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NUMERICAL SOLUTION OF NAVIER-STOKES EQUATIONS IN A TWO-DIMENSIONAL COMPLEX DOMAIN

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The paper is devoted to the numerical solution of the Navier-Stokes equations in the case of plane motion when the area for which we are seeking the solution can be distributed over adjacent to rectangles. For the rectangles one can find the "partial" solutions by the A. A. Dorodnitzyn method, from which one can construct the solution for the whole area. Numerical examples of the application of this method are given.

1. Introduction

In many cases it is easier and faster to find the solution of a complex problem as a limit of the solutions of simpler problems. Sometimes this is the only possible way. When partial differential equations are solved numerically, the relative simplicity of solving second-order equations in standard domains (rectangle, circle) is well known. In this case the already well investigated finite-difference schemes may be used. The numerical solution is reduced to solving three-diagonal linear algebraic systems, where the number of the required arithmetic operations is proportional to the first power of the system's order and not to the third power as in the case of ordinary linear algebraic systems.

For that reason it is desirable to construct the iteration process which is to be used because of the nonlinearity of the Navier-Stokes equations, in such a way that during each step only partial differential equations of second order in standard domains are solved. This can be achieved by making use of the small parameter method proposed by Dorodnitzyn for the numerical solution of equations of mathematical physics [1]. The effectiveness of this approach has already been shown in Dorodnitzyn's and Meller's papers [2, 3]. Numerical results have been obtained for the plane flow of a viscous incompressible fluid in a widening channel and for flow around a semi-infinite plate. To approach simpler problems special boundary conditions were introduced at the rigid wall. In this manner, during each iteration the problem is reduced to solving the Helmholtz equation for the vorticity ω and the Poisson equation for the stream function ψ.

It will be shown how this method can be used for solving problems in complex domains.

2. Small parameter representation

Let us assume that it is possible to subdivide a given domain into several simpler domains (e.g.,rectangles). Thus we have the classical problem of transmission: the vector functions $\mathbf{u}=\{u_1, u_2\}$ satisfies the matrix equation $A\mathbf{u}=f$ in each sub-

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domain and the conjugate conditions (of the transmission) at the common boundary (the dashed line in Fig. 1).

Let us assume that the conjugate conditions are of the form

$$D_i u_1 = D_i u_2 , \qquad i=0, 1, 2, \dots, 2r-1 , \tag{2.1}$$

where u_1, u_2 are the solutions, in the subdomains 1 and 2 respectively, $D_i = \partial^i/\partial x^i$ is the i-th normal derivative at the common boundary and $2r$ is the order of the equation. It is natural to try to modify these conditions in such a way that each

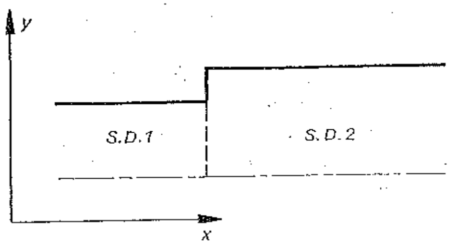

FIG. 1

iteration solves the given problem in each domain separately. This can be achieved by introducing a small parameter ε in the following manner. Instead of the $2r$ conditions of system (2.1) at the common boundary, we introduce r conditions for each subdomain:

s.d. 1

$$D_{2i+1} u_1 = \varepsilon [\alpha_i (D_{2i} u_1 - D_{2i} u_2) + D_{2i+1} u_1] , \tag{2.2}$$

s.d. 2

$$D_{2i+1} u_2 = \varepsilon [\alpha_i (D_{2i} u_1 - D_{2i} u_2) + D_{2i+1} u_2] ,$$

where α_i are relaxation parameters assuring the convergence of the iteration process. The introduction of the parameter ε allows one to seek a solution of the problem in the form of a series in this parameter:

$$u_1 = \sum_{n=0}^{\infty} u_{1,n} \varepsilon^n , \qquad u_2 = \sum_{n=0}^{\infty} u_{2,n} \varepsilon^n . \tag{2.3}$$

If, for the zero-order coefficients of the series, the values $D_{2i+1} u_{1,0}$; $D_{2i+1} u_{2,0}$; $i=0, 1, \dots, r-1$ are taken so that $D_{2i+1} u_{1,0} = D_{i+1} u_{2,0}$, then the equality between $D_{2i+1} u_{1,n}$ and $D_{2i+1} u_{2,n}$, $i=0, 1, \dots, r-1$ will be satisfied for each ε; moreover, at $\varepsilon=1$ the conditions $D_{2i} u_{1,n} = D_{2i} u_{2,n}$; $i=0, 1, \dots, r-1$ will also be satisfied. It is assumed above that the series are convergent, what evidently must be proved.

3. Convergence for the case of the Poisson equation

As an example, we will briefly describe the proof of convergence of the proposed method for second-order equations.

We wish to obtain the solution of the problem

$$\begin{aligned} Au &= f \quad \text{in} \quad G , \\ B_i u &= \varphi_i \quad \text{of} \quad \Gamma_i , \end{aligned} \tag{3.1}$$

where A is a second-order differential operator, B_i are differential operators which can be different on various parts of the boundary Γ_i. We assume that the two-dimensional domain G consists of two rectangles (see Fig. 1).

