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A COMPARATIVE ANALYSIS OF THEORETICAL MODELS OF GRAVITY MOVEMENTS OF COHESIONLESS GRANULAR MEDIA

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Three different theoretical models for the analysis of movements of granular media caused by the gravity forces only are critically discused. In each of them the motion is treated as a purely kinematical problem. It has been shown that in application to varoius practical problems, they lead to different displacements patterns (e.g. funnel or mass flow, formation of shear bands or a flow without such bands). Examples of applicaion illustrate the discussion.

1. Introduction

In numerous important practical problems, the movement of granular media is caused by gravity forces only. The movements in bins and hoppers, sand avalanches, subsidence of terrain caused by underground exploitation are the typical examples. It seems to be reasonable to treat such motions as purely kinematic processes. It is interesting to note that recently in physics, the movement of grains in sand piles is used as an illustration of the so-called "self-organized process" – see e.g. [1, 2]. In these papers the analysis was limited to the movements of grains in a pile of sand. More applications of engineering significance have been discussed in [3], among them an analysis of movement of granular materials in hoppers.

The method used in the mentioned papers is based on the "discrete cellular automata" concept, which is much simpler than continuous differential equations, as stated in [1]. To illustrate the idea of "self-organized criticality", the authors of [1] considered a "pile of sand" built by randomly adding a grain at a time. When the slope of the pile reaches a critical value called the "angle of repose", the added sand will slide down.

A theoretical model based on this concept will be used in the following Sec. 2 for the analysis of movements of loosely packed granular material in a hopper and for similar problems.

An analogous model for densely packed granular media will be discussed in Sec. 3. Section 4 is devoted to the discussion of a probabilistic treatment of the problem.

2. A model for loosely packed granular media

The theoretical model (cellular automaton) mentioned above is shown in Fig. 1. It is composed of a number of identical cicrular (spherical) "grains" forming a regular, loosely packed array. The movements of grain in the model are governed by two simple rules:

1. If an empty space is located under that occupied by a grain, the grain moves downwards filling the empty place – cf. Fig. 1b
2. The difference of the heights of two neighbouring columns of grains cannot be larger than the diameter of a single grain. When it is larger, the upper grain falls down to the lower position in the next column – cf. Fig. 1c.

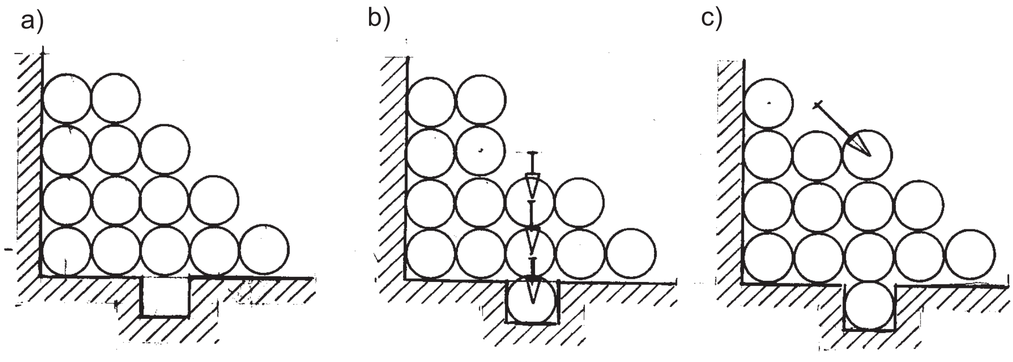

Fig. 1.

Using the two rules one can solve numerous problems of practical significance. As an example let us analyse displacements of grains in a bin. The initial stage of motion is shown in Fig. 2a when four grains have left the outlet. In Fig. 2b is presented a more advanced stage of the movement. The medium flows through a vertical funnel, while the rest of the bulk remains motionless. Thus the model predicts the well-known phenomen of the often observed so-called funnel flow in hoppers – see e.g. [4–7].

Let us note that this simple model does not predict the existence of any horizontal interaction (pressure) between the granular medium and the container's wall. The stress state in the medium reduces everywhere to uniaxial compression increasing towards the bottom. This unrealistic result is caused by the specific arrangement of grains in the model.

As a next example, let us consider the problem of terrain subsidence caused by partial tectonic translation of bedrock uder the sand-like layer resting on it – Fig. 3. The empty space formed by this translation must be filled by granular medium of the upper layer.

