ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 24, 4, 731–744, 1976
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

SOLUTION OF CERTAIN SYSTEMS OF DUAL INTEGRAL EQUATIONS WITH BESSEL KERNELS AND ITS APPLICATION IN THE THEORY OF ELASTICITY

J. GASZYŃSKI (KRAKÓW)

The paper deals with the construction of solutions for the systems of dual integral equations of the form

$$
(1)\qquad
\begin{aligned}
&\int_0^\infty G_{ij}(v)\,\Phi_j(v)\,J_\nu(uv)\,dv\,, && u<1;\\
&\int_0^\infty \Phi_i(v)\,J_\nu(uv)\,dv=0\,, && u>1;\\
&\nu=0,1\,,
\end{aligned}
$$

where the weight functions $G_{ij}(v)$ have the following forms

$$
G_{ij}(v)=v\,[\overset{*}{A}_{ij}+\overset{*}{H}_{ij}(v)]+v^{-1}\,[\overset{**}{A}_{ij}+\overset{**}{H}_{ij}(v)]
$$

or

$$
G_{ij}(v)=v^{2k-1}\,[A_{ij}+H_{ij}(v)]\,,\qquad k=0,1,\ldots\,.
$$

The method of reducing the systems (1) to the system of differential-integral equations of the second kind and second order is presented. The results obtained are illustrated by the solution of the contact problem of coupled thermoelasticity with mixed boundary conditions under the assumption of thermal insulation of the boundary.

1. Introduction

This paper deals with a solution of the systems of dual integral equations with Bessel kernels of zero and first order and first kind. A significant number of the mixed discontinuous boundary-value problems of mechanics of continuous media, when expressed in terms of cylindrical system of coordinates, may be reduced to the system of equations mentioned above.

The following systems of integral equations are the subject of considerations:

$$
(1.1)\qquad
\begin{aligned}
&\int_0^\infty G_{ij}(v)\,\Phi_j(v)\,J_0(uv)\,dv=f_i(u)\,, && u<1\,,\\
&\int_0^\infty \Phi_i(v)\,J_0(uv)\,dv=0\,, && u>1\,,
\end{aligned}
$$

and

$$(1.2)\qquad \begin{aligned} \int_0^\infty G_{ij}(v)\,\Phi_j(v)\,J_1(uv)\,dv &= f_i(u), \qquad u<1,\\ \int_0^\infty \Phi_i(v)\,J_1(uv)\,dv &= 0, \qquad u>1. \end{aligned}$$

The methods of solution of the systems (1.1) and (1.2) depend on the form and the properties of the weight functions $G_{ij}(v)$. G. Szefer [6] outlines a method of reducing the systems (1.1) and (1.2) with the Bessel functions of the arbitrary order to the Fredholm integral equations of the second kind, assuming the matrix of the weight functions in the form:

$$(1.3)\qquad G_{ij}(v)=\delta_{ij}+H_{ij}(v),$$

where δ_{ij} denotes the Kronecker symbol and $H_{ij}(v)$ is the matrix of the weight functions ensuring the existence and convergence of the integral:

$$(1.4)\qquad \int_0^\infty H_{ij}(v)\,[\sin uv, \cos uv]\,dv.$$

The same author presents in [7] the solutions of the systems of dual integral equations with the Bessel kernels of the order zero and with the matrix of the weight functions of the following form

$$(1.5)\qquad G_{ij}(v)=v\,[\delta_{ij}+H_{ij}(v)]$$

reducing, as previously, the problem to the system of the Fredholm integral equations of the second kind.

In the paper [5] one can find the approximate methods of reducing the systems of dual integral equations with the Bessel kernels of arbitrary order to the systems of algebraic equations.

The aim of this paper is to give the solution of the systems (1.1) and (1.2) in which the matrix of the weight functions is given by

$$(1.6)\qquad G_{ij}(v)=v\,[\overset{*}{A}_{ij}+\overset{*}{H}_{ij}(v)]+v^{-1}\,[\overset{**}{A}_{ij}+\overset{**}{H}_{ij}(v)]$$

and

$$(1.7)\qquad G_{ij}(v)=v^{2k-1}\,[A_{ij}+H_{ij}(v)],$$

where A denotes the matrix of the constants and H_{ij} is the matrix of the known functions possessing properties (1.4). The matrix of the weight functions in the form (1.6) occurs in the consideration of a certain group of the mixed boundary-value problems of thermoelasticity theory and theory of consolidation — described by a coupled system of partial differential equations of elliptic-parabolic type. Besides, the representation (1.6) permits to treat the boundary-value problems of punches and cracks as a joint problem with the particular forms of the matrices $G_{ij}(v)$ given by Eq. (1.6).

The results obtained in this paper are illustrated by the example of the contact problem for a thermoelastic halfspace with mixed conditions for thermal isolation of the boundary.

