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ON A ONE-DIMENSIONAL MODEL OF THE FRACTURE PROCESS

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An elastic rod is extracted from an elastic foundation by means of a longitudinal force applied to the free end of the rod (Fig. 1). In spite of the extreme simplicity of the model (axial forces in the rod, the foundation transmits only shearing stresses), the process reveals close analogy with the phenomena of fracture (crack propagation) in three-dimensional elastic solids.

Several aspects of that analogy are discussed: the possible fracture criteria, steady-state motion of the rod extracted from the medium at a constant velocity, sudden start and stopping of motion; in conclusion the paper presents a representation of steady-state solution in terms of superposition of the waves produced by "elementary fractures".

Notations

$[F, L, M, T]$ dimensions (force, length, mass, time),
$x\,[L]$, $\xi\,[1]$ space coordinates; $\xi = x/\kappa$,
$t\,[T]$, $\tau\,[1]$ time coordinates; $\tau = ct/\kappa$.

Rod characteristics

$A\,[L^2]$ cross-sectional area,
$S\,dx\,[L^2]$ element of the lateral surface,
$\rho\,[ML^{-3}]$ mass density,
$E\,[FL^{-2}]$ Young's modulus,
$P\,[F]$ longitudinal force,
$C\,[LT^{-1}]$ elastic wave propagation velocity, $c = \sqrt{E/\rho}$,
$u\,[L]$ longitudinal displacement,
$\sigma\,[FL^{-2}]$ longitudinal stress,
$\varepsilon\,[1]$ longitudinal strain.

Foundation characteristics

$\tau_s\,[FL^{-2}]$ shearing stress,
$k\,[FL^{-3}]$ foundation rigidity parameter,
$\kappa\,[L]$ foundation compliance parameter , $\kappa = \sqrt{EA/kS}$.

General notations

$v\,[LT^{-1}]$ fracture propagation velocity,
$\beta\,[1]$ dimensionless velocity parameter, $\beta^2 = 1 - v^2/c^2$,
$p\,[T^{-1}]$ Laplace transform parameter,
$\mathscr{L}, \mathscr{L}^{-1}$ Laplace and inverse Laplace transform oparator,
$\eta(t)$ Heaviside step function,
$\delta(t)$ Dirac's impulse function,
$J_n(x)$ Bessel function.

1. Introduction. Description of the model

The problems of propagation of cracks in solids attracts the attention of numerous specialists, theoreticians and experimentalists, from various fields of solid mechanics and physics, applied mathematics and technology. The technological and physical significance of crack propagation and fracture phenomena is obvious, and requires no emphasis here. From the purely mathematical point of view, the problem of non-topological motions of continua (following the definition by C. Truesdell [1]) is equally interesting in view of very serious difficulties in proper formulation of the corresponding initial and boundary value problems of the phenomena discussed (cf. [2]). The principal difficulty consists in defining the boundary of the medium: a propagating crack creats new boundaries at which certain dynamic conditions must be satisfied. However, the velocity at which the boundaries are created, and their form (crack path), depends on the physical decohesion properties of the material (fracture criterion), which are different for different materials and may vary considerably with physical conditions (temperature, humidity etc.). At the same time, the crack velocity obviously depends upon the actual states of displacement and stress in the medium, which cannot be determined unless its previous configurations are known. The main problem consists, then, in selecting from all geometrically possible crack motions the one which ensures the fulfillment of the fracture criterion at each instant of time. This vicious circle-type problem, combined with the fact that non-topological motions lead to violation of the fundamental assumptions of most of the existing solid body models (such as elasticity, [2]), makes the analysis of the crack propagation processes extremely difficult.

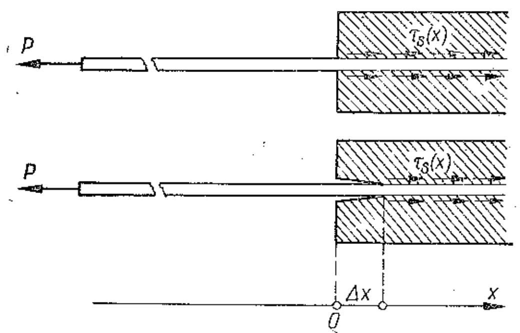

Fig. 1

The aim of this paper is to discuss the fracture process on the simplest possible, one-dimensional model resembling the situation sketched in Fig. 1. A rod (or wire) is pulled out (extracted) from a solid block (e.g. fiber-reinforced material, concrete). The shearing stresses τ_s acting at the rod-block interface keep the rod in equilibrium. Once the stresses $\tau_s(x)$ (presumably at $x=0$) — or the corresponding displacement or strains — attain certain limiting values, mutual bonds between the rod and the matrix are broken, the rod is partly pulled out from the block, and the origin of the contact area is displaced by a certain distance Δx; a new state of equilibrium is then established (Fig. 1).

A similar situation is encountered in considering the riveted (or welded) joint shown in Fig. 2. At a certain value of force P^{cr}, the first rivet suffers break, and the region of interaction between sheets a and b is displaced by a distance Δx to the right, which correspond to a crack propagating (in jumps) into a solid body.

The model to be considered in this paper, however, must be even simpler to ensure its one-dimensionality. The real stress and displacement distributions in the cases mentioned here are three-dimensional, the rod is subject to lateral contrac-

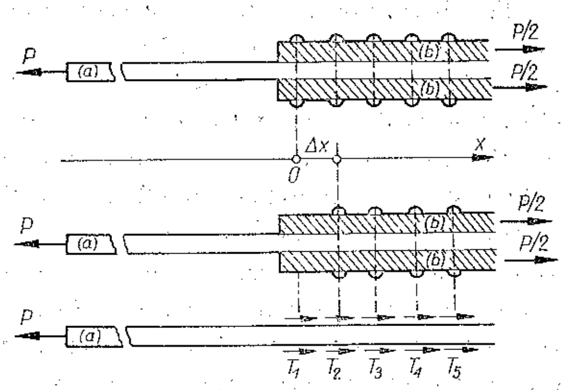

FIG. 2

tion, elastic waves are propagated and reflected in the transversal direction, infinite stress concentrations may be expected at the edge of the contact zone between the block at the rod in Fig. 1.

Hence, our attention will be focussed on a very simple, theoretical model shown in Fig. 3, defined by the following assumptions:

An infinitely long, prismatic elastic rod (of cross-section A bounded by the boundary curve S) is subject to a longitudinal force $P=P(t)$. The material of the rod is characterized by the Young modulus E, Poisson ratio $\nu=0$ and mass density ρ. The lateral surface of the rod $x<0$ is stress-free, and for $x>0$ the shearing stresses τ_s exerted by the foundation on the surface of the rod are proportional to the longitudinal displacements $u(x)$ of the rod, $\tau_s(x)=-ku(x)$, coefficient k characterizing the rigidity of the elastic foundation. The elastic foundation transmits no other forces to the rod (no normal stresses in particular), and it may be imagined as consisting of thin, weigtless, densely distributed elastic wires transmitting no elastic waves and able to react to shearing deformation only.

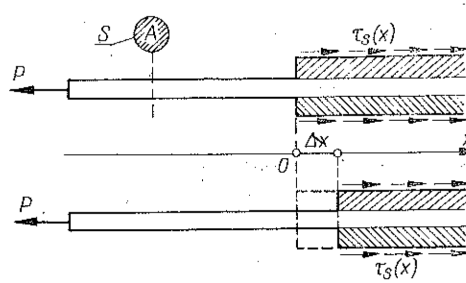

FIG. 3

These assumptions ensure the complete lack of interaction forces between individual elements of the foundation (except for those transmitted through the rod), and make it possible to reduce the entire problem to analysis of the longitudinal motion of the rod cross-sections, described by the single displacement component $u_x=u(x, t)$ and the single stress component $\sigma_{xx}=\sigma(x, t)$.

It is assumed, moreover, that at a certain critical value of displacement, $u(x, t) = u^{cr}$, the "wires" representing the foundation are broken, and the surface of the rod at that area becomes free (Fig. 3, dashed lines represent broken bonds).

It will be shown, that, in spite of the extreme simplicity of the assumptions, behaviour of the model retains certain characteristic features of the much more complex, three-dimensional fracture processes occurring in elastic bodies. In view of its conceptual and mathematical simplicity, the model may serve as a useful tool for predicting or explaining certain phenomena occurring in real fracture processes. This is the principal aim of the considerations to follow, apart from their possible direct applications, such as those indicated by Figs. 1 and 2.

The assumptions concerning the model may also be modified to embrace a wider class of phenomena and materials subject to fracture. Shearing waves may be assumed to propagate within the foundation; the corresponding "fracture criterion" may include terms depending on time derivatives of u; material of the rod may be viscoelastic or elastic-plastic, and the same applies to the foundation. In this paper, however, purely elastic material properties are assumed.