We introduce an additional boundary γ which divides the domain G into two rectangles G_i; $i=1,2$. A necessary and sufficient condition for representing the solution $u(x,y)$ of the problem (3.1) as a vector function

$$\mathbf{u}(x,y)=\begin{cases} u_1(x,y), & (x,y)\in\bar{G}_1, \\ u_2(x,y), & (x,y)\in\bar{G}_2, \end{cases}$$

where u_1, u_2 are solution of Eqs. (3.1) in each subdomain respectively, is the satisfaction of the conjugate conditions at γ

$$u_1=u_2, \tag{3.2}$$

$$\frac{\partial u_1}{\partial x}=\frac{\partial u_2}{\partial x}. \tag{3.3}$$

We obtain the following transmission problem:

$$Au_1=f \quad \text{in} \quad G_1, \qquad Au_2=f \quad \text{in} \quad G_2,$$

$$B_i u_1=\psi_i \quad \text{on} \quad \Gamma_{1i}, \quad i=1,2,4, \qquad B_i u_2=\psi_i \quad \text{on} \quad \Gamma_{2i}, \quad i=1,2,3,4,$$

$$u_{1|\gamma}=u_{2|\gamma},$$

$$\left.\frac{\partial u_1}{\partial x}\right|_\gamma=\left.\frac{\partial u_2}{\partial x}\right|_\gamma.$$

The introduction of a small parameter ε into the conditions of the common boundary γ leads to separate problems as follows:

$$\begin{aligned} &Au_1=f \quad \text{in} \quad G_1, && Au_2=f \quad \text{in} \quad G_2, \\ &B_i u_i=\psi_i \quad \text{on} \quad \Gamma_{1i}, \quad i=1,2,4, && B_i u_2=\psi_i \quad \text{on} \quad \Gamma_{2i}, \quad i=1,2,3,4, \\ &\left.\frac{\partial u_1}{\partial x}\right|_\gamma=h, && \left.\frac{\partial u_2}{\partial x}\right|_\gamma=h, \end{aligned} \tag{3.4}$$

$$h=\varepsilon\left[\alpha(u_1-u_2)+\frac{\partial u_1}{\partial x}\right]_\gamma.$$

The second conjugate condition (3.3) is satisfied for every value of the parameter ε and for $\varepsilon=1$ the first condition (3.2) is also satisfied.

We wish to obtain the solution of the problem in the form of a series

$$u_1(x,y;\varepsilon)=\sum_{n=0}^{\infty}u_1^{(n)}(x,y)\,\varepsilon^n, \qquad u_2(x,y;\varepsilon)=\sum_{n=0}^{\infty}u_2^{(n)}(x,y)\,\varepsilon^n, \tag{3.5}$$

In order to find the coefficients, we formally substitute Eqs. (3.5) into Eqs. (3.4). Equating the expressions in front of the respective powers of ε, we get

$$Au_1^{(0)}=f \quad \text{in} \quad G_1, \qquad Au_2^{(0)}=f \quad \text{in} \quad G_2,$$

$$B_i u_1^{(0)}=\psi_i \quad \text{on} \quad \Gamma_{1i}, \quad i=1,2,4, \qquad B_i u_2^{(0)}=\psi_i \quad \text{on} \quad \Gamma_{2i}, \quad i=1,2,3,4,$$

$$\left.\frac{\partial u_1^{(0)}}{\partial x}\right|_\gamma=0, \qquad \left.\frac{\partial u_2^{(0)}}{\partial x}\right|_\gamma=0.$$

For u_1^n and u_2^n the problems have the following form:

$$\text{(3.5')} \quad \begin{aligned} &Au_1^{(n)}=0 \quad \text{in} \quad G_1, && Au_2^{(n)}=0 \quad \text{in} \quad G_2,\\ &B_i u_1^{(n)}=0 \quad \text{on} \quad \Gamma_{1i}, \quad 1,2,4, && B_i u_2^{(n)}=0 \quad \text{on} \quad \Gamma_{2i}, \quad i=1,2,3,4,\\ &\left.\frac{\partial u_1^{(n)}}{\partial x}\right|_\gamma=h^{(n)}, && \left.\frac{\partial u_2^{(n)}}{\partial x}\right|_\gamma=h^{(n)}, \end{aligned}$$

where

$$h^{(n)}=\alpha\,(u_1^{(n-1)}-u_2^{(n-1)})+\frac{\partial u_1^{(n-1)}}{\partial x}.$$

Therefore, knowing the preceding terms, we can determine every following one in the series (3.5).