2. Solution of the Eqs. (1.1) and (1.2) with the condition (1.7)

We seek the solution of the Eq. (1.1) in the form

$$\Phi_i(v)=v\int_0^1 \varphi_i(\xi)\cos v\xi\, d\xi \tag{2.1}$$

with boundary conditions

$$\varphi_i^{(2s)}(1)=\varphi_i^{(2s+1)}(0)=0, \qquad s=0,1,\ldots,k-1 \tag{2.2}$$

and the solution of the Eq. (1.2) in a form

$$\Phi_i(v)=v\int_0^1 \psi_i(\xi)\sin v\xi\, d\xi \tag{2.3}$$

with boundary conditions

$$\psi_i^{(2s)}(0)=\psi_i^{(2s)}(1)=0; \qquad s=0,1,\ldots,k-1. \tag{2.4}$$

When $k=0$, the boundary conditions are superflous. Substitute now the Eq. (2.1) into Eq. $(1.1)_2$ adn transform the result obtained in the following way

$$\int_0^\infty v\int_0^1 \varphi_i(\xi)\cos v\xi\, d\xi J_0(uv)\, dv=\int_0^\infty \left[\sin v\xi\varphi_i(\xi)|_0^1-\int_0^1 \varphi_i'(\xi)\sin v\xi\, d\xi\right]\times$$
$$\times J_0(uv)\, dv=\varphi_i(1)\int_0^\infty \sin vJ_0(uv)-\int_0^1 \varphi_i'(\xi)\int_0^\infty \sin v\xi J_0(uv)\, dv\, d\xi.$$

Inserting the Eq. (2.3) into Eq. (1.2) one obtains

$$\int_0^\infty v\int_0^1 \psi_i(\xi)\sin v\xi\, d\xi J_1(uv)\, dv=-\frac{d}{du}\int_0^1 \psi_i(\xi)\int_0^\infty \sin v\xi J_0(uv)\, dv\, d\xi.$$

Using the following property of the Weber-Schafheitlin integral [3]

$$\int_0^\infty \sin v\xi J_0(uv)\, dv=\begin{cases} 0 & \text{for} \quad \xi<u, \\ \dfrac{1}{\sqrt{\xi^2-u^2}} & \text{for} \quad \xi>u, \end{cases} \tag{2.5}$$

one can see that the Eqs. $(1.1)_2$ and $(1.2)_2$ are satisfied identically for $\Phi_i(v)$ given by the Eqs. (2.1) and (2.3), respectively. After substitution of the Eq. (2.1) to the Eq. $(1.1)_1$ and the Eq. (2.3) to the Eq. $(1.2)_1$ one obtains

$$\begin{aligned} &\int_0^\infty [A_{ij}+H_{ij}(v)]v^{2k}\int_0^1 \varphi_i(\xi)\cos v\xi\, d\xi J_0(uv)\, dv=f_i(u), \\ &\int_0^\infty [A_{ij}+H_{ij}(v)]v^{2k}\int_0^1 \psi_i(\xi)\sin v\xi\, d\xi J_1(uv)\, dv=f_i(u). \end{aligned} \tag{2.6}$$

Integrating (2.6) by parts we find

$$\begin{aligned}\int_0^1 \varphi_j(\xi)\cos v\xi d\xi &= \frac{(-1)^k}{v^{2k}}\int_0^1 \varphi_j^{(2k)}(\xi)\cos v\xi\, d\xi,\\ \int_0^1 \psi_j(\xi)\sin v\xi d\xi &= \frac{(-1)^k}{v^{2k}}\int_0^1 \psi_j^{(2k)}(\xi)\sin v\xi\, d\xi.\end{aligned}\tag{2.7}$$

In the above relations the boundary conditions (2.2) and (2.4) have been taken into account. Introducing the Eqs. (2.7) into Eqs. (2.6) we get

$$\begin{aligned}\int_0^\infty [A_{ij}+H_{ij}(v)]\int_0^1 \varphi_j^{(2k)}(\xi)\cos v\xi\, d\xi J_0(uv)\,dv &= (-1)^k f_i(u),\\ \int_0^\infty [A_{ij}+H_{ij}(v)]\int_0^1 \psi_j^{(2k)}(\xi)\sin v\xi\, d\xi J_1(uv)\,dv &= (-1)^k f_i(u).\end{aligned}\tag{2.8}$$

The change of order of integration gives

$$A_{ij}\int_0^1 \varphi_j^{(2k)}(\xi)\int_0^\infty \cos v\xi J_0(uv)\,dv\,d\xi + \int_0^1\int_0^\infty H_{ij}(v)\cos v\xi J_0(uv)\,dv\,\varphi_j^{(2k)}(\xi)\,d\xi = (-1)^k f_i(u),$$

$$A_{ij}\int_0^1 \psi_j^{(2k)}(\xi)\int_0^\infty \sin v\xi J_1(uv)\,dv\,d\xi + \int_0^1\int_0^\infty H_{ij}(v)\sin v\xi J_1(uv)\,dv\psi_j^{(2k)}(\xi)\,d\xi = (-1)^k f_i(u).$$

Computing in the above equations the first integrals by means of the following Weber-Schaftheitlin integrals [3]

$$\begin{aligned}\int_0^\infty \sin v\xi J_1(uv)\,dv &= \begin{cases} 0 & \text{for} \quad \xi>u \\ \dfrac{\xi}{u\sqrt{u^2-\xi^2}}, & \text{for} \quad \xi<u \end{cases}\\ \int_0^\infty \cos v\xi J_0(uv)\,dv &= \begin{cases} 0 & \text{for} \quad \xi>u, \\ \dfrac{1}{\sqrt{u^2-\xi^2}} & \text{for} \quad \xi<u, \end{cases}\end{aligned}\tag{2.9}$$

we obtain

$$\begin{aligned}A_{ij}\int_0^u \frac{\varphi_j^{(2k)}(\xi)}{\sqrt{u^2-\xi^2}}\,d\xi + \int_0^1\int_0^\infty H_{ij}(v)\cos v\xi J_0(uv)\,dv\varphi_j^{(2k)}(\xi)\,d\xi &= (-1)^k f_i(u),\\ A_{ij}\int_0^u \frac{\xi\psi_j^{(2k)}(\xi)}{u\sqrt{u^2-\xi^2}}\,d\xi + \int_0^1\int_0^\infty H_{ij}(v)\sin v\xi J_1(uv)\,dv\psi_j^{(2k)}(\xi)\,d\xi &= (-1)^k f_i(u).\end{aligned}\tag{2.10}$$