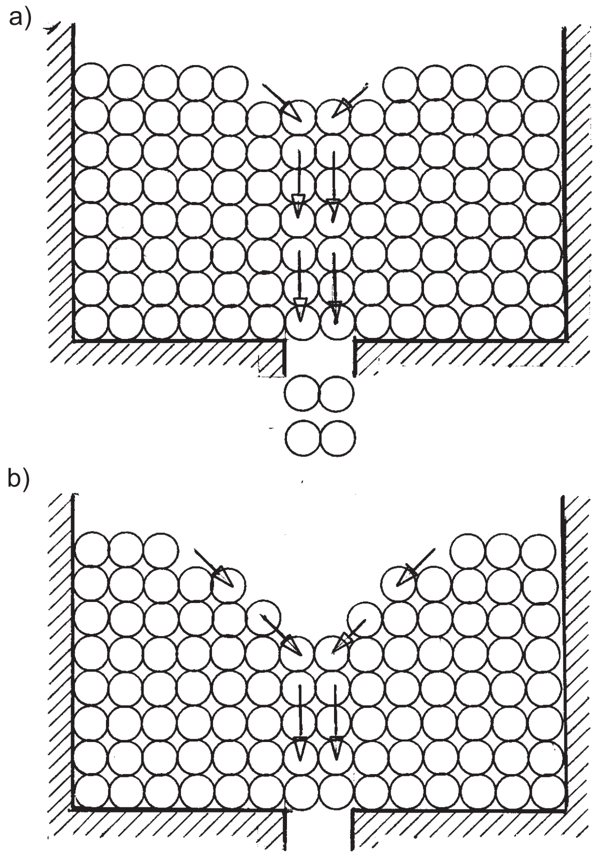

Fig. 2.

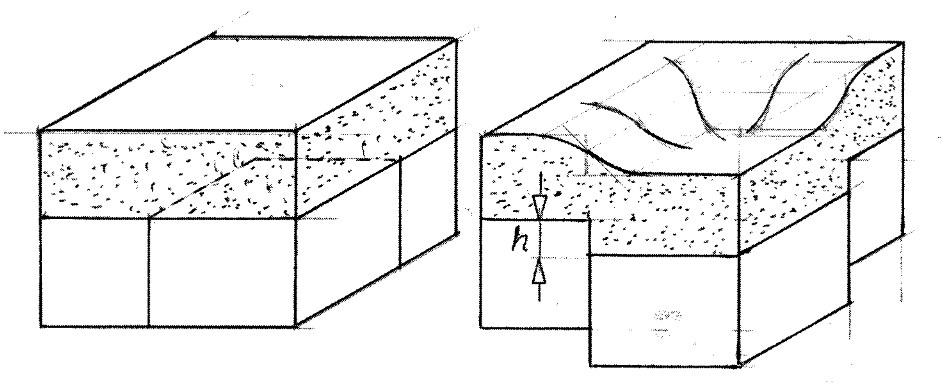

Fig. 3.

The solution is shown in Fig. 4. Elememts of granular medium have been presented in a cuboidal form. Such a presentation assures better visualization of the shape of deformed upper surface of the medium.

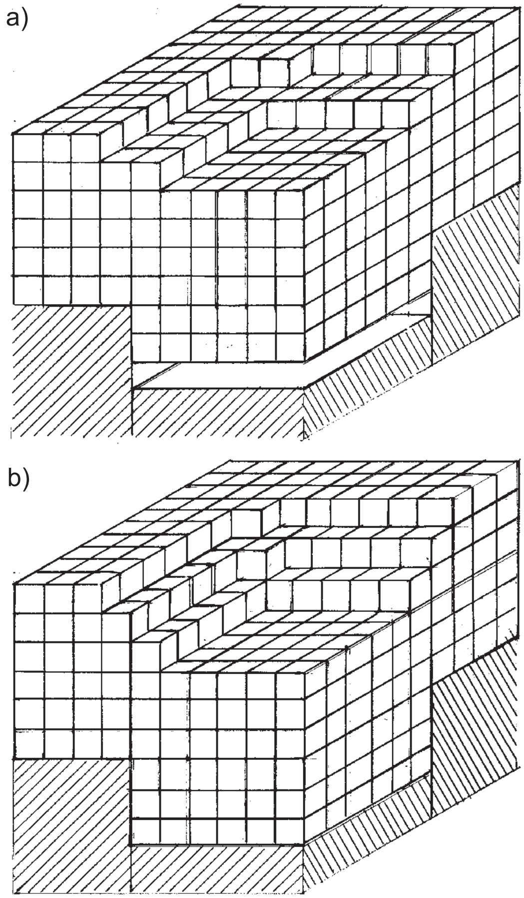

Fig. 4.

