition of the specific strain energy and of the constitutive equations.

The specific strain energy is a function of 27 independent variables $x^k_{;K}$ and $x^k_{;KL}$. Then, the system of three partial differential equations (6.24) admits $27-3=24$ independent integrals. We shall take the following

$$\begin{aligned} C_{KL} &= g_{kl} x^k_{;K} x^l_{;L}, \\ E_{KLM} &= G_{KN} X^N_{;k} x^k_{;LM} = E_{KML}, \end{aligned} \tag{6.25}$$

so that the system (6.24) has a general solution

$$w = w(C_{KL}, E_{KLM}). \tag{6.26}$$

Substituting now (6.26) into (6.23), and using (6.25), we obtain the non-linear constitutive equations

$$\begin{aligned} \tau^{ij} &= 2\rho \frac{\partial w}{\partial C_{KL}} x^i_{;K} x^j_{;L}, \\ h^{i(jk)} &= \rho \frac{\partial w}{\partial E^K_{.LM}} g^{il} X^K_{;l} x^j_{;L} x^k_{;M}, \end{aligned} \tag{6.27}$$

which are form-invariant with respect to the superposed rigid motion. These equations can be obtained from the constitutive equations (4.11) for microelastic materials, if, instead of $\chi^k_{.K}$, we write the deformation gradients $x^k_{;K}$, since then the tensor Σ_{KL} becomes G_{KL}, while the tensors F_{KL} and D_{KLM} become C_{KL} and E_{KLM}.

If we introduce the strain tensor

(6.28) $$2E_{KL}=C_{KL}-G_{KL},$$

the constitutive equations (6.27) become

(6.29) $$\tau^{ij}=\rho\frac{\partial w}{\partial E_{KL}}x^i_{;K}x^j_{;L},\qquad h^{i(jk)}=\rho\frac{\partial w}{\partial E^K_{.LM}}g^{il}X^K_{;l}x^j_{;L}x^k_{;M}.$$

These equations, also, can be obtained from the constitutive equations (4.13) for micro-elastic materials, since in this case $\varepsilon_{KL}=0$.

The deformation tensors E_{KL} and E_{KLM}, using (6.28), (6.25) and (4.18), may be expressed in the form

(6.30 $$2E_{KL}=u_{K,L}+u_{L,K}+u_{M,K}u^M_{,L},\qquad E_{KLM}=u_{K,LM}-u_{K,k}u^k_{;LM}.$$

In the linear theory, they are of the form

(6.31) $$E_{KL}=\frac{1}{2}(u_{K,L}+u_{L,K}),\quad E_{KLM}=u_{K;LM}.$$

For isotropic materials, we shall introduce the following spatial deformation tensors

(6.32) $$c_{kl}=G_{KL}X^K_{;k}X^L_{;l},\qquad e_{klm}=g_{rk}x^r_{;LM}X^L_{;l}X^M_{;m}=-g_{rk}x^r_{;K}X^K_{;lm}=e_{kml}.$$

The constitutive equations (6.23) then become

(6.33) $$\tau^{ij}=-2\rho\frac{\partial w}{\partial c_{jk}}c^i_k+\rho\frac{\partial w}{\partial e_{ilm}}e^j_{.lm}-2\rho\frac{\partial w}{\partial e_{kmj}}e^{\cdot\cdot i}_{km},\qquad h^{i(jk)}=\rho\frac{\partial w}{\partial e_{ijk}},$$

and the condition of objectivity

(6.34 $$\left(-2\frac{\partial w}{\partial c_{jk}}c^i_k+\frac{\partial w}{\partial e_{ilm}}e^j_{.lm}-2\frac{\partial w}{\partial e_{kmj}}e^{\cdot\cdot i}_{km}\right)_{[ij]}=0.$$

If we introduce the spatial strain tensor

(6.35) $$2e_{kl}=g_{kl}-c_{kl},$$

the constitutive equations will be

$$(6.36)\qquad \begin{aligned} \tau^{ij} &= \rho\left(\frac{\partial w}{\partial e_{ij}} - 2\frac{\partial w}{\partial e_{jk}} e^{i}_{k} + \frac{\partial w}{\partial e_{ilm}} e^{j}_{\cdot lm} - 2\frac{\partial w}{\partial e_{kmj}} e_{km}^{\cdot\,\cdot\, i}\right),\\ h^{i(jk)} &= \rho\frac{\partial w}{\partial e_{ijk}}, \end{aligned}$$

and the condition of objectivity

$$(6.37)\qquad \left(-2\frac{\partial w}{\partial e_{jk}} e^{i}_{k} + \frac{\partial w}{\partial e_{ilm}} e^{j}_{\cdot lm} - 2\frac{\partial w}{\partial e_{kmj}} e_{km}^{\cdot\,\cdot\, i}\right)_{[ij]} = 0.$$