After substitution of $\xi = u\sin\theta$ to the first integrals of the Eq. (2.10), and the integral representation of the Bessel functions

$$J_0(uv) = \frac{2}{\pi}\int_0^{\pi/2} \cos(uv\sin\theta)\,d\theta, \qquad J_1(uv) = \frac{2}{\pi}\int_0^{\pi/2} \sin(uv\sin\theta)\sin\theta\,d\theta,$$

to the second integrals of these equations, after some transformations one gets

$$\int_0^{\pi/2}\left[A_{ij}\varphi_j^{(2k)}(u\sin\theta) + \frac{2}{\pi}\int_0^1\int_0^\infty H_{ij}(v)\cos v\xi\cos(uv\sin\theta)\,dv\,\varphi_j^{(2k)}(\xi)\,d\xi\right]d\theta = (-1)^k f_i(u),$$

(2.11)

$$\int_0^{\pi/2}\left[A_{ij}\psi_j^{(2k)}(u\sin\theta) + \frac{\pi}{2}\int_0^1\int_0^\infty H_{ij}(v)\sin v\xi\sin(uv\sin\theta)\,dv\,\psi_j^{(2k)}(\xi)\,d\xi\right]\times \sin\theta\,d\theta = (-1)^k f_i(u).$$

Let us write the Eqs. (2.11) in a general form

$$\int_0^{\pi/2} F_i(u\sin\theta)\,d\theta = (-1)^k f_i(u), \qquad i = 1, 2, \ldots, n. \tag{2.12}$$

The equation (2.12) constitutes a system of Schlömilch equations the solution of which is

$$F_i(u) = \frac{2}{\pi}\left[(-1)^k f_i(0) + (-1)^k u\int_0^{\pi/2} f_i(u\sin\theta)\,d\theta\right], \tag{2.13}$$

where the functions $F_i(u\sin\theta)$ are determined in a following way

$$F_i(u\sin\theta) = \begin{cases} A_{ij}\varphi_j^{(2k)}(u\sin\theta) + \dfrac{2}{\pi}\displaystyle\int_0^1\int_0^\infty H_{ij}(v)\cos v\xi\cos(uv\sin\theta)\,dv\times \varphi_j^{(2k)}(\xi)\,d\xi & \text{for } (2.11)_1, \\ \left[A_{ij}\psi_j^{(2k)}(u\sin\theta) + \dfrac{2}{\pi}\displaystyle\int_0^1\int_0^\infty H_{ij}(v)\sin v\xi\sin(uv\sin\theta)\,dv\times \psi_j^{(2k)}(\xi)\,d\xi\right]\sin\theta & \text{for } (2.11)_2. \end{cases} \tag{2.14}$$

Using the above results, the following system of equations for determination of the functions $\varphi_j^{(2k)}(u)$ and $\psi_j^{(2k)}(u)$ is obtained

(2.15)

$$A_{ij}\varphi_j^{(2k)}(u) + \frac{2}{\pi}\int_0^1\int_0^\infty H_{ij}(v)\cos v\xi\cos vu\,dv\,\varphi_j^{(2k)}(\xi)\,d\xi = F_i(u),$$

$$A_{ij}\psi_j^{(2k)}(u) + \frac{2}{\pi}\int_0^1\int_0^\infty H_{ij}(v)\sin v\xi\sin vu\,dv\,\psi_j^{(2k)}(\xi)\,d\xi = F_i(u).$$

Thus the systems of dual integral equations (1.1) and (1.2) have been reduced to the systems of Fredholm integral equations of the second kind $(2.15)_1$ and $(2.15)_2$, respectively.

Introducing the notations

$$\varphi_j^{(2k)}(u)=\bar{\varphi}_j(u), \qquad \psi_j^{(2k)}(u)=\bar{\psi}_j(u), \tag{2.16}$$

$$K_{ij}(u,\xi)=\begin{cases} \int\limits_0^\infty H_{ij}(v)\left[\cos v(u+\xi)+\cos v(u-\xi)\right]dv & \text{for} \quad (2.15)_1, \\ \int\limits_0^\infty H_{ij}(v)\left[\cos v(\xi-u)-\cos v(\xi+u)\right]dv & \text{for} \quad (2.15)_2, \end{cases} \tag{2.17}$$

into Eqs. (2.15) we obtain finally

$$\begin{aligned} &A_{ij}\bar{\varphi}_j(u)+\frac{1}{\pi}\int\limits_0^1 K_{ij}(u,\xi)\bar{\varphi}_j(\xi)\,d\xi=F_i(u), \\ &A_{ij}\bar{\psi}_j(u)+\frac{1}{\pi}\int\limits_0^1 K_{ij}(u,\xi)\bar{\psi}_j(\xi)\,d\xi=F_i(u). \end{aligned} \qquad i=1,2,\ldots,n, \tag{2.18}$$

The analysis and solution of the Eqs. (2.18) allows for evaluation of the solution for dual equations (1.1) and (1.2).

3. Solution of the Eqs. (1.1) and (1.2) with the condition (1.6)

As previously, we seek the solutions for the Eqs. (1.1) in the form of Eq. (2.1) with boundary conditions (2.2) and for the Eqs. (1.2) in the form of Eq. (2.3) with boundary conditions (2.4).