Let us designate as R_1 and R_2 the operators of each function h, determined at γ, which are compared with the solutions of the problems (3.5) respectively, e.g., $u_1=R_1\,h$, $u_2=R_2\,h$. The values of the solutions u_i, $i=1,2$, at the common boundary γ will be designated as $\bar{u}_i$, $i=1,2$. The modified boundary conditions can then be expressed as

$$h^{(s)}=\alpha\,(\bar{u}_1^{(s-1)}-\bar{u}_2^{(s-1)})+h^{(s-1)},$$

or

$$h^{(s)}=\alpha\,(\bar{u}_1^{(s-1)}-\bar{u}_2^{(s-1)})+R_1^{-1}\,u_1^{(s-1)},$$

$$h^{(s)}=\alpha\,(\bar{u}_1^{(s-1)}-\bar{u}_2^{(s-1)})+R_2^{-1}\,u_2^{(s-1)}.$$

Thus for the solutions u_i we have the following relations:

$$u_1^{(s)}=R_1\,h^{(s)}=\alpha R_1\,(\bar{u}_1^{(s-1)}-\bar{u}_2^{(s-1)})+u_1^{(s-1)},$$

$$u_2^{(s)}=R_2\,h^{(s)}=\alpha R_2\,(\bar{u}_1^{(s-1)}-\bar{u}_2^{(s-1)})+u_2^{(s-1)}, \qquad s=2,3,\ldots,$$

$$u_1^{(1)}=\alpha R_1\,(\bar{u}_1^{(0)}-\bar{u}_2^{(0)}),$$

$$u_2^{(1)}=\alpha R_2\,(\bar{u}_1^{(0)}-\bar{u}_2^{(0)}).$$

Then, by induction

$$u_1^{(s)}=\alpha R_1\sum_{k=0}^{s-1}(\bar{u}_1^{(k)}-\bar{u}_2^{(k)}),$$

$$u_2^{(s)}=\alpha R_2\sum_{k=0}^{s-1}(\bar{u}_1^{(k)}-\bar{u}_2^{(k)})$$

we substitute the expressions obtained for the coefficients $u_1^{(s)}$, $u_2^{(s)}$ $i=1, 2$ in (3.5)

$$u_1=\sum_{n=0}^{\infty} u_1^{(n)}\,\varepsilon^n=u_1^{(0)}+\alpha R_1 \sum_{n=0}^{\infty}\left[\sum_{n=0}^{n-1}(\bar{u}_1^{(k)}-\bar{u}_2^{(k)})\right]\varepsilon^n,$$

$$u_2=\sum_{n=0}^{\infty} u_2^{(n)}\,\varepsilon^n=u_2^{(0)}+\alpha R_2 \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n-1}(\bar{u}_1^{(k)}-\bar{u}_2^{(k)})\right]\varepsilon^n.$$

From the recurrence relations (3.6) for the differences $\bar{u}_1^{(s)}-\bar{u}_2^{(s)}$, $s=1, 2, \ldots$, we get

$$\bar{u}_1^{(s)}-\bar{u}_2^{(s)}=[I+\alpha\,(R_1-R_2)]\,(\bar{u}_1^{(s-1)}-\bar{u}_2^{(s-1)}),$$

$$\bar{u}_1^{(1)}-\bar{u}_2^{(1)}=\alpha\,(R_1-R_2)\,(\bar{u}_1^{(0)}-\bar{u}_2^{(0)})$$

or, inductively,

$$\bar{u}_1^{(s)}-\bar{u}_2^{(s)}=T^{s-1}\,(\bar{u}_1^{(1)}-\bar{u}_1^{(1)}),$$

$$\bar{u}_1^{(1)}-\bar{u}_2^{(1)}=\alpha\,(R_1-R_2)\,(\bar{u}_1^{(0)}-\bar{u}_2^{(0)}),$$

where we have introduced the operator $T=I+\alpha\,(R_1-R_2)$. Summing over to k, we obtain for the differences of the s partial sums of the series (3.5) at the common boundary with $\varepsilon=1$, we get

$$\sum_{k=0}^{s}(\bar{u}_1^{(k)}-\bar{u}_2^{(k)})=T^s\,(\bar{u}_1^{(0)}-\bar{u}_2^{(0)}).$$

In this manner the solutions have been represented in the form of infinite series depending upon the initial approximation

$$u_1=u_1^{(0)}+\alpha R_1\,(I+\varepsilon T+\varepsilon^2\,T^2+\ldots)\,(\bar{u}_1^{(0)}-\bar{u}_2^{(0)}),$$

$$u_2=u_2^{(0)}+\alpha R_2\,(I+\varepsilon T+\varepsilon^2\,T^2+\ldots)\,(\bar{u}_1^{(0)}-\bar{u}_2^{(0)}).$$

The convergence of these series depends upon the behaviour of the operator T.

So far we have not used the specific form of the domain G. If $A=\Delta$, i.e., if we solve the Poisson equation in a domain G composed of two rectangles, it is easy to prove the convergence of the iteration process. For a sufficiently small negative value of the norm of the operator T in L_2-space is smaller than 1 and the series (3.7) are convergent. The value of the parameter α depends upon the dimensions of the rectangles and has the form

$$-\pi\sqrt{2}\Bigg/\left((b_1\,\mathrm{cth}\frac{2\sqrt{2}\,\pi a_1}{\sqrt{16b_1^2+\pi^2\,a_1^2}}+b_2\,\mathrm{cth}\,\frac{2\sqrt{2}\,\pi a_2}{\sqrt{16b_2^2+\pi^2\,a_2^2}}\right)<\alpha<0,$$

where a_i are the dimensions in the x direction, b_i are the dimensions in the y direction.