2. Equations of motion and equilibrium

Let us consider the longitudinal motion of the rod shown in Fig. 3. An element of the rod of length dx is acted on by normal tractions, $-A\sigma(x)$ and $A(\sigma + \partial\sigma/\partial x)\,dx$, at its ends, by shearing forces $\tau_s(x)\,S\,dx$ on its lateral surface, and by the force of inertia — $\rho A\,(\partial^2 u/\partial x^2)$ at its center of mass (Fig. 4). Summing up all the forces, denoting by k the foundation stiffness parameter (coefficient of proportionality of τ_s and u), and using Hooke's law in its simplest form,

$$\sigma = E\frac{\partial u}{\partial x}, \tag{2.1}$$

we obtain the equation:

$$AE\frac{\partial^2 u}{\partial x^2} - kSu = \rho A\frac{\partial^2 u}{\partial^2 t^2}. \tag{2.2}$$

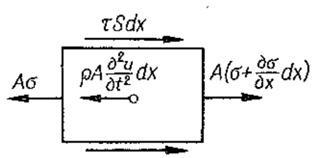

FIG. 4

By introducing a new parameter κ (foundation compliance coefficient, of the dimension of length),

$$\kappa = \sqrt{\frac{EA}{kS}}, \tag{2.3}$$

the following equation of motion of the rod immersed in the particular type of elastic medium is obtained:

$$\frac{\partial^2 u}{\partial x^2} - \frac{u}{\kappa^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{2.4}$$

This is a well-known Klein-Gordon wave equation (see, e.g., P. MORSE, H. FESHBACH [3], chapter 2). Motion of the portion of the rod outside the eleastic medium (where $\tau_s \equiv 0$) is governed by the usual wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}. \tag{2.5}$$

In dimensionless coordinates $\xi = x/\kappa$, $\tau = ct/\kappa$, Eqs. (2.4) and (2.5) assume the simple forms:

$$\frac{\partial^2 u}{\partial \xi^2} - u = \frac{\partial^2 u}{\partial \tau^2}, \tag{2.6}$$

and

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial^2 u}{\partial \tau^2}. \tag{2.7}$$

In the static case, $u(x)$ is a function of the only space coordinate x (or ξ) and hence Eq. (2.4) transforms to

$$\frac{d^2 u(x)}{dx^2} - \frac{1}{\kappa^2} u(x) = 0, \tag{2.8}$$

while Eq. (2.5) is reduced to:

$$\frac{d^2 u(x)}{dx^2} = 0. \tag{2.9}$$

Let us solve the particular static problem shown in Fig. 5; l denotes here the length of the free ($x<0$) portion of the rod, which for $x>0$ is immersed in the medium and extends to $+\infty$. Solution of Eq. (2.9) valid for $-l \leqslant x \leqslant 0$ is denoted by $u_{\mathrm{I}}(x)$ and for $x>0$ the solution of Eq. (2.8) is $u_{\mathrm{II}}(x)$. It is easily found that

$$u_{\mathrm{I}} = C_1 x + C_2, \qquad u_{\mathrm{II}} = B_1 e^{-x/\kappa} + B_2 e^{x/\kappa}.$$

Constants C_1, C_2, B_1, B_2 are then found from the boundary (at $x=-l$) and continuity (at $x=0$) conditions of $u(x)$ and $\sigma(x)$, and from the condition of vanishing of u at $x \to \infty$. Thus,

$$C_1 = \frac{\sigma_0}{E}, \qquad C_2 = -\frac{\sigma_0}{E}\kappa, \qquad B_1 = -\frac{\sigma_0}{E}\kappa, \qquad B_2 = 0,$$

and the solutions for $u(x)$, $\sigma(x)$ and $\tau_s(x)$ are:

$$u_{\mathrm{I}}(x) = -\frac{\sigma_0}{E}(\kappa - x), \qquad \sigma_{\mathrm{I}}(x) = \sigma_0 \qquad (-l \leqslant x \leqslant 0), \tag{2.10}$$

and

$$\begin{aligned} u_{\mathrm{II}}(x) &= -\frac{\sigma_0}{E}\kappa e^{-x/\kappa}, \qquad \sigma_{\mathrm{II}}(x) = \sigma_0 e^{-x/\kappa}, \\ \tau_s(x) &= \frac{\sigma_0}{E} k\kappa e^{-x/\kappa} \qquad (x>0). \end{aligned} \tag{2.11}$$

The results are illustrated by graphs in Fig. 5.

An analogous solution may be derived in the case of an infinite rod supported at the points $x_n = nl$, $n = 0, 1, 2, \ldots,$ by uniformly spaced, discrete supports satisfying the condition:

$$P_n = k^* u_n .$$

P_n is horizontal reaction, u_n — displacement of the rod measured at the n-th support, and k^* — the stiffness coefficient independent of n. Stresses σ_n and strains

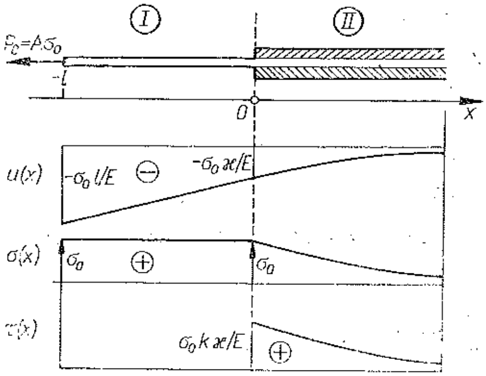

FIG. 5

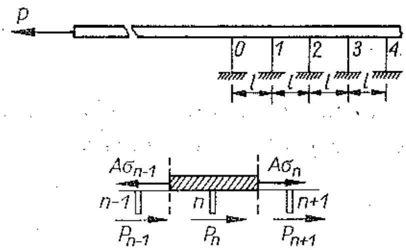

FIG. 6

ε_n are constant between the supports n and $n+1$, (cf. the notation indicated in Fig. 6), and the corresponding stress-displacement relation has the form:

$$\sigma_n = E\varepsilon_n, \qquad \varepsilon_n = \frac{u_{n+1} - u_n}{l}.$$

The condition of equilibrium of the element shaded in Fig. 6 yields the difference equation for displacements u_n, to be satisfied for all natural n (i.e. for positive values of x):

$$u_{n-1} - (2+q)\, u_n + u_{n+1} = 0 .$$

Here, $q = k^* l/EA$.

Under the assumption that $\lim u_n = 0$ for $n \to \infty$, the solution is an exponential function of n,

$$u_n = u_0 r^n, \tag{2.12}$$

with the notation

$$r = \frac{1}{2}\left[(2+q) - \sqrt{4q+q^2}\right].$$

With decreasing values of l, the ratio k^*/l should remain finite, and may be replaced by the previously introduced parameter $k^*/l = kS$; then $q = l^2/\kappa^2$ and, with $l \to 0$, also $q \to 0$. Parameter r tends at small values of q, to

$$1 - \sqrt{q} = 1 - l/\kappa .$$

and, since

$$u(x)=u_{n=x/l},$$

the displacement u_n may be expressed as a function of x:

$$u(x)=u_0 \lim_{l\to 0}(1-l/\kappa)^{x/l}=u_0\, e^{-x/\kappa}.$$

This result coincides with the Eq. (2.11).

Another case in which the equations of motion (2.4), (2.5) may easily by solved by elementary methods is the case of steady-state motion of the rod drawn out from the elastic medium by a constant force $P_0=A\sigma_0$. The velocity at which the bonds between the rod and the medium are broken is denoted by v; it may also be viewed as the velocity of propagation of the stress-free surface of the rod, which creates a clear analogy to the process of propagation of a crack in a continuous, two- ot three-dimensional medium. For the sake of brevity, v will be called the fracture propagation velocity. The analogy will be confirmed by subsequent derivation and analysis.

In order to analyze the process of steady-state motion of the system, the following assumptions have to be made:

a) The rod extends from $-\infty$ to $+\infty$.

b) The fracture process has started sufficiently long ago to become steady; the question of such a state may be achieved (if it can be achieved at all) will not be discussed here.

c) At time $t=0$, the elastic foundation extends from $x=0$ to $+\infty$, the lateral surface of the rod being stress-free for all negative values of x.

d) No physical conditions of detaching the rod from the foundation (i.e. no fracture criterion) has as yet been introduced. It is tacitly assumed that the force P_0 is large enough to maintain the steady-state rod--foundation separation process at the prescribed velocity v.