As it results from the considerations of section 2, the equations $(1.1)_2$ and $(1.2)_2$ are satisfied identically.

Substituting the Eq. (2.1) in Eq. $(1.1)_1$ and the Eq. (2.3) in Eq. $(1.2)_1$ and taking into account the condition (1.6) we obtain

$$\begin{aligned} &\int\limits_0^\infty [\overset{*}{A}_{ij}+\overset{*}{H}_{ij}(v)]v^2\int\limits_0^1 \varphi_j(\xi)\cos v\xi J_0(uv)\,dv+ \\ &\qquad +\int\limits_0^\infty [\overset{**}{A}_{ij}+\overset{**}{H}_{ij}(v)]\int\limits_0^1 \varphi_j(\xi)\cos v\xi\,d\xi J_0(uv)\,dv=f_i(u), \\ &\int\limits_0^\infty [\overset{*}{A}_{ij}+\overset{*}{H}_{ij}(v)]v^2\int\limits_0^1 \psi_j(\xi)\sin v\xi\,d\xi J_1(uv)\,dv+ \\ &\qquad +\int\limits_0^\infty [\overset{**}{A}_{ij}+\overset{**}{H}_{ij}(v)]\psi_j(\xi)\sin v\xi\,d\xi J_1(uv)\,dv=f_i(u). \end{aligned} \tag{3.1}$$

From the relations (2.7) we have

$$\begin{aligned} &\int\limits_0^1 \varphi_j(\xi)\cos v\xi\,d\xi=-v^{-2}\int\limits_0^1 \varphi_j''(\xi)\cos v\xi d\xi, \qquad \varphi_j(1)=\varphi_j'(0)=0, \\ &\int\limits_0^1 \psi_j(\xi)\sin v\xi\,d\xi=-v^{-2}\int\limits_0^1 \psi_j''(\xi)\sin v\xi d\xi, \qquad \psi_j(0)=\psi_j(1)=0. \end{aligned} \tag{3.2}$$

Inserting Eqs. (3.2) into Eqs. (3.1), after the simple transformations, we obtain

(3.3)
$$-\int_0^\infty [\overset{*}{A}_{ij}+\overset{*}{H}_{ij}(v)]\int_0^1 \varphi_j''(\xi)\cos v\xi J_0(uv)\,dv+ +\int_0^\infty [\overset{**}{A}_{ij}+\overset{**}{H}_{ij}(v)]\int_0^1 \varphi_j(\xi)\cos v\xi J_0(uv)\,d\xi dv=f_i(u),$$
$$-\int_0^\infty [\overset{*}{A}_{ij}+\overset{*}{H}_{ij}(v)]\int_0^1 \psi_j''(\xi)\sin v\xi J_1(uv)\,d\xi dv+ +\int_0^\infty [\overset{**}{A}_{ij}+\overset{**}{H}_{ij}(v)]\int_0^1 \psi_j(\xi)\sin v\xi J_1(uv)\,d\xi dv=f_i(u).$$

Changing in the above formulae the order of integration and evaluating the integrals by means of the Eq. (2.9) we have

(3.4)
$$-\overset{*}{A}_{ij}\int_0^u \frac{\varphi_j''(\xi)}{\sqrt{u^2-\xi^2}}\,d\xi-\int_0^1\int_0^\infty \overset{*}{H}_{ij}(v)\cos v\xi J_0(uv)\,dv\varphi_j''(\xi)\,d\xi+ +\overset{**}{A}_{ij}\int_0^u \frac{\varphi_j(\xi)}{\sqrt{u^2-\xi^2}}\,d\xi+\int_0^1\int_0^\infty \overset{**}{H}_{ij}(v)\cos v\xi J_0(uv)\,dv\varphi_j(\xi)\,d\xi=f_i(u),$$
$$-\overset{*}{A}_{ij}\int_0^u \frac{\psi_j''(\xi)}{\sqrt{u^2-\xi^2}}\,d\xi-\int_0^1\int_0^\infty \overset{*}{H}_{ij}(v)\sin v\xi J_1(uv)\,dv\psi_j''(\xi)\,d\xi+ +\overset{**}{A}_{ij}\int_0^u \frac{\psi_j(\xi)}{\sqrt{u^2-\xi^2}}\,d\xi+\int_0^1\int_0^\infty \overset{**}{H}_{ij}(\sigma)\sin v\xi J_1(uv)\,dv\psi_j(\xi)\,d\xi=f_i(u).$$

Reducing (as in section 2) the system of equations (3.4) to the systems of the Schlömilch equations and using of the solutions (2.13), the following system of equations for determining the functions $\varphi_i(u)$ and $\psi_i(u)$ is obtained

(3.5)
$$-\overset{*}{A}_{ij}\varphi_j''(u)-\frac{1}{\pi}\int_0^1 \overset{*}{K}_{ij}(u,\xi)\,\varphi_j''(\xi)\,d\xi+ +\overset{**}{A}_{ij}\varphi_j(u)+\frac{1}{\pi}\int_0^1 \overset{**}{K}_{ij}(u,\xi)\,\varphi_j(\xi)\,d\xi=F_i(u),$$
$$-\overset{*}{A}_{ij}\psi_j''(u)-\frac{1}{\pi}\int_0^1 \overset{*}{K}_{ij}(u,\xi)\,\psi_j''(\xi)\,d\xi+ +\overset{**}{A}_{ij}\psi_j(u)+\frac{1}{\pi}\int_0^1 \overset{**}{K}_{ij}(u,\xi)\,\psi_j(\xi)\,d\xi=F_i(u),$$