The deformation tensors e_{kl} and e_{klm}, using (6.35), (6.32) and (4.18), may be expressed in the form

$$(6.38)\qquad \begin{aligned} 2e_{kl} &= u_{k,l} + u_{l,k} - u_{m,k} u^{m}_{,l},\\ e_{klm} &= u_{k,lm} + u_{k;K} u^{K}_{;lm}. \end{aligned}$$

In the linear theory, these tensors are

$$(6.39)\qquad e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \qquad e_{klm} = u_{k,lm}.$$

In the linear theory, disregarding non-linear terms in (6.36), we obtain the constitutive equations

$$(6.40)\qquad \tau^{ij} = \rho\frac{\partial w}{\partial e_{ij}}, \qquad h^{i(kj)} = \rho\frac{\partial w}{\partial e_{ijk}}.$$

These equations, as well as the spatial deformation tensors, may be obtained from the corresponding expressions for micro-elastic materials.

In the equations (6.40) and (6.36), the specific strain energy is a function of the tensors e_{kl} and e_{klm}. If we suppose that there are no initial stresses, in the linear theory it is a quadratic polynomial of the form

$$(6.41)\qquad \rho w = \frac{1}{2} A^{ijkl} e_{ij} e_{kl} + \frac{1}{2} B^{ijklmn} e_{ijk} e_{lmn},$$

where

$$(6.42)\qquad A^{ijkl} = \lambda g^{ij} g^{kl} + \mu(g^{ik} g^{jl} + g^{il} g^{jk}),$$

$$(6.43)\qquad \begin{aligned} B^{ijklmn} = {} & \gamma_1 g^{jk} g^{il} g^{mn} + \gamma_2 (g^{lj} g^{mk} g^{nl} + g^{lk} g^{mj} g^{ni} + g^{lj} g^{nk} g^{mi} + g^{lk} g^{nj} g^{mi}) +\\ & + \gamma_3 (g^{li} g^{jm} g^{kn} + g^{il} g^{jn} g^{km}) + \gamma_4 (g^{in} g^{lm} g^{jk} + g^{im} g^{ln} g^{jk} + g^{lj} g^{ki} g^{mn} +\\ & + g^{kl} g^{ij} g^{mn}) + \gamma_5 (g^{ij} g^{kn} g^{lm} + g^{jn} g^{lm} g^{ki} + g^{ij} g^{km} g^{ln} + g^{jm} g^{ln} g^{ki}), \end{aligned}$$

are isotropic tensors, and $\lambda, \mu, \gamma_1, \gamma_2, \dots, \gamma_5$ material constants.

Substituting now (6.41) into (6.40), and using (6.42) and (6.43), we finally obtain the linear constitutive equations

$$(6.44)\qquad \tau^{ij} = \lambda e_{1} g^{ij} + 2\mu e^{ij},$$

$$h^{i(jk)}=h_1 g^{jk} e^{il}_{..l}+h_2 (e^{jki}+e^{kji})+h_3 e^{ijk}+ \\ +h_4 (2g^{jk} e^{l.i}_{.l}+g^{ki} e^{jl}_{..l}+g^{ji} e^{kl}_{..l})+h_5 (g^{ij} e^{l.k}_{.l}+g^{ik} e^{l.j}_{.l}), \tag{6.45}$$

where

$$\begin{aligned} &\gamma_1=h_1, \quad 2\gamma_2=h_2, \quad 2\gamma_3=h_3, \\ &\gamma_4=h_4, \quad 2\gamma_5=h_5. \end{aligned} \tag{6.46}$$

In the linear theory of dipolar elastic materials, the total number of material constants is 7.

7. Polar elastic materials

A polar elastic material, or so-called elastic material of grade two, may be considered as an elastic Cosserat continuum with constraint rotations. In this case, the rotations of the triads of directors are not independent of displacements of the material points of a body. Therefore, the motion of a body is completely determined by equations

$$x^k=x^k(X^K, t) \tag{7.1}$$

and rotations of the triads of directors are constrained by the relations

$$\dot{d}_{i(\alpha)} d^{(\alpha)}_{.\,j}=v_{[i,j]}=\omega_{ij}. \tag{7.2}$$

Since the motion of directors is a rigid motion, it is clear that the directors $d^k_{.(\alpha)}$ *are not material vectors.*

