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THE PROPAGATION OF ELASTIC WAVES DUE TO THE ACTION OF FLUID SOURCES IN A CONSOLIDATING MEDIUM

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In this study the propagation of elastic waves due to the action of fluid sources in a fluid-saturated porous elastic solid is investigated. The basis of these considerations are the equations of motion formulated by M. A. Biot. In the first part of this work the equations governing the propagation of dilatational waves for the case of the action of fluid sources in such a medium are obtained. In the next section of this paper the action of the point source varying harmonically with time is considered. These considerations are restricted to the lower radian frequency range where the assumption of Poiseuille flow is valid. The fluid source is due to the propagation of two dilatational waves. A detailed discussion of the properties of these waves is presented in the last part of this work.

1. Introduction

In this paper the propagation of elastic waves due to the action of fluid sources in a consolidating medium is investigated. The basis of our considerations are the equations of motion of a fluid-saturated porous elastic solid formulated by M. A. Biot [1]. Using Biot's equations, many problems of wave propagation in such a medium were extensively studied, both theoretically and experimentally (literature in the paper [2]). The equation of fluid flow for the case of the action of fluid sources in the medium was obtained by W. Derski [3]. This equation is valid for quasistatic problems. Some problems of the action of fluid sources in a consolidating medium were solved in the papers [4–7]. To the author's knowledge, the dynamic effects of the action of fluid sources in such a medium have not been considered so far.

2. System of equations of the theory of consolidation

Let us introduce the notations: u_i — the components of the displacement vector of the solid skeleton, U_i — the components of the displacement vector of the fluid, ε_{ij} — the components of the solid skeleton strain tensor, ε — the dilatation of the solid skeleton, θ — the dilatation of the fluid, σ_{ij} — the components of the stress tensor in the solid skeleton, $\sigma = -fp$ — the reduced pressure of the fluid (p is the pressure in the fluid and f is the porosity of the medium).

The equations of motion have the following form [1]:

$$\sigma_{ij,\,j} = \rho_{11}\,\ddot{u}_i + \rho_{12}\,\ddot{U}_i + b\,(\dot{u}_i - \dot{U}_i), \tag{2.1}$$

$$\sigma_{,i} = \rho_{12}\,\ddot{u}_i + \rho_{22}\,\ddot{U}_i - b\,(\dot{u}_i - \dot{U}_i), \tag{2.2}$$

where the parameters ρ_{11}, ρ_{22} and ρ_{12} are the dynamic coefficients which take into account the inertia effect of the moving fluid. These parameters are related to mass densities of the solid ρ_s and the fluid ρ_f by the equations

$$\begin{aligned}\rho_{11}+\rho_{12}&=(1-f)\rho_s,\\ \rho_{12}+\rho_{22}&=f\rho_f\end{aligned}\tag{2.3}$$

and satisfy the inequalities

$$\rho_{11}>0,\quad \rho_{22}>0,\quad \rho_{12}<0,\quad \rho_{11}\rho_{22}-\rho_{12}^2>0,\tag{2.4}$$

where ρ_{12} is the coupling parameter. The physical significance of these parameters has discussed by BIOT [1] and will not be repeated here.

Finally, the parameter b represents the damping coefficient between the fluid and the solid. In analogy to Darcy's equation, the coefficient b may be related to the permeability k, the porosity f, and the fluid viscosity by the equation

$$b=\frac{\mu f^2}{k}.\tag{2.5}$$

Realizing that the viscosity μ may be dependent on the frequency and geometry of the pores, a method for the correction of μ was developed by BIOT [1].

The remaining equations of the theory of consolidation have the following form, [1]:

the geometric relations

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\quad \varepsilon=u_{k,k},\quad \theta=U_{k,k}\tag{2.6}$$

and the physical relations

$$\sigma_{ij}=2N\varepsilon_{ij}+(A\varepsilon+Q\theta)\delta_{ij},\tag{2.7}$$

$$\sigma=Q\varepsilon+R\theta,\tag{2.8}$$

where the parameters A, N, Q, R represent the mechanical properties of the fluid-saturated porous medium. These parameters satisfy the inequalities

$$PR-Q^2>0,\quad P\rho_{22}+R\rho_{11}-2Q\rho_{12}>0.$$

A detailed discussion of these parameters was carried out by BIOT and WILLIS [8].