where the following notations were used

(3.6)
$$F_i(u)=\frac{2}{\pi}\left[f_i(0)+u\int_0^{\pi/2} f_i'(u\sin\theta)\,d\theta\right].$$

$$\overset{*}{K}_{ij}(u,\xi)=\begin{cases}\int\limits_0^\infty \overset{*}{H}_{ij}(v)\left[\cos v(u+\xi)+\cos v(u-\xi)\right]dv & \text{for} \quad (3.5)_1,\\ \int\limits_0^\infty \overset{*}{H}_{ij}(v)\left[\cos v(u+\xi)-\cos v(u-\xi)\right]dv & \text{for} \quad (3.5)_2,\end{cases} \tag{3.6 [cont.]}$$

$$\overset{**}{K}_{ij}(u,\xi)=\begin{cases}\int\limits_0^\infty \overset{**}{H}_{ij}(v)\left[\cos v(u+\xi)+\cos v(u-\xi)\right]dv & \text{for} \quad (3.5)_1,\\ \int\limits_0^\infty \overset{**}{H}_{ij}(v)\left[\cos v(u+\xi)-\cos v(u-\xi)\right]dv & \text{for} \quad (3.5)_2.\end{cases}$$

Thus the systems of dual integral equations (1.1) and (1.2) with the general form of the weight function (1.6) have been reduced to the system of the differential equations of the second kind.

4. Example

Consider the quasistatic axially-symmetric problem of a punch pressed into thermoelastic halfspace. Assume the punch to be plane and smooth (the shear stresses are then neglected). The contact region is thermally insulated. On the remaining parts of the boundary the temperature is prescribed. The state of stress and strain is determined by means of the equations of coupled thermoelasticity. In cylindrical system of coordinates r, φ, z the displacement equations (in which the body forces are neglected) and the equation of heat conduction assume the form [4]

$$\begin{gathered}\mu\left(\nabla^2 u-\frac{u}{r^2}\right)+(\mu+\lambda)\varepsilon_{,r}=\gamma\theta_{,r},\\ \mu\nabla^2 w+(\mu+\lambda)\varepsilon_{,z}=\gamma\theta_{,z},\\ \nabla^2\theta-\frac{1}{k}\dot{\theta}-\eta\dot{\varepsilon}=0,\end{gathered} \tag{4.1}$$

where u and w are the components of the displacement vector, θ is the temperature, $\nabla=\frac{\partial}{\partial r}+\frac{1}{r}+\frac{\partial}{\partial z}$, $(\dot{\ })=\frac{\partial}{\partial t}$: λ and μ are Lamé constants, $\gamma=(3\lambda+2\mu)\alpha_t$, α_t is a coefficient of linear thermal expansion, $\kappa=\frac{\lambda_0}{c_\varepsilon}$, $\eta=\frac{T_0}{\lambda_0}\gamma$ and c_ε, λ_0 and T_0 denote the specific heat, coefficient of heat conduction and temperature in the natural state, respectively. Besides we have the geometrical relations

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \tag{4.2}$$

and the Duhamel-Neumann equations

$$\sigma_{ij}=2\mu\varepsilon_{ij}+(\lambda\varepsilon-\gamma\theta)\delta_{ij}, \tag{4.3}$$

which combine the states of stress and strain.

The boundary conditions for the problem considered are as follows:

$$
(4.4)\qquad \begin{aligned}
&w(r,0,t)=c(t) && \text{for}\quad r<R,\\
&\sigma_z(r,0,t)=0 && \text{for}\quad r>R,\\
&\sigma_{rz}(r,0,t)=0 && \text{for}\quad r>0,\qquad t>0,\\
&\theta_{,z}(r,z,t)_{z=0}=0 && \text{for}\quad r<R,\\
&\theta(r,0,t)=\theta_0(r,t)=0 && \text{for}\quad r>R;
\end{aligned}
$$

$$
(4.5)\qquad u(r,z,0)=w(r,z,0)=\theta(r,z,0)=0.
$$

Without any loss of generality we can assume that $\theta_0(r,t)=0$ for $r>R$. Using the Hankel transforms with respect to the variable r and the Laplace transforms with respect to the variable t, the Laplace transforms for the functions sought for are obtained [4]

$$
\bar{u}(r,z,s)=\int_0^\infty \left[C_1\frac{\mu+\lambda}{2\mu}\left(\frac{1}{k\eta}+\frac{\gamma}{\mu+\lambda}\right)ze^{-\omega z}-C_2\frac{\gamma\omega}{s\beta(2\mu+\lambda)}e^{-mz}+\right.
$$
$$
\left.+C_3e^{-\omega z}\right]\omega J_1(\omega r)\,d\omega,
$$

$$
(4.6)\qquad \bar{w}(r,z,s)=\int_0^\infty\left\{C_1\left[\left(\frac{\mu+\lambda}{k\eta}+\gamma\right)\frac{z}{2\mu}+\left(\frac{3\mu+\lambda}{k\eta}+\gamma\right)\frac{1}{2\mu\omega}\right]e^{-\omega z}-\right.
$$
$$
\left.-C_2\frac{\gamma_m}{s\beta(2\mu+\lambda)}e^{-mz}+C_3e^{-\omega z}\right\}\omega J_0(\omega r)\,d\omega,
$$

$$
\bar{\theta}(r,z,s)=\int_0^\infty [C_1e^{-\omega z}+C_2e^{-mz}]\omega J_0(\omega r)\,d\omega,
$$