3. A model for densely packed granular material

A regular dense packing of elements of a granular medium is shown in Fig. 5a. As an illustrating example let us assume that at the bottom, the container is equipped with a feeder moving downwards. When the bottom of the feeder moves, the grains of the medium above it also move until each of them reaches the position assuring the smallest possible potential energy (the lowest possible position). Using this simple rule we can find the momentary translation vectors for all particles. They are shown in Fig. 5b and c for two subsequent positions of the bottom of the feeder. Comparing these patterns of translations with those shown previously in Fig. 2, one can see that the present model leads to a solution with the "mass flow" instead of the funnel motion predicted by the previous model.

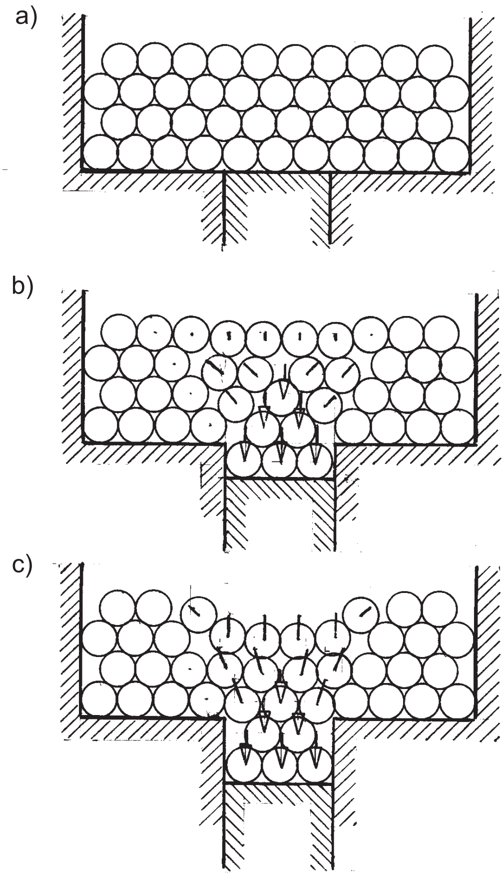

Fig. 5.

A more advanced example of momentary movement in a hopper is shown in Fig. 6. When calculating the momentary translation vectors of particular elements in an assembly of grains, we must often arbitrarily decide which of the adjacent grains moves towards the empty place below. Thus the procedure of determining the translation vectors is influenced by certain random factors. Slightly different translation patterns may result from the calculation procedure. In Fig. 6 is presented one of the possible solutions. It corresponds to the stage when fourteen elements have left the outlet of the bin.

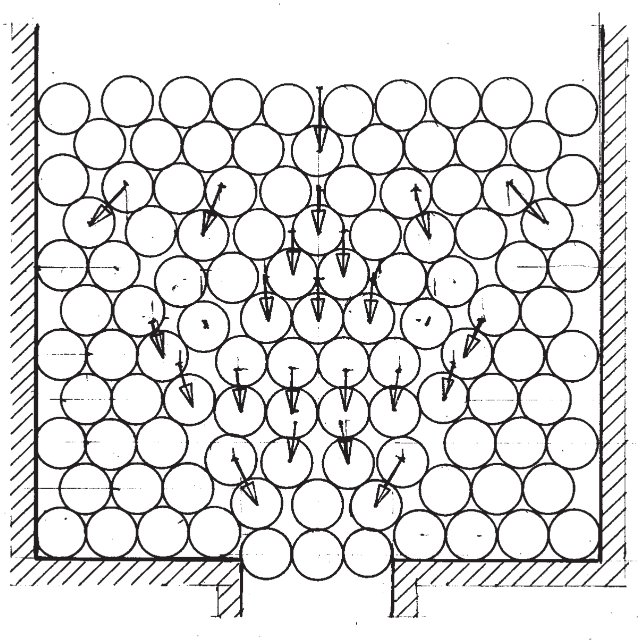

Fig. 6.

Let us note that during the flow of grains the packing of them becomes less dense. Such a loosening of the packing has been observed in deformed granular materials in experiments performed with the use of X-ray method – see e.g. [8–11].