4. Formulation of the problem in terms of Navier-Stokes equations

The proved convergence for the simple model equation — the Laplace equation — shows that we may also hope to use the proposed method successfully in more complex cases of nonlinear systems.

Let us consider the Navier-Stokes equation for plane flows of an incompressible viscous fluid in a channel with a cavity (Fig. 2) written in dimensionless cooridnates:

$$\Delta\Delta\psi=2\left(\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}\right). \tag{4.1}$$

This is an equation of fourth order for the stream function ψ in a complex domain. The channel walls are parallel to the coordinates axes and extend to infinity in the x-direction. We assume the channel width at infinity H_1 ($H_1=H_3$) to correspond to the volume flow rate ψ_0. The channel has a horizontal axis of symmetry and therefore we consider its upper half only. Along this axis the boundary conditions have the form

$$\psi=0, \qquad \Delta\psi=0. \tag{4.2}$$

At the rigid walls the conditions for the stream function are given as

$$\psi=\text{const.}, \qquad \frac{\partial\psi}{\partial\mathbf{n}}=0 \tag{4.3}$$

and at infinity

$$\frac{\partial\psi}{\partial x}\to 0, \qquad \frac{\partial\Delta\psi}{\partial x}\to 0. \tag{4.4}$$

We will construct an iteration procedure for determining the solution of the Navier-Stokes equation as a limit of the solutions of second-order equations for simple domains.

Introducing the vorticity $\omega=\Delta\psi$ we can write Eq. (4.1) as a system of two-second--order equations:

$$\Delta\omega=2\left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right), \qquad \Delta\psi=\omega. \tag{4.5}$$

The complete reduction of this system to separate second-order equations is hindered by the boundary conditions at the rigid walls, Eqs. (4.3). There is no boundary condition for the vorticity ω. Leaving one of the conditions in Eqs. (4.3) unchanged, we introduce into the other one a small parameter ε. Therefore, two types of boundary conditions are possible:

a) the condition for wall impermeability is unchanged $\psi=\text{const.}$,

$$\omega=\varepsilon\left[\mu\frac{\partial\psi}{\partial\mathbf{n}}+\omega\right];$$

b) the condition for no slip is unchanged

$$\frac{\partial\psi}{\partial\mathbf{n}}=0;$$

$$\frac{\partial\omega}{\partial\mathbf{n}}=\varepsilon\left[\beta(\psi-\psi_0)+\frac{\partial\omega}{\partial\mathbf{n}}\right].$$

Here μ and β are relaxation parameters introduced to assure the convergence of the iteration process.

We have assigned boundary conditions of the type a) at the rigid horizontal walls Γ_{i4}, $i=1, 2, 3$, and of the type b) at the rigid vertical walls Γ_{2j}, $j=1, 3$ (Fig. 2).

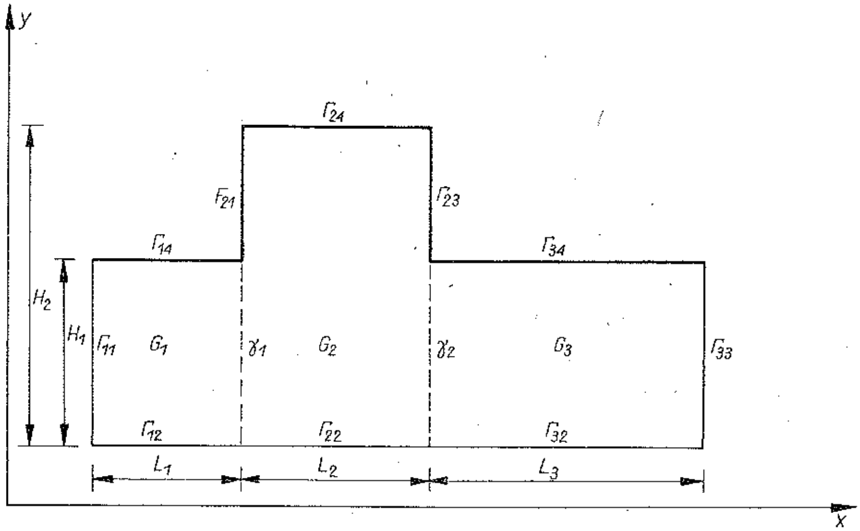

FIG. 2.

The next step toward the simplification of the intermediate problems was solving the problems in simpler domains.