Fig. 7 shows the configurations of the system at $t=0$ and $t=t_1>0$ measured in two reference frames: ox and OX; the first is fixed in space, and the second moves together with the free surface region thus satisfying the condition:

$$X=x-vt.$$

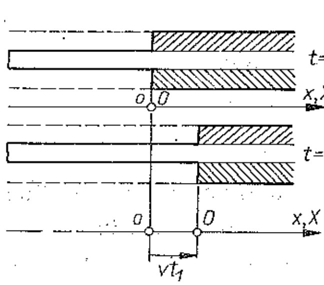

FIG. 7

The condition of steady-state motion requires the functions $u(x, t)$ and $\sigma(x, t)$ to be time-independent as viewed from the convective coordinate system OX — that is:

$$u(x, t)=u(x-vt)=u(X),$$

$$\sigma(x, t)=\sigma(x-vt)=\sigma(X).$$

Consequently, since

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}, \qquad \frac{\partial u}{\partial t}=-v\frac{\partial u}{\partial X},$$

the two equations of motion (2.4), (2.5) are simplified to the ordinary differential equations:

$$(2.13)\qquad \frac{\partial^2 u_{\rm I}}{\partial X^2}=0 \quad \text{for} \quad X<0, \qquad \frac{\partial^2 u_{\rm II}}{\partial X^2}-\frac{u_{\rm II}}{\beta^2\kappa^2}=0 \quad \text{for} \quad X>0.$$

Here, β is the velocity parameter,

$$\beta=\sqrt{1-v^2/c^2}$$

well-known from papers dealing with crack propagation phenomena. The solutions of Eqs. (2.13) have the simple forms:

$$(2.14)\qquad u_{\rm I}=C_1X+C_2, \qquad u_{\rm II}=B_1e^{-X/\beta\kappa}+B_2e^{X/\beta\kappa}.$$

As in the static case, constants C_1, C_2, B_1, B_2 are determined from the boundary and continuity conditions. The final results may now be written separately for positive and negative values of X that is, for $x>vt$ and $x<vt$.

For $X<0$:

$$(2.15)\qquad u_{\rm I}(X)=-\frac{\sigma_0}{E}(\kappa\beta-X), \qquad \sigma_{\rm I}(X)=\sigma_0.$$

For $X>0$:

$$(2.16)\qquad u_{\rm II}(X)=-\frac{\sigma_0}{E}\beta\kappa e^{-X/\beta\kappa}, \qquad \sigma_{\rm II}(X)=\sigma_0e^{-X/\beta\kappa}$$

and in addition

$$\tau_s(X)=\frac{\sigma_0}{E}k\kappa\beta e^{-X/\beta\kappa}.$$

The solutions differ from those concerning the static case by the constant β, multiplying some of the terms and appearing in the exponents of the functions (2.16).

The velocity of motion of material particles (material cross-sections of the rod) measured in coordinates x, t,

$$\frac{\partial u(x,t)}{\partial t}=-v\frac{\partial u}{\partial X}$$

is equal to

$$(2.17)\qquad \dot u_{\rm I}=-v\frac{\sigma_0}{E} \quad \text{for} \quad x<vt$$

(i.e., it is constant), and

$$(2.18)\qquad \dot u_{\rm II}=-v\frac{\sigma_0}{E}e^{-X/\kappa\beta} \quad \text{for} \quad x>vt.$$

Since in most materials the ratio $\sigma_0/E \ll 1$, the particle velocities (2.17), (2.18) are much less than the fracture propagation velocity v.

Fig. 8 presents the comparison of static and steady-state solutions for u and σ, calculated at the instant $t=0$, and with $\beta=1/2$. Owing to the factor $1/\beta$ appearing in the exponents, the displacements and stresses decrease with increasing (positive) x more rapidly than in the static case. With v approaching c, the parameter β tends to zero, and so $u(x)$ and $\sigma(x)$ for $x>0$ are practically equal to zero except for the immediate vicinity of the origin 0. The disturbances produced by breaking the bonds between the rod and the found ation are transmitted along the rod at the elastic wave velocity c, and with $v \to c$ they are unable to overtake the propagating fracture.

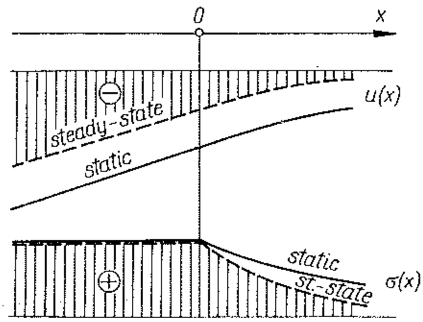

FIG. 8

3. Fracture criteria

From the solution of the static case derived in Sect. 2 it is seen that the maximum horizontal displacement of the foundation occurs at $x=0$ and equals $-\sigma_0 \kappa/E$; stress $\tau_s(x)$ attains its maximum value at the same point. With increasing values of $\sigma_0=P_0/A$, it is only natural to assume that the fracture process should start also at that point. The simplest possible criterion of initiation of the process of fracture seems to be the assumption that there exists a certain critical value of displacement, u^{cr}, which cannot be exceeded at any point of the rod within the elastic medium (foundation). Then the critical value of stress σ_0 is found:

$$\sigma_0^{cr} = \frac{Eu^{cr}}{\kappa}. \tag{3.1}$$

If this assumption remains unaltered in the conditions of steady-state motion (that is, if the foundation is perfectly elastic-brittle and fracture does not depend on the velocities $\dot{u}$ or v), then the formula (3.1) is transformed to

$$\sigma_0^{cr} = \frac{Eu^{cr}}{\kappa\beta}. \tag{3.2}$$

The $\sigma_0^{cr}=\sigma_0^{cr}(v)$ diagram (solid line shown in Fig. 9) may be compared with analogous diagrams obtained from the analysis of brittle crack propagation problems in two- or three-dimensional problems. The dashed line shown in Fig. 9 illustrates the variation of critical loads P^{cr} in the antiplane strain case of an elastic layer containing a semi-infinite crack, [4]. Both σ_0^{cr} and P^{cr} tend to infinity with v approaching the elastic wave speed c_e (which in the antiplane strain case equals $c_T=\sqrt{G/\rho}$, the shear wave velocity). The limiting values, c or c_T, can never be achieved in this way, since they require infinite loads.

The notions of critical values of displacements, strains or stresses cannot be used in classical fracture analysis owing to the well-known fact of infinite stress and strain concentrations encountered in elastic media in the vicinity of crack tips. The fracture criterion as proposed by GRIFFITH [5] was based on the energy balance evaluation: the elastic energy released in the crack propagation process must be equal to the surface energy required for the creation of new boundaries.

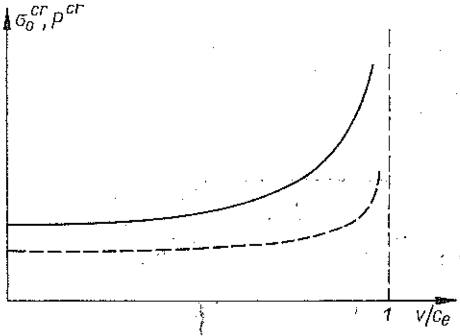

FIG. 9

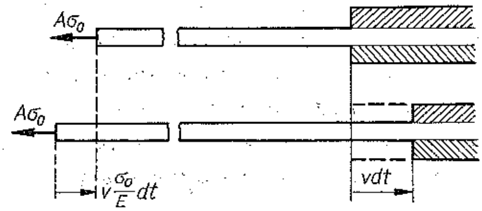

FIG. 10

In order to compare those two approaches in the one-dimensional case under consideration, let us analyze the power balance of the steady-state fracture propagation discussed in Sect. 2. The following, physically obvious assumptions must be made:

a) The work W done by the external load $P_0 = A\sigma_0$ applied to the left-hand end of the rod moving at a constant velocity $-v\sigma_0/E$, is transformed into three types of energies: kinetic energy K of the rod, its elastic strain energy U, and the dissipated energy D which is expended solely on breaking the bonds between the rod and the elastic foundation.

b) The amount of energy D needed for detaching a unit lateral surface of the rod from the foundation is equal to e_s (surface energy), and is independent of the fracture propagation velocity.

Let us calculate the power balance of the process of steady-state fracture propagation. Fig. 10 presents the configurations of the system at two instants of time, t and $t+dt$.

The work done by P_0 is equal to

$$dW = (A\sigma_0)\left(v\frac{\sigma_0}{E}dt\right), \tag{3.3}$$

$v\sigma_0/E$ being the constant velocity of motion of all rod cross-sections for $x<vt$. The total value of work done by the shearing forces $\tau_s(x)$ on the respective displacements $(x>vt)$ is easily found to be equal to zero in the process of steady-state motion.