The equation of fluid flow for the case of the action of fluid source in the medium was obtained by DERSKI [3] in the form

$$\nabla^2\sigma=-b(\dot{\varepsilon}-\dot{\theta}-W).\tag{2.9}$$

W is the discharge of the fluid source. This equation is valid for the quasi-static problems.

Introducing, in accordance with Biot's theory, the inertia terms into Eq. (2.9), and using the physical relations (2.2) and (2.8), we obtain the following equation of fluid flow:

$$Q\nabla^2\varepsilon+R\nabla^2\theta=\rho_{12}\ddot{\varepsilon}+\rho_{22}\ddot{\theta}-b(\dot{\varepsilon}-\dot{\theta}-W).\tag{2.10}$$

This equation is valid for the dynamic case. Applying the divergence operation to Eqs. (2.1) and (2.2), substituting the physical relations and taking into account Eq. (2.10) we get

(2.11)
$$D_{11}\,\varepsilon+D_{12}\,\theta=0,$$
$$D_{12}\,\varepsilon+D_{22}\,\theta=bW,$$

where the operators D_{11}, D_{12} and D_{22} have the form

(2.12)
$$D_{11}=P\nabla^2-\rho_{11}\frac{\partial^2}{\partial t^2}-b\frac{\partial}{\partial t},\qquad P=A+2N,$$
$$D_{12}=Q\nabla^2-\rho_{12}\frac{\partial^2}{\partial t^2}+b\frac{\partial}{\partial t},$$
$$D_{22}=R\nabla^2-\rho_{22}\frac{\partial^2}{\partial t^2}-b\frac{\partial}{\partial t}.$$

Equations (2.11) govern the propagation of dilatational waves for the case of action of the fluid sources in the fluid-saturated porous medium. The uncoupling of the system of Eqs. (2.11) leads to the equations

(2.13)
$$(D_{11}\,D_{22}-D_{12}^2)\,\varepsilon=-bD_{12}\,W,$$

(2.14)
$$(D_{11}\,D_{22}-D_{12}^2)\,\theta=bD_{11}\,W,$$

in which

$$D_{11}\,D_{22}-D_{12}^2=a_1\,\nabla^2\,\nabla^2-\left(a_2\frac{\partial^2}{\partial t^2}+a_3\frac{\partial}{\partial t}\right)\nabla^2+a_4,$$

where

$$a_1=PR-Q^2,\quad a_2=R\rho_{11}+P\rho_{22}-2Q\rho_{12},\quad a_3=b\,(P+2Q+R),$$
$$a_4=\rho_{11}\,\rho_{22}-\rho_{12}^2,\quad a_5=b\,(\rho_{11}+\rho_{22}+2\rho_{12}).$$

3. Point source of fluid in a consolidating space

Let us consider the action of a point source of fluid having an intensity

(3.1)
$$W=W_0\,\delta\,(R)\,e^{i\omega t},\quad R=x_k\,x_k,\quad k=1,2,3$$

varying harmonically with time, where ω is the radian frequency of oscillation. The dilatation of the solid skeleton and the fluid can be written in the form

(3.2)
$$\varepsilon=\varepsilon^*\,(R)\,e^{i\omega t},\quad \Theta=\Theta^*\,(R)\,e^{i\omega t}.$$

Substituting the dilatations (3.2) into Eqs. (2.13) and (2.14), we obtain

(3.3)
$$a_1\,(\nabla^2-\alpha_1^2)\,(\nabla^2-\alpha_2^2)\,\varepsilon^*=-bW_0\,(Q\nabla^2+\rho_{12}\,\omega^2+ib\omega)\,\delta\,(R),$$

(3.4)
$$a_1\,(\nabla^2-\alpha_1^2)\,(\nabla^2-\alpha_2^2)\,\Theta^*=bW_0\,(P\nabla^2+\rho_{11}\,\omega^2-ib\omega)\,\delta\,(R),$$

where α_1 and α_2 are the roots of the equation

$$a_1 \alpha^4 + (a_2 \omega^2 - ia_3 \omega) \alpha^2 + (a_4 \omega^4 + - ia_5 \omega^3) = 0. \tag{3.5}$$