We divide domain G into three rectangles (see Fig. 2). Such a division provides two additional boundaries — sew-up lines — γ_1 and γ_2, along which are satisfied the conjugate conditions

$$\psi_L=\psi_R, \qquad \omega_L=\omega_R,$$

$$\left(\frac{\partial\psi}{\partial x}\right)_L=\left(\frac{\partial\psi}{\partial x}\right)_R, \qquad \left(\frac{\partial\omega}{\partial x}\right)_L=\left(\frac{\partial\omega}{\partial x}\right)_R,$$

where the indices L and R designate the function's boundary values assigned to the left and to the right of the sew-up lines. By introducing the small parameter ε, the system of four conditions is replaced by new boundary conditions replaced in each subdomain:

$$\left(\frac{\partial\psi}{\partial x}\right)_L=\left(\frac{\partial\psi}{\partial x}\right)_R=\varepsilon\left[\alpha\,(\psi_L-\psi_R)+\frac{\partial\psi}{\partial x}\right],$$

$$\left(\frac{\partial\omega}{\partial x}\right)_L=\left(\frac{\partial\omega}{\partial x}\right)_R=\varepsilon\left[\alpha\,(\omega_L-\omega_R)+\frac{\partial\omega}{\partial x}\right],$$

where α is the relaxation parameter.

For $\varepsilon=0$ the boundary conditions are expressed in the following form: at the rigid horizontal walls

$$\psi=\psi_0, \qquad \omega=0,$$

at the rigid vertical walls

$$\frac{\partial\psi}{\partial x}=0, \qquad \frac{\partial\omega}{\partial x}=0,$$

at the sew-up lines

$$\left(\frac{\partial\psi}{\partial x}\right)_L=\left(\frac{\partial\psi}{\partial x}\right)_R=0, \qquad \left(\frac{\partial\omega}{\partial x}\right)_L=\left(\frac{\partial\omega}{\partial x}\right)_R=0.$$

The solutions to the problems for „ideal" flow are $\psi_i=\frac{\psi_0}{H_i}y$, $i=1, 2, 3$ in each of the rectangles G_1, which we also took as an initial approximation. At $\varepsilon=1$ we return to the basic problem.

The introduction of the parameter ε makes it possible to use the method of representing the solution as a series in ε or the method of successive approximations. The second method is more convenient for a numerical solution. Substituting for the stream function ψ in the system (4.5) the complementary stream function $\varphi_i=\psi_i-\frac{\psi_0}{H_i}y$ and then setting $\varepsilon=1$ in the boundary conditions, we obtain the following system of equations for the $(n+1)$th iteration for φ_s and ω_s:

(4.6)
$$\Delta\omega_s^{(n+1)}-2\frac{\psi_0}{H_s}\frac{\partial\omega_s^{(n+1)}}{\partial x}=2\left(\frac{\partial\varphi_s^{(n)}}{\partial y}\frac{\partial\omega_s^{(n)}}{\partial x}-\frac{\partial\varphi_s^{(n)}}{\partial x}\frac{\partial\omega_s^{(n)}}{\partial y}\right),$$
$$\Delta\varphi_s^{(n+1)}=\omega_s^{(n+1)}.$$

with the following boundary conditions:

at the rigid horizontal walls Γ_{s4}, $s=1, 2, 3$:

$$\varphi_s^{(n+1)}=0, \qquad \omega_s^{(n+1)}=\mu_s\left(\frac{\psi_0}{H_s}+\frac{\partial\varphi_s^{(n)}}{\partial y}\right)+\omega_s^{(n)};$$

at the rigid vertical walls Γ_{2i}, $i=1, 3$:

(4.7)
$$\frac{\partial\varphi_2^{(n+1)}}{\partial x}=0, \qquad \frac{\partial\omega_2^{(n+1)}}{\partial x}=\beta_i\left(\frac{\psi_0}{H_2}y-\psi_0+\varphi_2^{(n)}\right)+\frac{\partial\omega_2^{(n)}}{\partial x};$$

at the sew-up lines γ_s, $s=1, 2$:

$$\left.\frac{\partial\varphi_2^{(n+1)}}{\partial x}\right|_{\gamma s}=\alpha_s\left[\left(\frac{\psi_0}{H_2}-1\right)y+\varphi_{s+1}^{(n)}-\varphi_s^{(n)}\right]+\frac{\partial\varphi_2^{(n)}}{\partial x},$$

$$\left.\frac{\partial\omega_2^{(n+1)}}{\partial x}\right|_{\gamma s}=\alpha_s\,(\omega_{s+1}^{(n)}-\omega_s^{(n)})+\frac{\partial\omega_2^{(n)}}{\partial x},$$

where $s=1, 2, 3$ is the number of the respective rectangle. Here we have not written the boundary conditions (of the symmetry axis and at infinity) which are independent of ε.

Thus in each of the rectangles G_i we have to solve the classical equations: the Helmholtz equation for $\omega_s^{(n)}$ and the Poisson equation for $\varphi_s^{(n)}$. Everywhere we have boundary conditions of the same type: at the horizontal boundaries the values of the functions are assigned and at the vertical boundaries the values of the first derivatives. Therefore it is possible to organize a standard computing procedure for each subdomain.