$$
C_i=C_i(\omega,s),\qquad i=1,2,3,
$$

where

$$
(\bar{u},\bar{w},\bar{\theta})=\int_0^\infty (u,w,\theta)e^{-st}\,dt,
$$

$$
m^2=\omega^2+s\beta,
$$

$$
\beta=\frac{1}{k}+\frac{\gamma\eta}{2\mu+\lambda}.
$$

The mixed boundary conditions (4.4) lead to the following system of dual integral equations

$$
(4.7)\qquad \begin{aligned}
&\int_0^\infty C_1J_0(\omega r)\,d\omega=\frac{2\mu}{2\mu+\lambda}\left(\frac{1}{k\eta}+\frac{\gamma}{2\mu+\lambda}\right)^{-1}\bar{c}(s) && \text{for}\quad r<R,\\
&\int_0^\infty\left[C_1\left(\frac{\mu+\lambda}{k\eta}+\gamma\right)+C_2\frac{2\gamma\mu\omega(m-\omega)}{s\beta(2\mu+\lambda)}\right]\omega J_0(\omega r)\,d\omega=0 && \text{for}\quad r>R,
\end{aligned}
$$

(4.7) [cont.]

$$\int_0^\infty (\omega C_1 + mC_2)\,\omega J_0(\omega r)\,d\omega = 0 \qquad \text{for} \quad r<R,$$

$$\int_0^\infty (C_1 + C_2)\,\omega J_0(\omega r)\,d\omega = 0 \qquad \text{for} \quad r>R.$$

Inserting in the above equations the following relations

$$r = uR, \qquad \omega = vR^{-1},$$

(4.8)

$$\Phi_1(v,s) = C_1 v\left(\frac{\mu+\lambda}{k\eta} + \gamma\right) + C_2 v\,\frac{2\gamma\mu v(m-v)}{s\beta(2\mu+\lambda)R^2},$$

$$\Phi_2(v,s) = C_1 v + C_2 v, \qquad C_i = C_i(v,s),$$

we get

(4.9)

$$\int_0^\infty \left[\frac{\Phi_1(v,s)}{vM^*(v,s)} - \frac{2\gamma\mu v(m-v)}{s\beta(2\mu+\lambda)R^2}\,\frac{\Phi_2(v,s)}{vM^*(v,s)}\right] J_0(uv)\,dv = \bar{f}(s) \qquad \text{for} \quad u<1,$$

$$\int_0^\infty \Phi_1(v,s) J_0(uv)\,dv = 0 \qquad \text{for} \quad u>1,$$

$$\int_0^\infty \left[v(m-v)\frac{\Phi_1(v,s)}{vM^*(v,s)} - \left(\frac{\mu+\lambda}{k\eta} + \gamma\right) v(m-v)\frac{\Phi_2(v,s)}{vM^*(v,s)} - v\Phi_2(v,s)\right] J_0(u,v)\,dv = 0 \qquad \text{for} \quad u<1,$$

$$\int_0^\infty \Phi_2(v,s) J_0(uv)\,dv = 0 \qquad \text{for} \quad u>1,$$

where

(4.10)

$$M^*(v,s) = \frac{\mu+\lambda}{k\eta} + \gamma - \frac{2\gamma\mu v(m-v)}{s\beta(2\mu+\lambda)R^2},$$

$$\bar{f}(s) = \frac{2\mu R}{2\mu+\lambda}\left(\frac{1}{k\eta} + \frac{\gamma}{2\mu+\lambda}\right)^{-1} \bar{c}(s),$$

$$\bar{c}(s) = \int_0^\infty c(t)e^{-st}\,dt.$$

After simple algebraic transformations the system of equations (4.9) may be written in the form of the Eq. (1.1) with the condition (1.6)

(4.11)

$$\int_0^\infty \{[\overset{**}{A}_{11} + \overset{**}{H}_{11}(v,s)]v^{-1}\Phi_1(v,s) + [\overset{**}{A}_{12} + \overset{**}{H}_{12}(v,s)]v^{-1}\Phi_2(v,s)\}\times \times J_0(u,v) = \bar{f}(s) \qquad \text{for} \quad u<1,$$

$$\int_0^\infty \Phi_1(v,s) J_0(u,v)\,dv = 0 \qquad \text{for} \quad u>1,$$

(4.11) [cont.]

$$\int_0^\infty \{[\overset{**}{A}_{21}+\overset{**}{H}_{21}(v,s)]v^{-1}\Phi_1(v,s)+[\overset{**}{A}_{22}+\overset{**}{H}_{22}(v,s)]v^{-1}\Phi_2(v,s)-$$
$$-v\Phi_2(v,s)\}J_0(u,v)\,dv=0 \quad \text{for} \quad u<1,$$
$$\int_0^\infty \Phi_2(v,s)J_0(uv)\,dv=0 \quad \text{for} \quad u>1,$$

where

(4.12)
$$\overset{**}{A}_{11}=(\mu+\lambda)^{-1}\left(\frac{1}{k\eta}+\frac{\gamma}{2\mu+\lambda}\right)^{-1},$$
$$\overset{**}{A}_{12}=-\frac{\gamma\mu}{2\mu+1}\overset{**}{A}_{11},$$
$$\overset{**}{A}_{21}=\frac{1}{2}\overset{**}{A}_{11},$$
$$\overset{**}{A}_{22}=-\frac{1}{2}\left(\frac{\mu+\lambda}{k\eta}+\gamma\right)\overset{**}{A}_{11},$$
$$\overset{*}{A}_{22}=-1,$$