The strain energy increment dU is calculated as the product of the uniform elastic energy density, $\sigma_0\,\varepsilon_0/2$, and the volume of the element extracted from the medium,

$$dU = \left(\frac{1}{2}\sigma_0\,\varepsilon_0\right)(Av) = \frac{1}{2}Av\frac{\sigma_0^2}{E}dt. \tag{3.4}$$

The kinetic energy of the system increases by the amount:

$$dK=\left(\frac{1}{2}\rho\dot{u}^2\right)Av=\frac{1}{2}A\rho\frac{\sigma_0^2}{E^2}v^3\,dt\,, \tag{3.5}$$

$\rho\dot{u}^2/2$ representing the constant kinetic energy density of the rod. Note that the strain and kinetic energies of the right-hand portion of the rod immersed in the elastic medium remain unchanged during the steady-state motion of the system.

Finally, the fracture energy is calculated as the product of e_s and the free lateral surface of the rod created in the time interval dt:

$$dD=Sve_s\,dt\,. \tag{3.6}$$

On combining the Eqs. (3.3)—(3.6), we obtain the power balance equation

$$\dot{W}=\dot{U}+\dot{K}+\dot{D}\,,$$

in the form:

$$A\frac{\sigma_0^2}{2}v=\frac{1}{2}Av\frac{\sigma_0^2}{E}+\frac{1}{2}A\rho\frac{\sigma_0^2}{E^2}v^3+Se_sv\,. \tag{3.7}$$

Rearranging the terms in Eq. (3.7), and making use of the notation

$$1-v^2\rho/E=\beta^2\,,$$

we obtain the formula expressing σ_0 in terms of the fracture propagation velocity v:

$$\sigma_0=\frac{1}{\beta}\sqrt{\frac{2ESe_s}{A}}\,. \tag{3.8}$$

Let us now return to the original concept of a critical displacement criterion. The specific surface energy e_s could also be expressed, in terms of u^{cr}, as the amount of energy necessary to displace a unit surface area of the foundation by the distance u^{cr}. Strain energy of the perfectly elastic foundation equals $(1/2)ku^2\times$ (surface element), and hence

$$e_s=\frac{1}{2}k\,(u^{cr})^2\,.$$

Substituting this value into Eq. (3.8), we obtain a relation identical with the result (3.2). Hence the two criteria, the critical displacement criterion and the power balance criterion prove to be entirely equivalent.

4. Wave propagation

The equation of longitudinal wave propagation in a rod supported on elastic foundation (2.4),

$$\frac{\partial^2 u}{\partial^2 x}-\frac{u}{\kappa^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\,, \tag{4.1}$$

may be solved by the method of Laplace transforms. Let us denote by $\bar{u}(x, p)$ the Laplace transform of $u(x, t)$;

$$\bar{u}(x, p) = \mathscr{L}\{u(x, t)\} = \int_0^\infty u(x, t) e^{-pt} dt.$$

The inverse transform is given by the formula:

$$u(x, t) = \mathscr{L}^{-1}\{\bar{u}(x, p)\} = \frac{1}{2\pi i} \int_\Gamma \bar{u}(x, p) e^{pt} dp,$$

the contour of integration representing a vertical line in the complex plane and extending from $\gamma - i\infty$ to $\gamma + i\infty$ ($\gamma > 0$, cf. e.g. [6]). Using the well-known formula

$$\int_0^\infty \frac{\partial^2 u(x, t)}{\partial t^2} e^{-pt} dt = p^2 \bar{u}(x, p) - pu(x, 0) - \dot{u}(x, 0),$$

where $u(x, 0)$, $\dot{u}(x, 0)$ denote the initial displacements and velocities, respectively, Eq. (4.1) may be written after transformation as

$$\frac{\partial^2 \bar{u}(p)}{\partial x^2} - \frac{r^2}{c^2} \bar{u}(p) = -\frac{1}{c^2}[pu(0) + \dot{u}(0)]. \tag{4.2}$$

Here the abbreviated notations are used

$$\bar{u}(p) = \bar{u}(x, p), \quad u(0) = u(x, 0),$$

and

$$r^2 = p^2 + \frac{c^2}{\kappa^2}. \tag{4.3}$$

Let us now assume that a semi-infinite rod extending from $x = 0$ to $x = \infty$ is supported on an elastic foundation and is at rest for $t < 0$: $u(0) = \dot{u}(0) = 0$. At $t = 0$, the stress $\sigma_0(t) = P_0(t)/A$ is applied to the free end of the rod at $x = 0$. The solution of Eq. (4.2), written for homogeneous initial conditions,

$$\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{r^2}{c^2} \bar{u} = 0 \tag{4.4}$$

has the form:

$$\bar{u}(x, p) = B(p) e^{-rx/c}, \tag{4.5}$$

the other term containing the exponential function $\exp(rx/c)$ being disregaded due to the requirement of finiteness of the solution at $x \to \infty$. From the boundary condition

$$E \left. \frac{\partial \bar{u}(x, p)}{\partial x} \right|_{x=0} = \bar{\sigma}_0(p),$$

we determine the constant $B(p)$ and obtain:

$$\bar{u}(x, p) = -\frac{c}{E} \frac{\bar{\sigma}_0(p)}{r} e^{-rx/c}.$$

After inversion:

$$u(x,t) = -\frac{c}{E}\,\mathscr{L}^{-1}\left\{\frac{\bar{\sigma}_0(p)}{r}\,e^{-rx/c},\right\}, \tag{4.6}$$

and

$$\sigma(x,t) = \mathscr{L}^{-1}\{\bar{\sigma}_0(p)\,e^{-rx/c}\}. \tag{4.7}$$

Application of the convolution theorem enables us to write the solution (4.6) in the form:

$$u(x,t) = \begin{cases} -\dfrac{c}{E}\displaystyle\int_{x/c}^{t} \sigma_0(t-\theta)\,J_0\left(\frac{c}{\kappa}\sqrt{\theta^2-\frac{x^2}{c^2}}\right)d\theta, & x<ct, \\ 0, & x>ct, \end{cases} \tag{4.8}$$

which may further by simplified by using the dimensionless variables introduced in Sect. 2

$$u(\xi,\tau) = \begin{cases} -\dfrac{\kappa}{E}\displaystyle\int_{\xi}^{\tau} \sigma_0(\tau-\theta)\,J_0(\sqrt{\theta^2-\xi^2})\,d\theta, & \xi<\tau, \\ 0, & \xi>\tau. \end{cases} \tag{4.9}$$

The speed of propagation of all disturbances transmitted along the rod immersed in an elastic medium is equal to c, as in the unsupported rod; however, the form and amplitude of such disturbances are not preserved in the process of their propagation.

Assuming that $0<\tau-\xi\ll 1$, and that $\sigma_0(t)$ may be expanded in the vicinity of $t=0$ into a power series

$$\sigma_0(t) = \sigma_0^{(0)} + t\sigma_0^{(1)} + t^2\sigma_0^{(2)} + \ldots, \tag{4.10}$$

the integral in Eq. (4.9) may be replaced by the approximate expression:

$$\int_{\xi}^{\tau} \sigma_0(\tau-\theta)\,d\theta \approx (\tau-\xi)\,\sigma_0(0),$$

and thus, for $\tau-\xi\to 0$ (close to the wave front), the following approximate formulae hold true:

$$\begin{aligned} u(\xi,\tau) &\approx -\frac{\kappa}{E}(\tau-\xi)\,\sigma_0(0) = -\frac{c}{E}\left(t-\frac{x}{c}\right)\sigma_0(0); \\ \sigma(\xi,\tau) &\approx \sigma_0(0). \end{aligned} \tag{4.11}$$

This means that the value of stress (or stress jump) at the wave front remains constant in the process of propagation; the corresponding displacement is always zero, and satisfies the continuity requirements.

Two particular cases of wave propagation will be of special interest. If $\sigma_0(t) = \sigma_0 \eta(t)$ ($\eta(t)$ — Heaviside function), then Eqs. (4.8), (4.9) yield the solution:

$$u(x,t) = -\frac{c\sigma_0}{E}\int_{x/c}^{t} J_0\left(\frac{c}{\kappa}\sqrt{\theta^2 - \frac{x^2}{c^2}}\right)d\theta,$$

$$u(\xi,\tau) = -\frac{\kappa\sigma_0}{E}\int_0^{\sqrt{\tau^2-\xi^2}} \frac{\theta J_0(\theta)\,d\theta}{\sqrt{\xi^2+\theta^2}}, \tag{4.12}$$

$$\sigma(\xi,\tau) = \sigma_0\left[1 - \xi\int_0^{\sqrt{\tau^2-\xi^2}} \frac{J_1(\theta)\,d\theta}{\sqrt{\xi^2+\theta^2}}\right].$$

Here, again, the stress discontinuity at the wave front $\xi=\tau$ is constant, and equal to the original value σ_0.