The pattern of particle translation and loosening of the packing was also experimentally investigated in [12]. In a simulation test an assembly of coins has been used. The coins were located on a glass plate in the initially horizontal position. Then the plate was inclined with respect to the horizontal plane. The coins slid downwards due to gravity forces. The obtained translation pattern of coins was similar to that shown in Fig. 6.

Using this model we can calculate the forces of interaction between the medium and a retaining wall. This can be done when the friction between the grains and the surface of the wall is neglected. In Fig. 7 are presented the calculated interaction forces. It was assumed that the weight G of a single grain is equal to unity ($G = 1$). To the right of the row $A - B$ of grains, the system of interaction forces reduces to the uniaxial compression in vertical direction. Within the triangle $A - B - C$ this system is disturbed by contact of grains with the wall. It is seen that the influence of the contact is strongly localized.

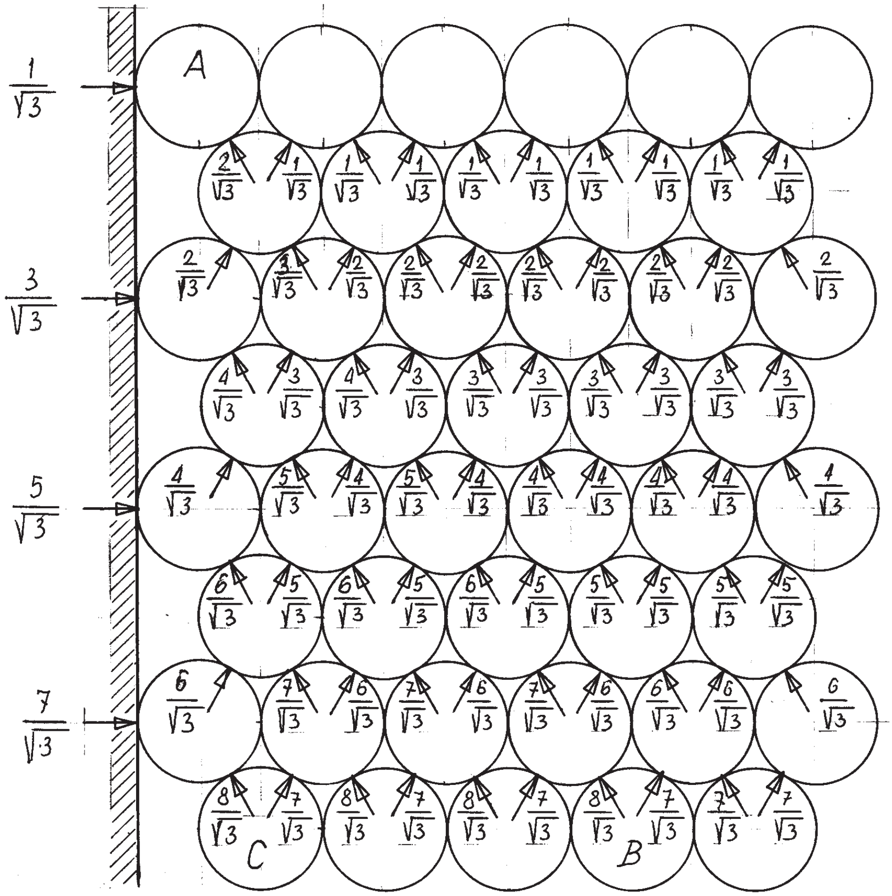

FIG. 7.

Using the model discussed in this section we can solve numerous two-dimensional problems. However, application of the model to the analysis of three-dimensional problems may prove to be difficult.

The model allows us to determine a general layout of the system of slip-lines (displacements discontinuity). Two simple examples are presented in Fig. 8. They concern the movements of elements of the model caused by displacement of the retaining wall. In Fig. 8a the wall slightly rotates about the point at the bottom. In the triangle ABC there appear several slip-lines. Such an effect has been confirmed in numerous experiments – cf.e.g. [9–10]. In the case shown in Fig. 8b, when the wall has been slightly shifted horizontally, the movement of the medium reduces to a displacement of the triangle ABC as a rigid body along the single slip-line $B-C$.