5. Difference equations

Let us introduce in the domain G a difference net so that the straight lines forming the boundary Γ belong to the family of lines which form the net. The whole interval along $x(L=L_1 UL_2 UL_3)$ is divided into N equal parts and y (H_2), into M equal parts. Thus for every rectangle G_s we get $\{x_1=ih,\ i=0, 1, ..., N_s,\ y_j=jk,\ j=0, 1, ..., M\}$ (relative to the lower left angle to G_s). The derivatives in the basic system of equations (4.6) are approximated to second degree accuracy on a five-point „cross" templet. For the first derivatives in the vertical boundary conditions we use one-sided difference approximations to first degree accuracy. When approximating the boundary conditions for ω at the rigid horizontal wall, one should not neglect the second derivatives $\partial^2 \varphi/\partial y^2$, because they are included in the formula determining the vorticity. Therefore, an asymmetric one-sided approximation of second-order accuracy was employed for the first derivative $\partial\varphi/\partial y$:

$$\left(\frac{\partial\varphi}{\partial y}\right)_{i,M} = \frac{4\varphi_{i,M-1}-\varphi_{i,M-2}-3\varphi_{i,M}}{2k} + O(k^2).$$

Let us designate by $\mathbf{\Omega}_i$ and $\mathbf{\Phi}_i$ the vectors which have as components the values $\omega_s^{(n+1)}$ and $\varphi_s^{(n+1)}$ on the line $x=x_i=ih$, $i=0, 1, ..., N_s$, $s=1, 2, 3$:

$$\mathbf{\Omega}_i=\{\omega_s^{(n+1)}(x_i, y_1), \omega_s^{(n+1)}(x_i, y_2), ..., \omega_s^{(n+1)}(x_i, y_{M_s-1})\},$$

$$\mathbf{\Phi}_i=\{\varphi_s^{(n+1)}(x_i, y_1), \varphi_s^{(n+1)}(x_i, y_2), ..., \varphi_s^{(n+1)}(x_i, y_{M_s-1})\}.$$

We have not included here the values $\omega_s^{(n+1)}$ and $\varphi_s^{(n+1)}$ at $y=O_s$ and $y=H$ because they are already given. Using the vectors $\mathbf{\Omega}_i$ and $\mathbf{\Phi}_i$ the difference equations are expressed in vector form

$$\begin{aligned}(1-\sigma h)\,\mathbf{\Omega}_{i+1}+B\mathbf{\Omega}_i+(1+\sigma h)\,\mathbf{\Omega}_{i-1}&=\mathbf{F}_i,\\ \mathbf{\Omega}_{i+1}+B\mathbf{\Phi}_i+\mathbf{\Phi}_{i-1}&=h^2\,\mathbf{\Omega}_i,\end{aligned} \tag{5.1}$$

where $\sigma=\psi_0/H_s$, B is a matrix of (M_s-1)-th order which depends only upon the steps of the net, and $\mathbf{F}_i$ is a vector defined by the solutions of the preceding iteration.

To these relations we add the interconnections between the terminal vector values which are derived from the boundary conditions (4.7):

$$\begin{aligned}\mathbf{\Omega}_0&=\mathbf{\Omega}_1-h\,\mathbf{P}_{0_s}, &\quad \mathbf{\Omega}_{N_s-1}&=\mathbf{\Omega}_{N_s}+h\,\mathbf{P}_{N_s},\\ \mathbf{\Phi}_0&=\mathbf{\Phi}_1-h\,\mathbf{Q}_{0_s}, &\quad \mathbf{\Phi}_{N_s-1}&=\mathbf{\Phi}_{N_s}+h\,\mathbf{Q}_{N_s}.\end{aligned} \tag{5.2}$$

Only three adjacent vector values are interconnected in the difference relations (5.1) and (5.2). Therefore it is convenient to solve the system using the matrix factorization method, initially for the first equation and then for the second one.

6. Numerical convergence of the iteration procedure

The convergence of the iteration procedure for the sewing-up of domains when using the Navier-Stokes equation was numerically demonstrated for the following simple problem. We consider an abruptly widening channel (Fig. 1) of a width of $2\psi = 4$, which corresponds to a Reynolds number of 8. The difference in width at $+\infty$ and $-\infty$ is equal to the step-size of the difference net. Here we have only two relaxation parameters μ_1 and α_1. The investigations of A. A. Dorodnitzyn and N. A. Meller have shown that the optimal value of the parameter μ_1 is $-3/\psi_0$. Figure 3 shows the convergence of the method $(\psi_1 - -\psi_2)_y \to 0$ in the mid-point of the sew-up line at $\alpha_1 = 0.1$ and $\alpha_1 = 0.5$. It is evident that at $\alpha = 0.5$ the convergence is fast and oscillatory and at $\alpha_1 = 0.1$ it is slow and one-sided.