In the case of an impulse excitation $\sigma_0(t) = \sigma_0\,\delta(t)$, the corresponding displacements are expressed explicity by a single Bessel function J_0:

$$u(\xi,\tau) = -\frac{\kappa\sigma_0}{E} J_0(\sqrt{\tau^2-\xi^2}). \tag{4.13}$$

This solution is of a rather formal character: the function $\sigma_0(t)$ cannot, in this case, be expanded into the power series (4.10) and, consequently, the condition of continuity of $u(x)$ is violated at the wave front; the displacement suffers a jump $\kappa\sigma_0/E$. However, this solution proves to be very useful in discussing the problems of wave superposition, presented in Sect. 6 of this paper.

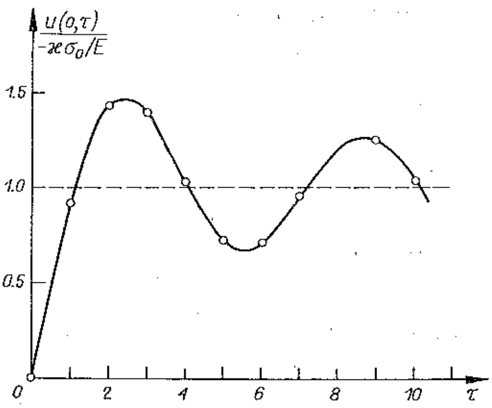

FIG. 11

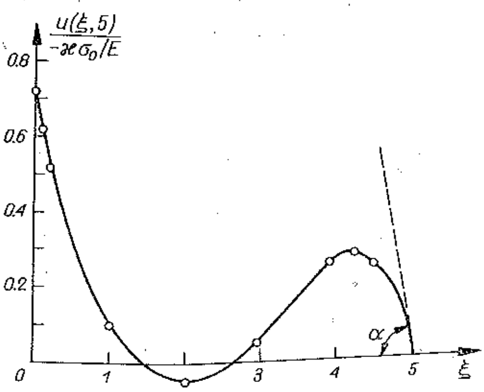

FIG. 12.

Some of the solutions derived here are illustrated by Figs. 11—13. Fig. 11 presents the variation of displacement (4.12) at the free end of the rod, $\xi=0$; $u(0,\tau)$ oscillates and tends asymptotically to the limiting (static) value of $-\kappa\sigma_0/E$ (cf. Eq. (2.10)). The distribution of $u(\xi,\tau)$ at a certain fixed value of τ ($\tau=5$) is shown in Fig. 12. The slope of this curve at the wave front (here $\xi=5$) is constant and independent of τ, which reflects the property of a constant stress jump propagated along the rod.

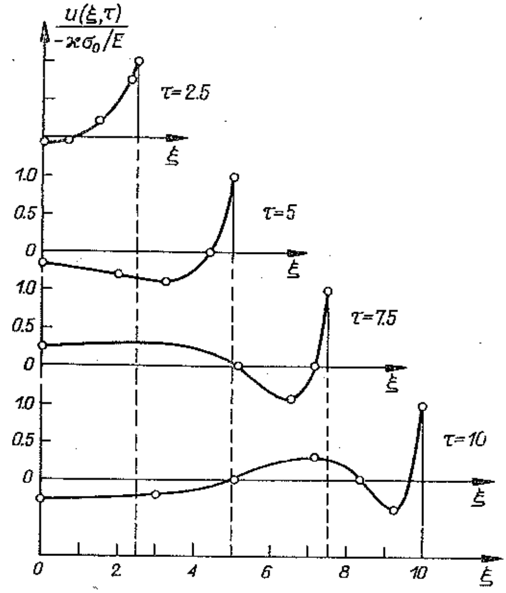

FIG. 13

Fig. 13 presents four consecutive phases of propagation of the $u(\xi, \tau)$ — wave produced by the stress-pulse, Eq. (4.13). The displacement jump at the wave front remains constant here.

5. Sudden stopping of thf fracture propagation

One of the most typical problems frequently discussed in papers dealing with non-uniform crack propagation phenomena is the crack arrest problem. Immediately after a propagating crack is stopped (which might be caused, e.g., by certain inhomogeneities located on the crack path), the stress intensity factor usually increases and then, more or less rapidly, approaches a certain limiting value; the SIF-variation may also involve oscillations. An example of such analysis is given in a paper by F. Nilsson [7]. Due to considerable mathematical difficulties induced by non-uniform crack propagation, closed-form solutions of such problems are extremely difficult to achieve.

Let us consider an analogous problem for the model considered in this paper (Fig. 7). Let the system undergo the steady-state motion as analyzed in Sect. 2, Eqs. (2.15), (2.16), due to the action of a constant force $P_0=A\sigma_0$. The process, which is assumed to have started at $t=-\infty$, is suddenly stopped at the instant $t=0$, exactly at the common origin of the two coordinate systems ox and OX (upper configuration shown in Fig. 7). Force P_0 is assumed to remain constant, and the reason for stopping the fracture propagation may be, e.g., the increased strength of the portion of the foundation adjacent to 0.

The solution may be derived from the analysis of the following initial and boundary value problem for $u(x, t)$ and $\sigma(x, t) = E\,\partial u(x, t)/\partial x$.

Determine the displacements $u_{\rm I}(x, t)$, $u_{\rm II}(x, t)$ and the corresponding stresses $\sigma_{\rm I}(x, t)$, $\sigma_{\rm II}(x, t)$, satisfying the differential equations (2.5), (2.4):

$$(5.1)\qquad \frac{\partial^2 u_{\rm I}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u_{\rm I}}{\partial t^2}, \qquad \frac{\partial^2 u_{\rm II}}{\partial x^2} - \frac{u_{\rm II}}{\kappa^2} = \frac{1}{c^2}\frac{\partial^2 u_{\rm II}}{\partial t^2},$$

the boundary and continuity conditions:

$$(5.2)\qquad \begin{aligned} &\sigma_{\rm I}(-\infty, t) = -\sigma_0, \qquad u_{\rm II}(\infty, t) = 0,\\ &u_{\rm I}(0, t) = u_{\rm II}(0, t), \qquad \sigma_{\rm I}(0, t) = \sigma_{\rm II}(0, t), \end{aligned}$$

and the initial conditions:

$$(5.3)\qquad \begin{aligned} &u_{\rm I}(x, 0) = -\frac{\sigma_0}{E}(\kappa\beta - x), \qquad u_{\rm II}(x, 0) = -\frac{\sigma_0}{E}\kappa\beta e^{-x/\kappa\beta},\\ &\dot u_{\rm I}(x, 0) = -\frac{\sigma_0}{E}v, \qquad \dot u_{\rm II}(x, 0) = -\frac{\sigma_0}{E}v e^{-x/\kappa\beta}. \end{aligned}$$

The latter conditions are formulated on the basis of the solutions (2.15), (2.16) by substituting $x - vt$ for X.

Applying the Laplace transforms to Eqs. (5.1) and (5.3), and using the formula (4.2), we arrive at the system of ordinary differential equations for $\bar u_{\rm I}(x, p)$ and $\bar u_{\rm II}(x, p)$:

$$\begin{aligned} &\frac{\partial^2 \bar u_{\rm I}}{\partial x^2} - \frac{p^2}{c^2}\bar u_{\rm I} = \frac{\sigma_0}{Ec^2}[v - p(x - \kappa\beta)], \qquad x < 0,\\ &\frac{\partial^2 \bar u_{\rm II}}{\partial x^2} - \frac{r^2}{c^2}\bar u_{\rm II} = \frac{\sigma_0}{Ec^2}(v + p\kappa\beta)e^{-x/\kappa\beta}, \qquad x > 0. \end{aligned}$$

Solutions of these non-homogeneous equations are written in the form:

$$(5.4)\qquad \begin{aligned} &\bar u_{\rm I} = Ae^{px/c} - \frac{\sigma_0}{E}\frac{v + p(\kappa\beta - x)}{p^2},\\ &\bar u_{\rm II} = Be^{-rx/c} - \frac{\sigma_0}{E}\frac{\kappa\beta}{p - h}e^{-x/\kappa\beta}. \end{aligned}$$

Here, A, B are constants to be determined from the conditions (5.2), r is given by Eq. (4.3), and h is an additional parameter:

$$(5.5)\qquad h = \frac{v}{\kappa\beta}.$$

Let us now consider the right-hand portion of the rod shown in Fig. 7 — i.e., use the solutions u_{II} and σ_{II}. The Eqs. (5.2) yield:

$$B = \frac{\sigma_0 h(v+c)}{E} \frac{1}{p(p-h)(p+r)},$$

and hence

$$\bar{u}_{II} = \frac{\sigma_0}{E}\left[\frac{h(v+c)}{p(p-h)(p+r)} e^{-rx/c} - \frac{\kappa\beta}{p-h} e^{-x/\kappa\beta}\right].$$

Inversion of this formula is possible by means of the simple decomposition:

$$\frac{1}{p(p-h)(p+r)} = \frac{1}{p-h}\left[\frac{1}{pr} - \frac{1}{(p+r)r}\right],$$

by application of the formulae given in [6], and by the convolution theorem:

$$\mathscr{L}^{-1}\{g_1(p) g_2(p)\} = \int_0^t f_1(\theta) f_2(t-\theta)\, d\theta .$$

Here, $f_i(t) = \mathscr{L}^{-1}\{g_i(p)\}$, $i = 1, 2$.