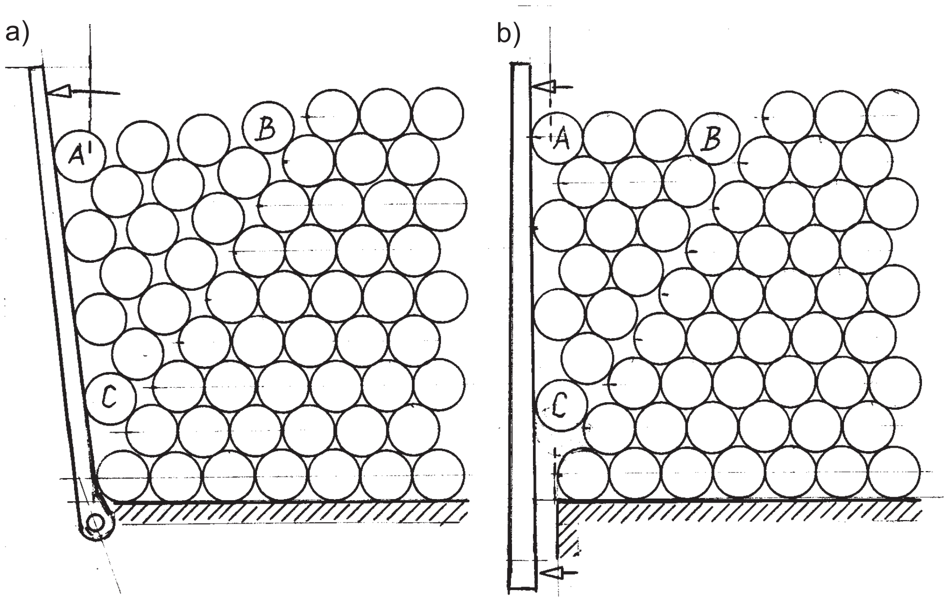

Fig. 8.

4. Stochastic finite cells model for granular media

The stochastic model is based on the concept of J. Litwiniszyn [13–15]. According to this concept displacements in a granular medium caused by gravity forces are of the mass character of random changes of mutual contacts between the particles. Consequently, the displacements of particles are random.

As the starting point let us imagine a device composed of a number of plates (layers) resting one on the other. Each plate is formed by a regular array of cuboidal cells with square holes (Fig. 9). The cells in subsequent plates are arranged with respect to each other in such a manner that central axes of the holes in a plate coincide with the common line of four corners of cells in the plate located just below or above.

Let us assume, similarly as in the so-called Galton's board (cf.e.g. [16]) for two-dimensional cases, that small balls falling down from a particular cell in a plate and striking the common vertical edges of four cells in the plate below are randomly directed into one of these cells with the probability equal to $1/4$. The random path of consecutively falling balls is repeated for each plate below. Finally the balls fall at random into one of the separate containers at the bottom. The distribution of the number of balls in these containers shown at the bottom of Fig. 9 approaches the circular normal distribution.

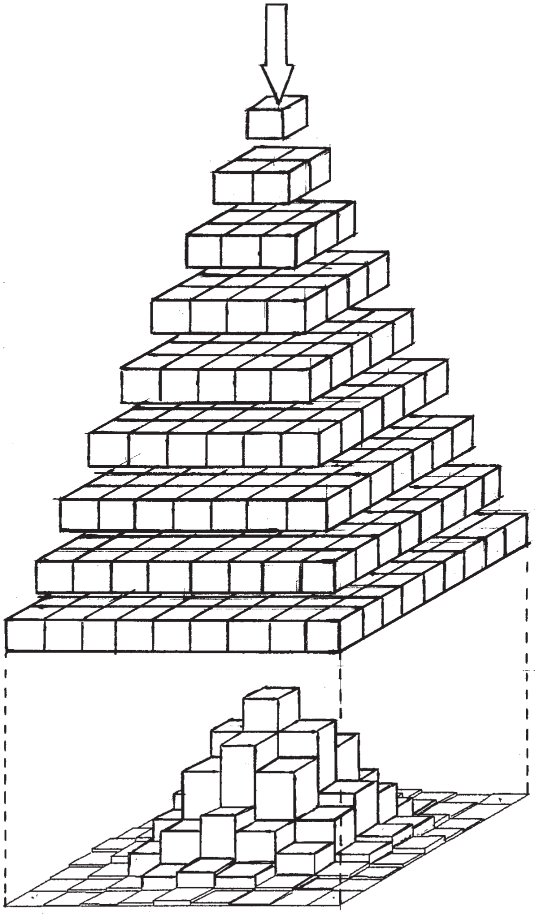

Fig. 9.