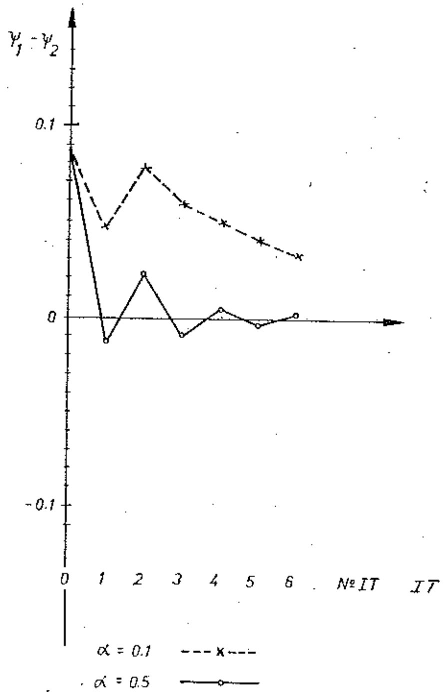

Fig. 3

7. Numerical results

For the numerical solution of the basic problem the flow of a viscous incompressible fluid in a channel with a cavity, it was necessary to choose six relaxation parameters. In the case of a square cavity (with a width to depth ratio of $\lambda = 1$) and Re$=8$, various groups of parameters were investigated to achieve fast convergence of an oscillatory character. After a sufficiently large number of iterations, the iteration process became divergent (Version 4, Table 1), which testifies to the strong interdependence between the parameters. Slowest convergence was obtained at the rigid vertical walls; this evidently is connected with a large change in the cavity flow pattern with respect to the initial approximation. Such a convergence behaviour of the iteration process might also be explained by the specific form of the boundary conditions assigned to the rigid vertical walls.

Version 5 (Table 1) was used as a base to investigate the behaviour of the parameters as a function of the Reynolds number and the relative depth of the cavity. The groups of parameters for which convergence was obtained for various λ are given in Table 2.

Table 1

Version	Re	μ_1	μ_2	α_1	α_2	β_1	β_2	
1	8	-1.5	-0.375	0.3	0.8	-0.5	-0.5	Divergence
2				0.3	0.3	-0.5	-0.5	Divergence
3				0.1	0.3	-0.5	-0.5	
4				0.2	0.3	-0.5	-0.5	Divergence
5				0.2	0.2	-0.3	-0.3	Convergence
6				0.2	0.3	-0.3	-0.3	Convergence
7	16	-0.75	-0.18	0.1	0.15	-0.0375	-0.0375	Divergence
8				0.05	0.075	-0.018	-0.018	Divergence
9						-0.0047	-0.0047	Convergence

Table 2

λ	μ_1	μ_2	α_1	α_2	β_1	β_2
0.5	-1.5	-0.67	0.4	0.6	-0.35	-0.53
1.0	-1.5	-0.375	0.2	0.3	-0.3	-0.3
2.0	-1.5	-0.17	0.1	0.15	-0.02	-0.03

Figures 4–6 show the streamlines. The values assigned to the stream function ψ are related are the results obtained from the machine computations by the relaxation $\psi = (\psi - \psi_0)10^4$. For comparison, in the upper part of Figs. 4 and 5 are

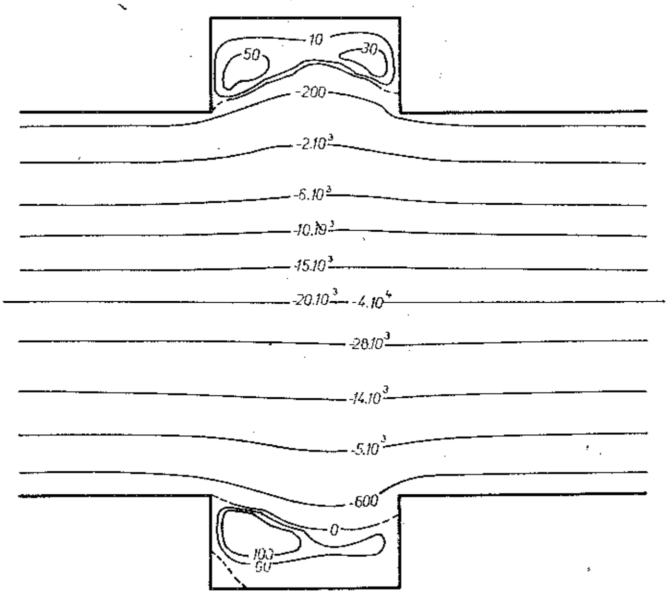

FIG. 4

shown the flows for Re$=8$ and in the lower part for Re$=16$. The presence of dead-water regions is common to all flow patterns. For a shallow cavity $\lambda=0.5$ and Re$=8$, the fluid flow in the cavity occurs along closed stream-lines and internal