The result is written in dimensionless variables, $\xi = x/\kappa$, $\tau = ct/\kappa$:

$$u_{II}(\xi, \tau) = \frac{\sigma_0 \kappa}{E}\left[\left(1 + \frac{v}{c}\right) F_1(\xi, \tau)\eta(\tau - \xi) - \beta F_2(\xi, \tau)\right],$$

(5.6) $$F_1(\xi, \tau) = \int_\xi^\tau \left\{[e^{w(\tau-\theta)} - 1] J_0(\sqrt{\theta^2 - \xi^2}) - w\sqrt{\frac{\theta - \xi}{\theta + \xi}}\, e^{w(\tau-\theta)} J_1(\sqrt{\theta^2 - \xi^2})\right\} d\theta ,$$

$$F_2(\xi, \tau) = \exp\left[-\left(\xi - \frac{v}{c}\tau\right)\Big/\beta\right],$$

with the additional notation $w = v/c\beta$. The formula (4.6) is the exact, closed-form and complete (for $x > 0$) solution of the fracture stopping problem. The solution consists of two terms: the exponential term $F_2(\xi, \tau)$, which is different from zero in the entire range of $x > 0$ from the very beginning ($t = 0$) of the process. The other term, $F_1(\xi, \tau)$, represents a wave radiated from the point of fracture arrest and propagating along the rod at the usual velocity c.

Setting $\tau = 0$ in the formula (5.6), we obtain the obvious result:

(5.7) $$u_{II}(\xi, 0) = -\frac{\sigma_0}{E} \kappa\beta e^{-\xi/\beta},$$

which coincides with the initial condition (5.3).

Particular attention should be paid to the behaviour of u (ξ, τ) at the point $\xi = 0$, the counterpart of the crack tip in crack propagation processes. On substituting $\xi = 0$ into the Eq. (4.6), and performing simple transformations, we obtain:

$$u_{II}(0, \tau) = -\frac{\sigma_0}{E} \kappa F(\tau),$$

(5.8) $$F(\tau) = \left(1 + \frac{v}{c}\right)\int_0^\tau \frac{J_1(\theta)\, d\theta}{\theta} + \left[\beta - \left(1 + \frac{v}{c}\right)\int_0^\tau \frac{J_1(\theta)}{\theta} e^{-w\theta} d\theta\right] e^{w\tau}.$$

The diagrams presented in Fig. 14 illustrate the variation of $F(\tau)$ in two particular cases: a) $v/c=0.6$ ($\beta=0.8$), and b) $v/c=0.8$ ($\beta=0.6$). At the beginning of the process ($\tau=0$), $F(\tau)$ assumes the value β in compliance with Eq. (5.7). However, an interesting result of the analysis of Eq. (5.8) and Fig. 14 is the limiting value of $u(0, \tau)$ at $\tau\to\infty$. Instead of the static value $-\sigma_0 \kappa/E$ (that is, instead of $\lim F(\tau)=1$), a greater value is obtained:

$$\lim_{\tau\to\infty} u(0, \tau) = -\frac{\sigma_0 \kappa}{E}\left(1+\frac{v}{c}\right). \tag{5.9}$$

The same result may be derived directly from Eq. (5.8); with $\tau\to\infty$, this formula yields:

$$u_{\mathrm{II}}(\xi, \tau) \approx -\frac{\sigma_0 \kappa}{E}\left(1+\frac{v}{c}\right)e^{-\xi}, \qquad \tau\to\infty, \tag{5.10}$$

i.e. a result differing from the static solution (2.11) by a constant factor $(1+v/c)$.

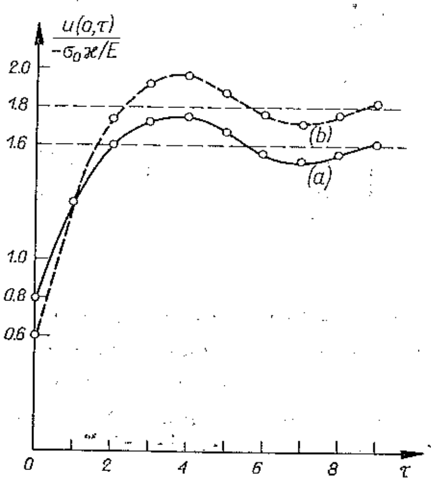

FIG. 14

From the graphs shown in Fig. 14, it is seen that the greatest concentration of $u(0, t)$ (and, consequently, the greatest τ_s-stress concentration, since $\tau_s=ku$) occurs at the approximate time $t=3.5\ \kappa/c$ after stopping, the stress concentration factors (referred to the static values) being 1.75 in case a), and 1.95 in case b).

This result seems to be due to the inertia effect of the infinitely long portion of the rod $x<0$, which is assumed to move at constant speed at the instant preceding the fracture arrest. A similar effect is observed in the case of a longitudinal impact of a long elastic bar striking an elastic support. Let us assume that a bar of length l, compressed by forces $P_0=A\sigma_0$, strikes at time $t=0$ a support characterized by the elastic constant $\bar{k}$ — i.e., at the point of impact $u=\bar{k}\sigma/E$. Simple analysis shows that deflection of the support will be equal to

$$u(0, t) = \frac{\bar{k}\sigma_0}{E}\left[1+\frac{v}{c}\left(1-e^{-ct/\bar{k}}\right)\right], \tag{5.11}$$

valid for all times $t<t_r=2l/c$ — that is, before the elastic wave produced by the impact is reflected from the free end of the bar and returns to the supported end. If t_r is large enough, Eq. (5.11) yields the static deflection $\bar{k}\sigma_0/E$, multiplied by the factor $1+v/c$.

It may be supposed that the result (5.10) follows from the assumption of infinite length of the rod under consideration. If the left-hand portion of the rod were finite, multiple reflections of elastic waves generated at $\xi=0$, and travelling along the rod would reduce the deflection (4.10) to its static value:

$$u_{\mathrm{I}}^{\mathrm{stat}}(0)=u_{\mathrm{II}}^{\mathrm{stat}}(0)=-\frac{k\sigma_0}{E}.$$

6. Start of the fracture propagation. Wave superposition

Let us consider the infinite rod resting partly (for $x>-l$) on an elastic foundation, and loaded by quasistatically increasing force P_0 (Fig. 15). Assume that at the instant $t=0$ the maximum displacement of the foundation $u(0)=-\sigma_0\kappa/E$

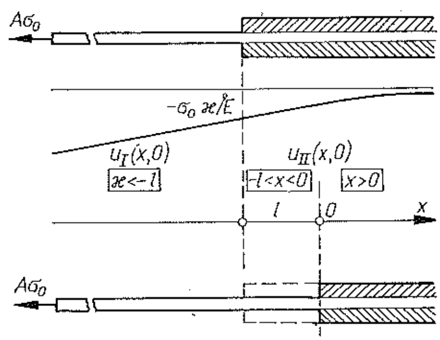

Fig. 15

exceeds the critical value u^{cr}. As a result, the bonds between the rod and the foundation are broken over a small portion Sl of the lateral surface of the rod. The resulting motion $u(x,t)$ may be determined as the solution of the following initial and boundary value problem:

Determine the displacements $u_{\mathrm{I}}(x,t)$, $u_{\mathrm{II}}(x,t)$ and stresses $\sigma_{\mathrm{I}}(x,t)$, $\sigma_{\mathrm{II}}(x,t)$ satisfying the differential equations:

(6.1)
$$\frac{\partial^2 u_{\mathrm{I}}(x,t)}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u_{\mathrm{I}}(x,t)}{\partial t_2}, \qquad x<0,$$
$$\frac{\partial^2 u_{\mathrm{II}}(x,t)}{\partial x^2}-\frac{u_{\mathrm{II}}(x,t)}{\kappa^2}=\frac{1}{c^2}\frac{\partial^2 u_{\mathrm{II}}(x,t)}{\partial t^2}, \qquad x>0.$$

the boundary and continuity conditions:

(6.2)
$$\sigma_{\mathrm{I}}(-\infty,t)=\sigma_0, \qquad u_{\mathrm{II}}(\infty,t)=0,$$
$$u_{\mathrm{I}}(0,t)=u_{\mathrm{II}}(0,t), \quad \sigma_{\mathrm{I}}(0,t)=\sigma_{\mathrm{II}}(0,t),$$

and the initial conditions (cf. Eqs. (2.10), (2.11)): $\dot{u}(x, 0)=0$,

$$(6.3)\qquad \begin{aligned} u_{\rm I}(x, 0) &= -\frac{\sigma_0}{E}[\kappa-(x+l)], && x<-l,\\ u_{\rm I}(x, 0) &= -\frac{\sigma_0}{E}\kappa e^{-(x+l)/\kappa}, && -l<x<0,\\ u_{\rm II}(x, 0) &= -\frac{\sigma_0}{E}\kappa e^{-(x+l)/\kappa}, && x>0. \end{aligned}$$

In order to simplify the initial conditions (6.3), we may make use of the assumption that l is small with respect to κ. Since

$$e^{-(x+l)\kappa}\approx e^{-x/\kappa}-\frac{l}{\kappa}e^{-x/\kappa},$$

Eqs. (6.3) may be approximately replaced by

$$(6.4)\qquad \begin{aligned} u_{\rm I}(x, 0) &= -\frac{\sigma_0}{E}(\kappa-x)+\varepsilon, && x<0,\\ u_{\rm II}(x, 0) &= -\frac{\sigma_0}{E}\kappa e^{-x/\kappa}+\varepsilon e^{-x/\kappa}, \end{aligned}$$

with the notation

$$\varepsilon=\sigma_0 l/E.$$

Note that the first right-hand terms in the Eqs. (6.4) represent exactly the static solutions, and hence they can be excluded from further considerations. The Eqs. (6.1), (6.2), and the dynamic components of conditions (6.4)

$$(6.5)\qquad \begin{aligned} u_{\rm I}^{\varepsilon}(x, 0) &= \varepsilon, && x<0,\\ u_{\rm II}^{\varepsilon}(x, 0) &= \varepsilon e^{-x/\kappa}, && x>0 \end{aligned}$$

(superscripts ε are refferred to the second right-hand terms in (6.4)) are now solved by the Laplace transform method. The transformed version of Eqs. (6.1), (6.5) is:

$$\frac{\partial^2 \bar{u}_{\rm I}^{\varepsilon}}{\partial x^2}-\frac{p^2}{c^2}\bar{u}_{\rm I}^{\varepsilon}=-\frac{p\varepsilon}{c^2},$$

$$\frac{\partial^2 \bar{u}_{\rm II}^{\varepsilon}}{\partial x^2}-\frac{r^2}{c^2}\bar{u}_{\rm II}^{\varepsilon}=-\frac{p\varepsilon}{c^2}e^{-x/\kappa}.$$

On using the conditions (6.2), we obtain:

$$(6.6)\qquad \begin{aligned} \bar{u}_{\rm I}^{\varepsilon}(x, p) &= \varepsilon\left[\frac{1}{p}-\frac{c}{\kappa}\,\frac{\exp(px/c)}{p(p+r)}\right],\\ \bar{u}_{\rm II}^{\varepsilon}(x, p) &= \varepsilon\left[\frac{\exp(-x/\kappa)}{p}-\frac{c}{\kappa}\,\frac{\exp(-rx/c)}{p(p+r)}\right]. \end{aligned}$$

The inverse transforms of (6.6) are derived by means of the decomposition:

$$\frac{1}{p(p+r)}=\frac{1}{pr}-\frac{1}{r(p+r)},$$

and by applying the convolution theorem and formulae given in [6]

$$\text{(6.7)}\qquad \begin{aligned} u_{\mathrm{I}}^{\varepsilon}(\xi,\tau)&=\varepsilon\left[1-\eta(\tau+\xi)\int_0^{\tau+\xi}\frac{J_1(\theta)\,d\theta}{\theta}\right],\\ u_{\mathrm{II}}^{\varepsilon}(\xi,\tau)&=\varepsilon\left[e^{-\xi}-\eta(\tau-\xi)\,U(\xi,\tau)\right],\end{aligned}$$

with the notation

$$\text{(6.8)}\qquad U(\xi,\tau)=\int_0^{\sqrt{\tau^2-\xi^2}}\frac{\theta J_0(\theta)\,d\theta}{\sqrt{\theta^2+\xi^2}}-\sqrt{\frac{\tau-\xi}{\tau+\xi}}\,J_1\left(\sqrt{\tau^2-\xi^2}\right).$$

A complete solution of the problem must include the static components of u (the first right-hand terms in Eqs. (6.4)).

E.g. for $\xi>0$ we obtain:

$$\text{(6.9)}\qquad u_{\mathrm{II}}(\xi,\tau)=-\frac{\sigma_0\kappa}{E}\left[\left(1-\frac{l}{\kappa}\right)e^{-\xi}+\frac{l}{\kappa}\eta(\tau-\xi)\,U(\xi,\tau)\right].$$

Thus the solution (6.9) consists of two parts: the static component containing the term $\exp(-\xi)$, and the wave-type component $U(\xi,\tau)\,\eta(\tau-\xi)$, transmitted through the rod with the speed of elastic waves c. For $\tau=0$, the latter term vanishes and $u_{\mathrm{II}}(\xi,0)$ satisfies the initial condition (6.4). For $\tau\to\infty$, on the basis of the formula [6]

$$U(\xi,\infty)=\int_0^{\infty}\frac{\theta J_0(\theta)\,d\theta}{\sqrt{\theta^2+\xi^2}},$$

we obtain

$$u_{\mathrm{II}}(\xi,\infty)=-\frac{\sigma_0}{E}\kappa e^{-\xi},$$

which represents a new static solution referred to the new position of the foundation.

Close to the wave front, for $(\tau-\xi)\to 0$, the approximate formula holds true:

$$U_f(\xi,\tau)\approx\frac{1}{2}(\tau-\xi),$$

leading to the conclusion that the wave front displacements are continuous, while the stresses $\sigma(x,t)$ suffer a jump of magnitude:

$$-\frac{\sigma_0}{E}l\frac{\partial U_f(\xi,\tau)}{\partial\xi}\frac{E}{\kappa}=\frac{1}{2}\sigma_0 l/\kappa,$$

independent of the time τ.

Fig. 16 illustrates the time-variation of $U(\xi, \tau)$ at $\xi=0$; it is seen that the maximum displacement occurs at the point at time $t\approx 3.5\,\kappa/c$ (similarly to the case shown in Fig. 14). The distribution of $U(\xi, \tau)$ at a certain fixed value of time ($\tau=5$) is shown in Fig. 17.

Let us now imagine the continuous process of fracture propagation as consisting of a series of consecutive "elementary fractures" similar to those described above. Let us assume that the "elementary fracture" occurring at time $t=0$ and point $x=0$ produces a wave,

$$u(x, t)=u_\Delta L(x, t)\eta(ct-x). \tag{6.10}$$

Here, u_Δ denotes a term proportional to σ_0/E, and $L(x, t)$ — an as yet unspecified function analogous to $U(x, t)$ but different from it; in the process of fracture propagation the initial conditions (6.3) vary with t and x; in particular, all cross-sections of the rod move at a certain velocity, and the actual distribution of $\dot{u}(x, 0)$ must influence the form of $L(x, t)$.

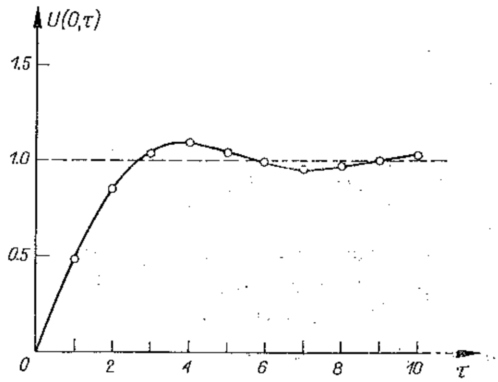

FIG. 16

The procedure of superposing the individual "elementary fractures" would be as follows. Assume that the system is at statical equilibrium at time $t<0$. The first fracture (at $t=0$ and $x=0$) produces the wave (6.10), in which $u_\Delta L(x, t)\eta(ct-x)$ should be replaced by the function (6.8). The second fracture occurs at time t_1, at which the displacement $u(0, t_1)$ reaches the critical value u^{cr}. This procedure must then be repeated, the time of the following fracture being determined from the condition that the sum of all waves produced by preceding fractures plus the static solution lead to critical displacements at the corresponding points of the foundation. The final solution (after n "elementary fractures") will have the form:

$$\begin{aligned} &u(x, t)=u^{\text{stat}}(x)+u^{\text{dyn}}(x; t), \\ &u^{\text{dyn}}(x, t)=\sum_{i=0}^{n} u_\Delta^{(i)} L^{(i)}(x-x_i, t-t_i)\eta[c(t-t)-(x-x_i)]. \end{aligned} \tag{6.11}$$

This procedure is evidently extremely complicated and cumbersome in view of the necessity of determining the consecutive values of t_i from transcedental equations of increasing complexity.