$$\overset{**}{H}_{11}(v,s)=-\overset{**}{A}_{11}\frac{sR^2\dfrac{\beta\gamma\mu}{2\mu+\lambda}}{(m+v)^2M^*(v,s)},$$
$$\overset{**}{H}_{12}(v,s)=-\frac{\gamma\mu}{2\mu+\lambda}\left[\overset{**}{H}_{11}(v,s)+\frac{s\beta R^2}{(m+v)^2}\overset{**}{A}_{11}+\frac{s\beta R^2}{(m+v)^2}\overset{**}{H}_{11}(v,s)\right],$$
$$\overset{**}{H}_{21}(v,s)=-\frac{2\mu+\lambda}{2\gamma\mu}\overset{**}{H}_{12}(v,s),$$
$$\overset{**}{H}_{22}(v,s)=\left(\frac{\mu+\lambda}{k\eta}+\gamma\right)\frac{2\mu+\lambda}{2\gamma\mu}\overset{**}{H}_{12}(v,s).$$

Taking into account the results of section 3, the solutions of the system of equations (4.11) assume the form

(4.13)
$$\Phi_1(v,s)=v\int_0^1 \varphi_1(\xi,s)\cos v\xi\,d\xi,$$
$$\Phi_2(v,s)=v\int_0^1 \varphi_2(\xi,s)\cos v\xi\,d\xi, \qquad \varphi_2(1,s)=\varphi_2'(0,s)=0.$$

The Eqs. $(4.11)_2$ and $(4.11)_4$ are satisfied identically by the Eqs. (4.13). Introducing the Eqs. (4.13) into Eqs. $(4.11)_1$ and $(4.11)_3$ and using the results of section 3 we obtain:

(4.14)
$$\overset{**}{A}_{11}\int_0^u \frac{\varphi_1(\xi.s)}{\sqrt{u^2-\xi^2}}d\xi+\int_0^1 \varphi_1(\xi,s)\int_0^\infty \overset{**}{H}_1(v,s)\cos v\xi J_0(uv)\,dv\,d\xi+$$
$$+\overset{**}{A}_{12}\int_0^u \frac{\varphi_2(\xi,s)}{\sqrt{u^2-\xi^2}}d\xi+\int_0^1 \varphi_2(\xi,s)\int_0^\infty \overset{**}{H}_{12}(v,s)\cos v\xi J_0(uv)\,dv d\xi=\bar{f}(s),$$

$$\text{(4.14) [cont.]}\quad \overset{**}{A}_{21}\int_0^u \frac{\varphi_1(\xi,s)}{\sqrt{u^2-\xi^2}}d\xi+\int_0^1 \varphi_1(\xi,s)\int_0^\infty \overset{**}{H}_{21}(v,s)\cos v\xi J_0(uv)\,dv\,d\xi+$$

$$+\overset{**}{A}_{22}\int_0^u \frac{\varphi_2(\xi,s)}{\sqrt{u^2-\xi^2}}d\xi+\int_4^1 \varphi_2(\xi,s)\int_0^\infty \overset{**}{H}_{22}(v,s)\cos v\xi J_0(u,v)\,dv\,d\xi-$$

$$-\int_0^u \frac{\varphi_2''(\xi,s)}{\sqrt{u^2-\xi^2}}d\xi=0\,.$$

After similar transformations as those performed in the section 3 we get

$$\text{(4.15)}\quad \begin{aligned}&\overset{**}{A}_{11}\varphi_1(u,s)+\frac{1}{\pi}\int_0^1 \overset{**}{K}_{11}(u,\xi,s)\varphi_1(\xi,s)\,d\xi+\overset{**}{A}_{12}\varphi_2(u,s)+\\&\qquad+\frac{1}{\pi}\int_0^1 \overset{**}{K}_{12}(u,\xi,s)\varphi_2(\xi,s)\,d\xi=0,\\&\overset{**}{A}_{21}\varphi_1(u,s)+\frac{1}{\pi}\int_0^1 \overset{**}{K}_{21}(u,\xi,s)\varphi_1(\xi,s)\,d\xi+\overset{**}{A}_{22}\varphi_2(u,s)+\\&\qquad+\frac{1}{\pi}\int_0^1 \overset{**}{K}_{22}(u,\xi,s)\varphi_2(\xi,s)\,d\xi-\varphi_2''(\xi,s)=0.\end{aligned}$$

where $\overset{**}{K}_{ij}(u,\xi,s)$ are given by the formulae (3.6).

Equations (4.15), as it is easily seen from the relations (4.12), constitute the system of differential-integral equations of the second kind with the continuous and bounded kernels. In the papers [1] and [2] a complete analysis and construction of the solution of the similar integral equations derived for the contact problems of the consolidation theory is presented. Therefore we omit in this paper the detailed discussion and treat the problem considered as the illustration of the method of solution presented in the section 3. Having obtained solution of the Eqs. (4.15) we are able to evaluate the all states of stress and strain interesting for us in the thermoelastic halfspace. In particular, with the help of functions $\varphi_1(u,s)$ and $\varphi_2(u,s)$ it is easy to determine the distribution of the contact stresses and the distribution of temperature under the punches since the Laplace transforms of these functions are

$$\text{(4.16)}\quad \bar{q}(u,s)=\underset{u<1}{\bar{\sigma}_z}(u,s)=\frac{1}{R^2}\int_0^\infty \Phi_1(v,s)\,J_0(uv)\,dv=$$