Applying Fourier-Hankel transformations to Eqs. (3.3) and (3.4) and using the relations (3.2), we get

$$\varepsilon = \frac{bW_0}{4\pi a_1 R} \left(Q + \frac{\rho_{12} \omega^2 + ib\omega}{\alpha_1^2 - \alpha_2^2} \right) (e^{-\alpha_1 R} - e^{-\alpha_2 R}) e^{i\omega t}, \tag{3.6}$$

$$\Theta = -\frac{bW_0}{4\pi a_1 R} \left(P + \frac{\rho_{11} \omega^2 + ib\omega}{\alpha_1^2 - \alpha_2^2} \right) (e^{-\alpha_1 R} - e^{-\alpha_2 R}) e^{i\omega t}. \tag{3.7}$$

These formulae show that the fluid source is due to the propagation of two dilatational waves with the phase velocities

$$v_l = \frac{\omega}{Im\alpha_l}, \qquad l = 1, 2. \tag{3.8}$$

Both have a dispersive character and are attenuated.
The attenuation coefficients are

$$\vartheta_l = \mathrm{Re}\alpha_l. \tag{3.9}$$

A detailed discussion of the properties of these waves is presented in the next part of this work.

4. Results of numerical calculations

In order to consider the properties of the two dilatational waves, numerical calculations were performed. The values of the material constants for kerosene-saturated sandstone were taken from data given by I. Fatt [10] and H. Deresiewicz, J. T. Rice [9]. The following data were used:

$$A = 0.44492 \cdot 10^{10}\ \mathrm{N/m^2} \qquad N = 0.27551 \cdot 10^{10}\ \mathrm{N/m^2}, \qquad Q = 0.07432 \cdot 10^{10}\ \mathrm{N/m^2}.$$
$$R = 0.03261 \cdot 10^{10}\ \mathrm{N/m^2}, \qquad \rho_s = 2.60 \cdot 10^3 \mathrm{kg/m^3}, \qquad \rho_f = 0.82 \cdot 10^3\ \mathrm{kg/m^3},$$
$$\mu = 2.00 \cdot 10^{-3}\ \mathrm{Ns/m^2},$$
$$f = 0.26, \qquad k = 7.90 \cdot 10^{-13}\ \mathrm{m^2}.$$

Since no measurement of the dynamical coupling coefficient was available, its values was taken as $\rho_{12} = -0.001$ [9].

We must remember that the equations of motion in the form (2.1) and (2.2) are valid for Poiseuille flow only. This flow takes place for a small Reynolds number [11]. From Biot's considerations [1] it follows that the assumption of Poiseuille flow breaks down if the radian frequency exceeds a certain value

$$\omega_t = \frac{\pi^2 \nu}{2d^2}, \tag{4.1}$$

where $\nu=\mu/\rho_f$ is the kinetic viscosity of the fluid, and d is the diameter of the pores. For the present case we may take $d=10^{-5}$ m [9], and we have $\omega_t=1.2\cdot 10^5$ s^{-1}. It is known that in engineering practice the radian frequency is lower. Therefore the equations of motion in the form (2.1) and (2.2) are considered in this paper.

Let us introduce:

a reference velocity

$$V_c=\left(\frac{H}{\rho}\right)^{\frac{1}{2}}, \qquad H=P+R+2Q, \qquad \rho=(1-f)\,\rho_s+f\rho_f, \tag{4.2}$$

a characteristic radian frequency

$$\omega_c=\frac{b}{f\rho_f}, \tag{4.3}$$

a characteristic length

$$L_c=\frac{2\pi V_c}{\omega_c}, \tag{4.4}$$

and a characteristic attenuation coefficient

$$\vartheta_c=\frac{1}{L_c}. \tag{4.5}$$

Now Eq. (3.7) takes the form

$$B_1\,x^4+\left(B_2\,\omega_0^2-i\frac{B_3}{\omega_c}\,\omega_0\right)x^2+\left(B_4\,\omega_0^4-i\frac{B_5}{\omega_c}\,\omega_0^3\right)=0, \tag{4.6}$$