Let us now assume that the process is running long enough to become steady; consecutive fractures occur at times $t=0, t, 2t, \ldots, it$ and at points $0, x, 2x, \ldots, ix$; all the functions $L^{(i)}(x, t)=L(x, t)$ and their coefficients $u^{(i)}=u_\Delta$ are equal; and, finally, that the force P_0 is large enough to produce fracture. The ratio $x_\Delta/t_\Delta=v$ is the fracture propagation velocity. A similar approach was proposed by Z. Wesołowski in a recent paper [8].

Under these conditions, the sum in Eq. (6.11) is considerably simplified to:

(6.12) $$u^{\rm dyn}(x, t)=u_\Delta \sum_{i=0}^{k} L(x-ix_\Delta, t-it_\Delta)\eta\,[c(t-it_\Delta)-(x-ix_\Delta)].$$

This formula is valid for $t>kt_\Delta$ and $x>kx_\Delta$.

At any time $t_s=st_\Delta>kt_\Delta$, the wave radiated from the point $x=0$ at $t=0$ has covered the distance ct_s, and the wave produced at the point $x_k=kx_\Delta$ and at time $t_k=kt_\Delta$ reaches the point $\bar{x}=k\dot{v}t_\Delta+c(s-k)t_\Delta$. The displacement $u(\bar{x}, t_s)$ is then equal to:

(6.13) $$u^{\rm dyn}(\bar{x}, t_s)=u_\Delta \sum_{i=0}^{k} L(x-ivt_\Delta, t_s-it_\Delta)$$

(all the corresponding η-functions are equal to 1). The last term of (6.13) has the form:

$$L[c(s-k)t_\Delta, (s-k)t_\Delta].$$

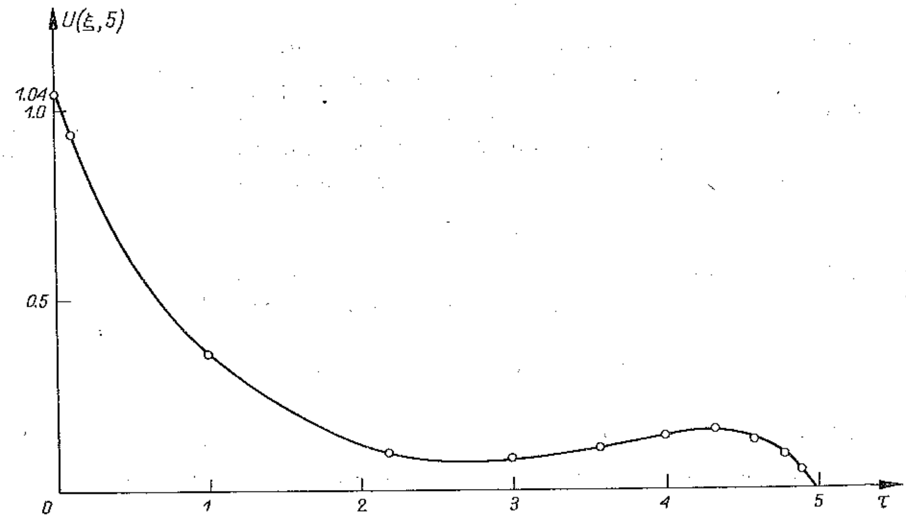

FIG. 17

Reversing the order of summation in Eq. (6.13), we obtain:

$$u^{\rm dyn}(\bar{x}, t_s)=u_\Delta \sum_{i=0}^{k} L[c(s-k)t_\Delta+ivt_\Delta, (s-k)t_\Delta+it_\Delta].$$

Denoting the difference $st_\Delta-kt_\Delta=\bar{t}$, we have:

(6.14) $$u^{\rm dyn}(\bar{x}, \bar{t}+kt_\Delta)=u_\Delta \sum_{i=0}^{k} (c\bar{t}+ivt_\Delta, \bar{t}+it_\Delta).$$

Under the usual continuity conditions, the sum (6.14) may now be replaced by the integral with t_Δ tending to zero:

$$u^{\mathrm{dyn}}(\bar{x}, t_s) = \frac{u_\Delta}{t_\Delta} \int_0^{t_k} L[c(t_s - t_k) + v\theta, (t_s - t_k) + \theta]\, d\theta. \tag{6.15}$$

This formula expresses the displacement measured at the distance $\bar{x} - vt_s$ from the actual position of the "crack tip" as a function of time t_s, elapsed from the beginning of the fracture process. Denoting the distance $\bar{x} - vt_s$ by X (this was the notation used in the analysis of steady-state motion in Sect. 2) and taking into account that $X = \bar{x} - vt_s = (c - v)(t_s - t_k)$, Eq. (6.15) may be rewritten in the form:

$$u^{\mathrm{dyn}}(X, t_s) = \frac{u_\Delta}{t_\Delta} \int_0^{t_s - X/(c-v)} L\left[\frac{cX}{c-v} + v\theta, \frac{X}{c-v} + \theta\right] d\theta. \tag{6.16}$$

Consider now a cross-section of the rod located far from the origin, and sufficiently large values of t_s. With $t_s \to \infty$, we obtain the time-independent formula:

$$u^{\mathrm{dyn}} = u^{\mathrm{dyn}}(X) = \frac{u_\Delta}{t_\Delta} \int_0^{\infty} L\left[\frac{cX}{c-v} + v\theta, \frac{X}{c-v} + \theta\right] d\theta. \tag{6.17}$$

Influence of the static component of the displacement in (6.17) is disregarded, since it decreases exponentially with the coordinate ξ according to the Eq. $(6.4)_2$.

The problem now consists in proper selection of the function $L(x, t)$. The function $U(x, t)$ given by Eq. (6.8) cannot be used here for the reasons already referred to. It may be verified, however, that the function (5.13) derived in Sect. 4.

$$u(\xi, \tau) = U_A J_0(\sqrt{\tau^2 - \xi^2})\, \eta(\tau - \xi), \tag{6.18}$$

solves the problem and leads to the result expected. Substituting the function (6.18) for $L(x, t)$ in Eq. (6.17), we obtain an integral of the type:

$$\Omega = \int_0^{\infty} J_0[\sqrt{A^2\theta(2B + \theta)}]\, d\theta,$$

where $A^2 = (c^2 - v^2)/\kappa^2$ and $B = cX/(c^2 - v^2)$. The integration is performed by means of several substitutions, to yield the result complying with Eq. $(2.16)_1$:

$$\Omega = \frac{1}{A} e^{-AB} = \frac{\kappa}{\sqrt{c^2 - v^2}} e^{-X/\kappa\beta}. \tag{6.19}$$

Comparing the coefficients of Eqs. (6.19) and (2.16), we obtain the required form of the "elementary fracture" wave:

$$u(x, t) = \frac{\sigma_0 c\beta^2}{E} J_0\left[\frac{1}{\kappa}\sqrt{(ct)^2 - x^2}\right] \eta(ct - x). \tag{6.20}$$

The questions as to whether the function (6.20) is the only function leading to the steady-state solution (2.16), and whether it could also be utilized in solving more complicated problems of variable fracture velocities, require further investigations.

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Streszczenie

O JEDNOWYMIAROWYM MODELU PROCESU PĘKANIA

Rozważono zagadnienie wyrywania pręta sprężystego ze sprężystego podłoża za pomocą siły osiowej przyłożonej do swobodnego końca pręta (rys. 1). Mimo maksymalnego uproszczenia modelu (w pręcie występują jedynie siły osiowe, podłoże przenosi tylko naprężenia styczne) okazuje się, że przebieg procesu jest podobny do zjawiska propagacji pęknięć (szczelin) w trójwymiarowych ośrodkach sprężystych. Omówiono możliwe hipotezy pękania, ruch ustalony pręta wyrywanego ze stałą prędkością z podłoża, przypadki nagłego zahamowania i przyspieszania procesu; na zakończenie przedstawiono rozwiązanie przypadku ruchu ustalonego w postaci superpozycji fal wypromieniowanych przez «pęknięcie elementarne».

Резюме

ОБ ОДНОМЕРНОЙ МОДЕЛИ ПРОЦЕССА РАЗРУШЕНИЯ

Рассмотрена задача срыва упругого стержня из упругого основания при помощи осевой силы приложенний к свободному концу стержня (рис. 1). Несмотря на максимальное упрощение модели (в стержне выступают только осевые силы, основание переносит только касательные напряжения) оказывается, что ход процесса аналогичен являнию распространения трещин (щелей) в трехмерных упругих средах. Обсуждены возможные гипотезы разрушения, установившаеся двужение стермня срываемого с постоянной скоростью из основания, случаи внезамного торможения и ускорения процесса; в заключении представлено решение случая установившегося движения в виде суперпозиции волн излучаемых через „элементраное разрушение".

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