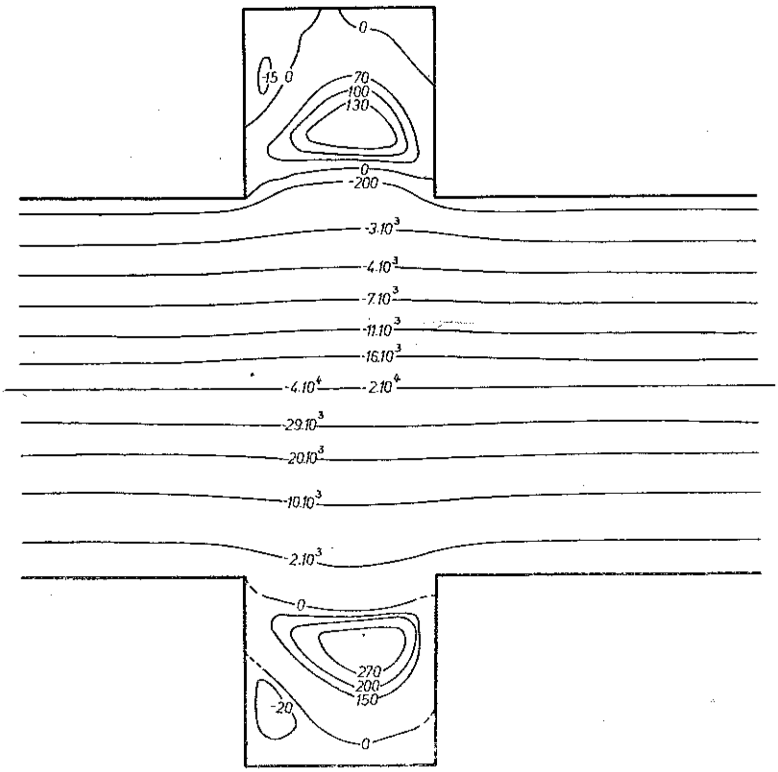
0
-15
0
70
100
130
0
-200
-3.10³
-4.10³
-7.10³
-11.10³
-16.10³
-4.10⁴
-2.10⁴
-29.10³
-20.10³
-10.10³
-2.10³
0
270
200
150
-20
0

Fig. 5

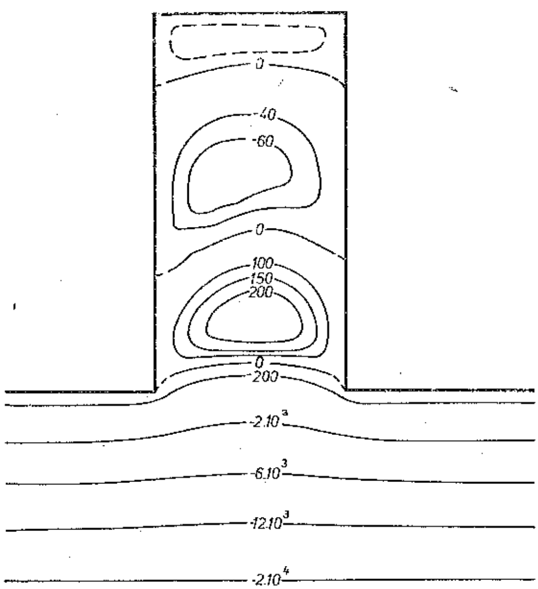
0
-40
-60
0
100
150
200
0
-200
-2.10³
-6.10³
-12.10³
-2.10⁴

Fig. 6

domains of smaller recirculatory regions can be observed. An increase in the Reynolds number makes the vortex intensity rise and in the left caved-in corner the stream function changes sign; this shows that a secondary vortex has occurred.

Figure 5 shows the constant value curves of the stream function in a square cavity for Re=8 and Re=16. The changes in the flow patterns due to an increase of the Reynolds number are as follows: 1) the vortex intensity is increased, 2) the centre moves in the direction of the flow, 3) a secondary vortex occurs.

It should be noted that an increase in the Reynolds number to twice its value led to a corresponding decrease in the convergence velocity of the iteration process to one-and-a-half time the value.

The number of dead-water regions depends on the relative cavity depth λ. In the case of the maximum possible depth $\lambda=2$ (Fig. 6) the vortex intensity decreases at the deeper part of the cavity. This reduction of the intensity can evidently be explained as a consequence of an increased influence of the cavity walls upon the fluid velocity.

The data obtained allow one to conclude that the method under consideration can, in principle, be used for the numerical solution of problems in complex domains.

The numerical results presented in this paper were obtained on the BESM-6 computer in the Computing Center of the USSR Academy of Sciences.

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Streszczenie

NUMERYCZNE ROZWIĄZANIE RÓWNAŃ NAVIERA-STOKESA W PRZYPADKU PEWNEGO OBSZARU DWUWYMIAROWEGO

Praca jest poświęcona numerycznemu rozwiązaniu równań Naviera-Stokesa w przypadku ruchu płaskiego, gdy obszar, dla którego poszukuje się rozwiązania, może być rozłożony na przylegające do siebie prostokąty. Dla prostokątów znajduje się rozwiązania „częściowo" metodą A. A. Dorodnicyna, z których buduje się rozwiązania dla całego obszaru. Podaje się również przykłady zastosowania tej metody.

Резюме

ЧИСЛЕННОЕ РЕШЕНИЕ УРАВНЕНИЙ НАВЬЕ-СТОКСА В СЛОЖНОЙ ДВУХМЕРНОЙ ОБЛАСТИ

Работа посвящена численному решению уравнений Навье-Стокса в случае плоского течения, когда область, для которой ищется решение может быть разделена на смежные друг к другу прямоугольники. Для прямоугольников находятся „частичные" решения методом А. А. Дородницына, из которых строится решение для целой области. Приведены тоже примеры численных расчетов этим методом.

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