where

$$x=\frac{\alpha}{\omega_c}, \qquad \omega_0=\frac{\omega}{\omega_c},$$

the phase velocities of the dilatational waves are

$$v_l=\frac{\omega_0}{\operatorname{Im} x_l}, \qquad l=1,2 \tag{4.7}$$

and the attenuation coefficients are

$$\vartheta_l=\omega_c\operatorname{Re} x_l. \tag{4.8}$$

The coefficients B_i in Eq. (4.6) have the form

$$\begin{gathered}
B_1=\kappa_{11}\,\kappa_{22}-\kappa_{12}^2, \qquad B_2=\frac{1}{V_c^2}(\kappa_{11}\,\gamma_{22}+\kappa_{22}\,\gamma_{11}-2\kappa_{12}\,\gamma_{12}),\\
B_3=\frac{b}{H}, \qquad B_4=\frac{1}{V_c^4}(\gamma_{11}\,\gamma_{22}-\gamma_{12}^2), \qquad B_5=\frac{B_3}{V_c^2},\\
\kappa_{11}=\frac{P}{H}, \quad \kappa_{12}=\frac{Q}{H}, \quad \kappa_{22}=\frac{R}{H}, \quad \gamma_{11}=\frac{\rho_{11}}{\rho}, \quad \gamma_{12}=\frac{\rho_{12}}{\rho}, \quad \gamma_{22}=\frac{\rho_{22}}{\rho}.
\end{gathered} \tag{4.9}$$

The dimensionless elastic and dynamic constants defined in Eq. (4.9) were found to be

$$\varkappa_{11}=0.8460, \quad \varkappa_{12}=0.0631, \quad \varkappa_{22}=0.0227,$$

$$\gamma_{11}=0.9012, \quad \gamma_{12}=-0.0010, \quad \gamma_{22}=0.1008.$$

Characteristic quantities are: $V_c=2347$ m/s, $\omega_c=8.0278\cdot 10^5$ s^{-1}, $\vartheta_c=54.44$m^{-1}. The results of the numerical calculations are showed in figures.

Figure 1 illustrates the variation of the dimensionless phase velocities

$$V_0^I=\frac{v_I}{V_c}$$

with the dimensionless radian frequency ω_0. It was mentioned above that our considerations are restricted to the lower frequency range

$$\omega_0\leqslant\frac{\omega_t}{\omega_c}\approx 0.15,$$

where the assumption of Poiseuille flow is valid. Fig. 2 shows the dimensionless attenuation coefficients as functions of the frequency ω_0. We noticed in Section 3 that there are two dilatational waves denoted as waves of the first and the second kind (Biot's terminology). The waves of the first kind are the fast waves (Fig. 1).

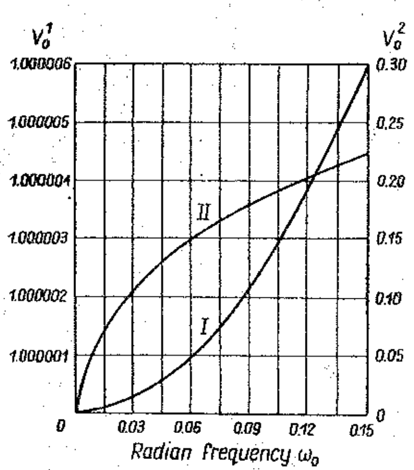

FIG. 1. Dispersion curves of the dilatational waves of the first and second kind.

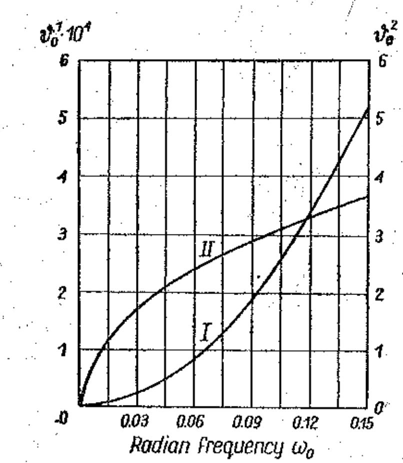

FIG. 2. Attenuation coefficients of the dilatational waves of the first and second kind.

The dispersion of these waves is practically negligible and the attenuation is small. The waves of the second kind (slow wave) are highly dispersive and highly attenuated.