$$=\frac{1}{R^2}\int_0^\infty v\int_0^1 \varphi_1(\xi,s)\cos v\xi\,d\xi\,J_0(uv)\,dv=$$

$$=\frac{\varphi_1(1,s)}{R^2\sqrt{1-u^2}}-\frac{1}{R^2}\int_u^1 \frac{\varphi_1'(\xi,s)}{\sqrt{\xi^2-u^2}}d\xi\,,$$

$$(4.17)\qquad \underset{u<1}{\bar{\theta}}(u,s)=\frac{1}{R^2}\int_0^\infty \Phi_2(v,s)J_0(u,v)\,dv=$$

$$=\frac{\varphi_2(1,s)}{R^2\sqrt{1-u^2}}-\frac{1}{R^2}\int_u^1\frac{\varphi_2'(\xi,s)}{\sqrt{\xi^2+u^2}}\,d\xi=-\frac{1}{R^2}\int_u^1\frac{\varphi_2'(\xi,s)}{\sqrt{\xi^2-u^2}}\,d\xi\,.$$

Vanishing of the singular term in the last formula results from the boundary conditions superimposed on the function $\varphi_2(u,s)$.

Thus, basing on the method presented, an effective exact solution of the mixed boundary-value problem is obtained. For evaluation of the final solution the system of equations (4.15) should be solved and the inverse transforms of the Eqs. (4.16) and (4.17) found. These problems are not discussed in this paper since, as it is seen from the results of the papers [1] and [2], they do not exhibit any essential difficulties.

References

1. J. Gaszyński, G. Szefer, *Axi-symmetric punch problem under condition of consolidation*, Arch. of Mech., **27**, 3, 1975.
2. J. Gaszyński, *On a certain solution of dual integral equations and its application to contact problems of theory of consolidation*, Arch. of Mech., **28**, 1, 1976.
3. J. S. Gradstein, J. N. Ryzik, *Tables of integrals, sum, series and products* [in Polish], PWN, Warszawa 1964.
4. W. Nowacki, *Theory of elasticity* [in Polish], PWN, Warszawa 1970.
5. J. N. Sneddon, *Mixed boundary value problems in potential theory*, John Wiley and Sons Inc., New York 1966.
6. G. Szefer, *Solution of a set of dual integral equations*, Arch. of Mech., **17**, 4, 1965.
7. G. Szefer, *On the solution of certain system of dual integral equations and its application in the theory of elasticity*, Bull. Acad. Polon. Sci., Série Sci. Mech., **13**, 3, 1965.
8. J. S. Ufland, *Integral transforms in problems of theory of elasticity* [in Russian], Leningrad 1968.

Streszczenie

ROZWIĄZANIE PEWNYCH UKŁADÓW DUALNYCH RÓWNAŃ CAŁKOWYCH Z JĄDRAMI BESSELA I ICH ZASTOSOWANIE W TEORII SPRĘŻYSTOŚCI

Przedmiot pracy stanowi konstrukcja rozwiązań układów dualnych równań całkowych postaci

$$(1)\qquad \begin{aligned}&\int_0^\infty G_{ij}(v)\,\Phi_j(v)\,J_\nu(uv)\,dv=f_i(u), && u<1,\\ &\int_0^\infty \Phi_i(v)\,J_\nu(vu)\,dv=0, && u>1,\quad \nu=0,1,\end{aligned}$$

gdzie funkcje wagowe $G_{ij}(v)$ są następujące:

$$G_{ij}(v)=v\,[\overset{*}{A}_{ij}+\overset{*}{H}_{ij}(v)]+v^{-1}\,[\overset{**}{A}_{ij}+\overset{**}{H}_{ij}(v)]$$

lub

$$G_{ij}(v)=v^{2k-1}\,[A_{ij}+H_{ij}(v)],\qquad k=0,1,\ldots\,.$$

W pracy przedstawiono metodę sprowadzenia układów (1) do układu równań różniczkowo-całkowych drugiego rodzaju i drugiego rzędu. Otrzymane rezultaty zilustrowano rozwiązaniem kontaktowego zadania sprzężonej teorii termosprężystości z mieszanymi warunkami termicznej izolacji brzegu.

Резюме

РЕШЕНИЕ НЕКОТОРЫХ СИСТЕМ ДУАЛЬНЫХ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ С ЯДРАМИ БЕССЕЛЯ И ИХ ПРИМЕНЕНИЕ В ТЕОРИИ УПРУГОСТИ

Предметом работы является построение решений систем дуальных интегральных уравнений вида:

$$(1)\qquad \begin{aligned} &\int_0^\infty G_{ij}(v)\,\Phi_j(v)\,J_\nu(uv)\,dv=f_i(u), && u<1,\\ &\int_0^\infty \Phi_i(v)\,J_\nu(uv)\,dv=0, && u>1, \quad \nu=0,1, \end{aligned}$$

где весовые функции $G_{ij}(v)$ принимают вид:

$$G_{ij}(v)=v\,[\overset{*}{A}_{ij}+\overset{*}{H}_{ij}(v)]+v^{-1}\,[\overset{**}{A}_{ij}+\overset{**}{H}_{ij}(v)]$$

или

$$G_{ij}(v)=v^{2k-1}\,[A_{ij}+H_{ij}(v)],\qquad k=0,1,\ldots.$$

В работе представлен метод сведения систем (1) к системе интегро-дифференциальных уравнений второго рода и второго порядка. Полученные результаты иллюстрированы решением контактной задачи сопряженной теории термоупругости со смешанными условиями термоизоляции границы.

POLITECHNIKA KRAKOWSKA

Received January 21, 1976.