Litwiniszyn analysed an inverse two-dimensional problem, in which cavities existing in the bulk of a loose medium migrated randomly upwards to the upper surface of the bulk. This idea has been generalized for three-dimensional cases. In this generalized procedure, systems of finite cells were used to calculate the formation of local depressions in the upper surfaces of granular media, in which there exist systems of cavities [17, 18].

As an example of application of the finite cells procedure, in Fig. 10 is shown a step-wise approximation of the deformed upper surface of the layer of a granular medium, resting on the bedrock with the initial narrow cuboidal cavity. The cavity has been divided into eighteen units. In the calculating procedure, these unit cavities are assumed to migrate randomly upwards through an assumed system of plates with cells such a those shown in Fig. 9.

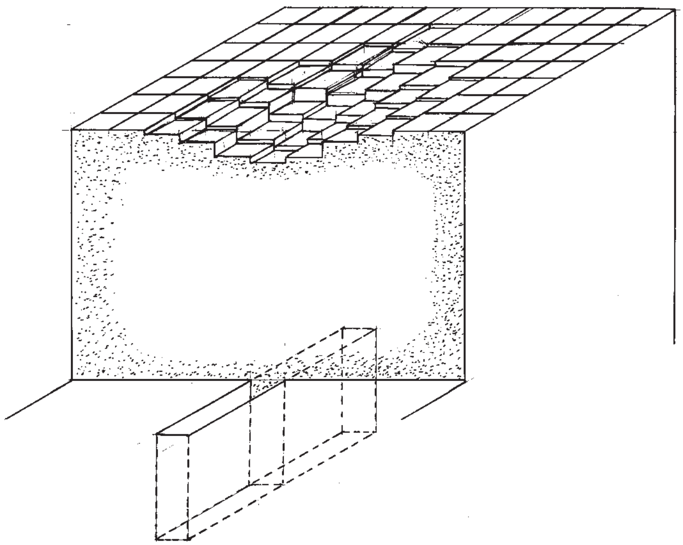

Fig. 10.

Using the stochastic finite cells procedure it is possible to calculate the displacement vectors of the particles of the medium – cf. [18]. The calculated vectors in the longer symmetry plane are shown in Fig. 11.

Let us notice that the finite cells model leads to realistic solutions even if the initial cavity is located very deeply. In such cases the model discussed in Sec. 2 gives unrealistic results concerning the funnel flow of the medium.

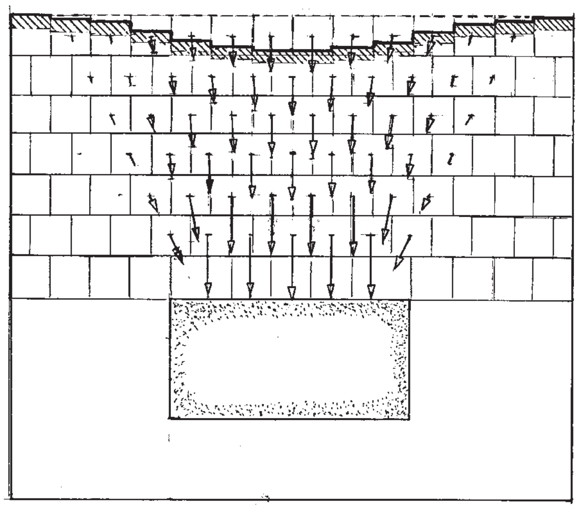

Fig. 11.

5. Concluding remarks

In all three theoretical models of granular media, their movements caused by gravity fores without any external action are treated as a purely kinematical problem. Application of each of these models is limited to a certain class of problems.

The model discussed in Sec. 2 predicts the so-called funnel flow observed sometimes in bins and hoppers. However, it cannot be used in the cases when the so-called mass flow is expected. As an example let us mention the motion in a bulk of granular medium leading to filling a deeply located empty space.

The model of a densely packed granular medium (Sec. 3) predicts mass flow of the medium. It gives good results when it is used for the analysis of two-dimensional problems. Its application to three-dimensional cases is difficult. However, using it in two-dimensional problems we can rationally analyses the pressure exerted by the medium on the retaining walls. Note that the previous model (Sec. 2) does not predict any pressure between the medium and the wall. Moreover, the model predicts formation of shear bands in the medium often observed in experimental tests.

The stochastic finite cells model (Sec. 4) predicts the mass flow of the medium and may be used for the analysis of any three-dimensional problem. However, it does not predict formation of shear bands in the granular medium.

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