From (7.2), we get

$$\dot{d}_{i(\alpha)}=\omega_{ij} d^j_{.(\alpha)}. \tag{7.3}$$

Consequently, we obtain

$$\delta d_{i(\alpha)}=\delta x_{[i,j]} d^j_{.(\alpha)}. \tag{7.4}$$

Using (7.4) in equation (3.17), we have

$$\int_v \rho(\dot{v}^i \delta x_i+\Gamma^{ij} \delta x_{[i,j]}) dv+\int_v \rho\delta w\, dv=\int_v \rho(f^i \delta x_i+l^{ij} \delta x_{[i,j]}) dv+ \\ +\int_v [t^{ij}_{\ ,j} \delta x_i+t^{ij} \delta x_{(i,j)}+(t^{[ij]}+m^{ijk}_{\ \ ,k}) \delta x_{[i,j]}+m^{ijk} \delta x_{[i,j]k}] dv. \tag{7.5}$$

For virtual translations, this equation gives

$$\rho\dot{v}^i=t^{ij}_{\ ,j}+\rho f^i, \tag{7.6}$$

and, next for virtual rigid displacements,

$$t^{[ij]}+m^{ijk}_{\ \ ,k}+\rho(l^{ij}-\Gamma^{ij})=0. \tag{7.7}$$

Equations (7.6) and (7.7) are Cauchy's first end second laws of motion.

Substituting now (7.6) and (7.7) into (7.5), we obtain the expression for variation of the specific strain energy,

$$\rho\delta w = t^{ij}\,\delta x_{(i,j)} + m^{ijk}\,\delta x_{[i,j]k}, \tag{7.8}$$

which is form-invariant with respect to the superposed rigid motion.

If we suppose w to be a function of the form

$$w = w(x^l_{;K}, x^l_{;KL}), \tag{7.9}$$

we get

$$\delta w = \frac{\partial w}{\partial x^l_{;K}}\,\delta x^l_{;K} + \frac{\partial w}{\partial x^l_{;KL}}\,\delta x^l_{;KL}. \tag{7.10}$$

According to

$$\delta x^l_{;K} = \delta x^l_{,m}\, x^m_{;K},$$
$$\delta x^l_{;KL} = \delta x^l_{,mn}\, x^m_{;K}\, x^n_{;L} + \delta x^l_{,m}\, x^m_{;KL},$$

we have

$$\delta w = g^{il}\left(\frac{\partial w}{\partial x^l_{;K}}\, x^j_{;K} + \frac{\partial w}{\partial x^l_{;KL}}\, x^j_{;KL}\right)\delta x_{i,j} + g^{il}\frac{\partial w}{\partial x^l_{;KL}}\, x^j_{;K}\, x^k_{;L}\,\delta x_{i,jk}. \tag{7.11}$$

Comparing (7.11) with (7.8), we obtain the constitutive equations

$$\begin{aligned} t^{(ij)} &= \rho g^{il}\left(\frac{\partial w}{\partial x^l_{;K}}\, x^j_{;K} + \frac{\partial w}{\partial x^l_{;KL}}\, x^j_{;KL}\right), \\ m^{i(jk)} &= \rho g^{il}\frac{\partial w}{\partial x^l_{;KL}}\, x^j_{;K}\, x^k_{;L}, \end{aligned} \tag{7.12}$$

and the conditions of objectivity

$$\begin{aligned} &\left(g^{il}\frac{\partial w}{\partial x^l_{;K}}\, x^j_{;K} + g^{il}\frac{\partial w}{\partial x^l_{;KL}}\, x^j_{;KL}\right)_{[ij]} = 0, \\ &\left(g^{il}\frac{\partial w}{\partial x^l_{;KL}}\, x^j_{;K}\, x^k_{;L}\right)_{(ijk)} = 0. \end{aligned} \tag{7.13}$$

Since w is a function of 27 independent variables $x^k_{;K}$ and $x^k_{;KL}$, the system of 13 partial differential equations (7.13) admits $27-13=14$ independent integrals.

Appendix

Equation (3.33) may be expressed in the equivalent form

$$\rho\dot{w} = t^{ij}(v_{i,j} - b_{ij}) + \tau^{ij}\, b_{(ij)} + h^{ijk}\, b_{ij,k}, \tag{1}$$

where

$$b_{ij} = \dot{d}_{i(\alpha)}\, d^{(\alpha)}_{\cdot\; j} = \dot{\chi}_{iK}\,\chi^K_{\cdot\; j}, \tag{2}$$

$$v_{i,j} = \dot{x}_{i;K}\, X^K_{;j}. \tag{3}$$

The tensors $v_{i,j} - b_{ij}$, $b_{(ij)}$ and $b_{ij,k}$ *are objective tensors.*

From (2), we get

$$b_{(ij)}=\frac{1}{2}(\dot{\chi}_{iK}\chi^{K}_{.j}+\dot{\chi}_{jK}\chi^{K}_{.i})=\frac{1}{2}(\dot{\chi}_{kK}\chi^{k}_{.L}\chi^{L}_{.i}\chi^{K}_{.j}+\dot{\chi}_{kK}\chi^{k}_{.L}\chi^{L}_{.j}\chi^{K}_{.i})=$$

$$=\frac{1}{2}(\dot{\chi}_{kK}\chi^{k}_{.L}+\dot{\chi}_{kL}\chi^{k}_{.K})\chi^{K}_{.j}\chi^{L}_{.i}=\frac{1}{2}\overline{\chi_{kK}\chi^{k}_{.L}}^{\bullet}\chi^{K}_{.j}\chi^{L}_{.i},$$

or

$$b_{(ij)}=\frac{1}{2}\dot{\Psi}_{KL}\chi^{K}_{.i}\chi^{L}_{.j}, \tag{4}$$

where

$$\Psi_{KL}=\chi_{kK}\chi^{k}_{.L}. \tag{5}$$

In the similar way, we obtain

$$v_{i,j}-b_{ij}=\dot{\Sigma}_{KL}\chi^{K}_{.i}X^{L}_{;j}, \tag{6}$$

$$b_{ij,k}=\dot{D}_{KLM}\chi^{K}_{.i}\chi^{L}_{.j}X^{M}_{;k}, \tag{7}$$

where

$$\Sigma_{KL}=\chi_{Kk}x^{k}_{;L}, \tag{8}$$

$$D_{KLM}=\chi_{Kk}\chi^{k}_{.L;M}. \tag{9}$$

Substituting now (4), (6) and (7) into (1), we get

$$\rho\dot{w}=t^{ij}\chi^{K}_{.i}X^{L}_{;j}\dot{\Sigma}_{KL}+\frac{1}{2}\tau^{ij}\chi^{K}_{.i}\chi^{L}_{.j}\dot{\Psi}_{KL}+h^{ijk}\chi^{K}_{.i}\chi^{L}_{.j}X^{M}_{;k}\dot{D}_{KLM}, \tag{10}$$

wherefrom we deduce that

$$w=w(\Psi_{KL},\Sigma_{KL},D_{KLM}). \tag{11}$$

It is clear that, as a function of the material tensors Ψ_{KL}, Σ_{KL} and D_{KLM}, the specific strain energy is an objective function. Therefore, the tensors Ψ_{KL}, Σ_{KL} and D_{KLM} represent a minimal integrity basis for the specific strain energy.

From (11), upon differentiating with respect to the time, we get

$$\dot{w}=\frac{\partial w}{\partial\Psi_{KL}}\dot{\Psi}_{KL}+\frac{\partial w}{\partial\Sigma_{KL}}\dot{\Psi}_{KL}+\frac{\partial w}{\partial D_{KLM}}\dot{D}_{KLM}. \tag{12}$$

Comparing (12) with (10), we obtain the constitutive equations (4.11), which are form-invariant with respect to the superposed rigid motion.

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Streszczenie

ZASADA PRACY WIRTUALNEJ A RÓWNANIA KONSTYTUTYWNE W UOGÓLNIONYCH TEORIACH SPRĘŻYSTOŚCI

W pracy rozpatruje się model materiału sprężystego z mikrostrukturą jako uogólniony ośrodek Cosseratów. Sformułowanie i zastosowanie zasady prac wirtualnych pozwala uzyskać równania ruchu i nieliniowe równania konstytutywne, które linearyzują się w przypadku materiałów izotropowych. Przeanalizowano warunki i metody otrzymania teorii mikropolarnej, dipolarnej i polarnej jako przypadków szczególnych.

Резюме

ПРИНЦИП ВИРТУАЛЬНОЙ РАБОТЫ И ОПРЕДЕЛЯЮЩИЕ УРАВНЕНИЯ В ОБОБЩЕННЫХ ТЕОРИЯХ УПРУГОСТИ

В работе рассматривается модель упругого материала с микроструктурой, как обобщенной среды Коссера. Формулировка и применение принципа виртуальных работ позволяет получить уравнения движения и нелинейные определяющие уравнения, которые линеаризуются в случае изотропных сред.

Проанализированы условия и методы получения микрополярной, диполярной и полярной теорий, как частных случаев.