The dilatation of the solid skeleton and the fluid can be described by the following formulae:

$$\varepsilon_l = \frac{bHW_0}{4\pi a_1}(-1)^l A_l(R,\omega_0)\exp\left[i\omega\left(t-\frac{R}{v_l}+\frac{\delta(\omega_0)}{\omega}\right)\right], \tag{4.10}$$

$$\Theta_l = -\frac{bHW_0}{4\pi a_1}(-1)^l \bar{A}_l(R,\omega_0)\exp\left[i\omega\left(t-\frac{R}{v_l}+\frac{\delta(\omega_0)}{\omega}\right)\right]. \tag{4.11}$$

The dimensionless amplitudes of the dilatation of the solid skeleton A_l and the fluid $\bar{A}_l$ are

$$A_l(R,\omega_0)=\xi_{12}(\omega_0)\frac{e^{-R\,\mathrm{Re}\,\alpha_l}}{R}, \tag{4.12}$$

$$\bar{A}_l(R,\omega_0)=\xi_{11}(\omega_0)\frac{e^{-R\,\mathrm{Re}\,\alpha_l}}{R}, \tag{4.13}$$

where

$$\xi_{pq}=|A_{pq}|, \qquad A_{pq}=\varkappa_{pq}+\frac{\dfrac{\rho_{pq}}{H}\omega_0^2+\dfrac{B_3}{\omega_c}\omega_0}{\alpha_1^2-\alpha_2^2}\omega_c^2,$$

$$\delta(\omega_0)=\arg A_{pq}.$$

From the formulae (4.12) and (4.13) it follows that the changes of the amplitudes of both waves are connected with the attenuation which depends on the mechanical parameters of the medium and have an exponential character. These changes

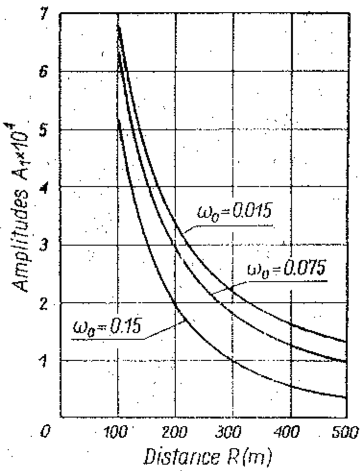

FIG. 3. Amplitude decay of the dilatational wave of the first kind.

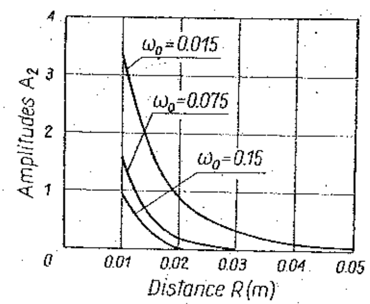

FIG. 4. Amplitude decay of the dilatational wave of the second kind

depend on the geometry of the waves, too. For the spherical waves this dependence is of the type R^{-1}.

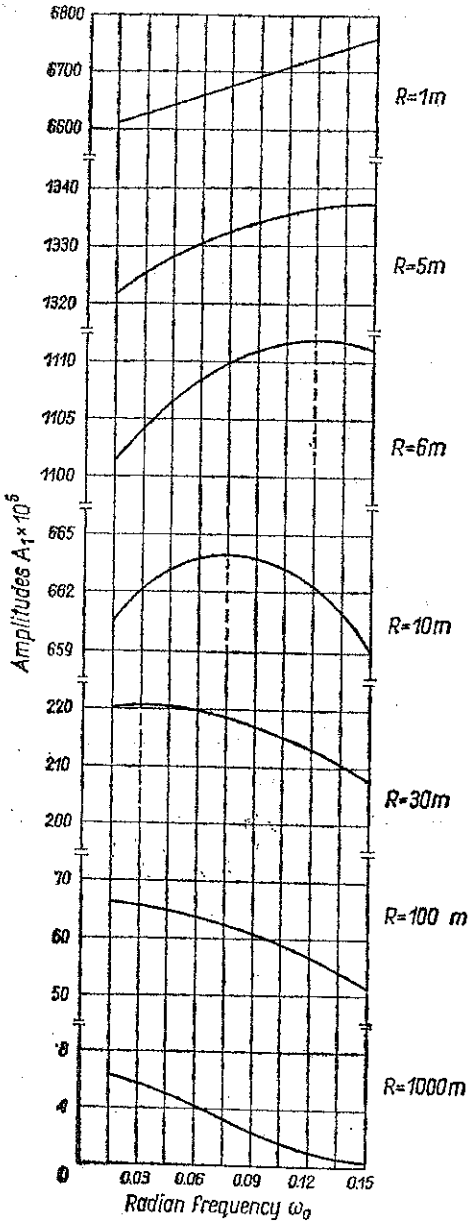

FIG. 5. Amplitudes of the solid skeleton dilatations of the first kind waves as functions of the frequency.

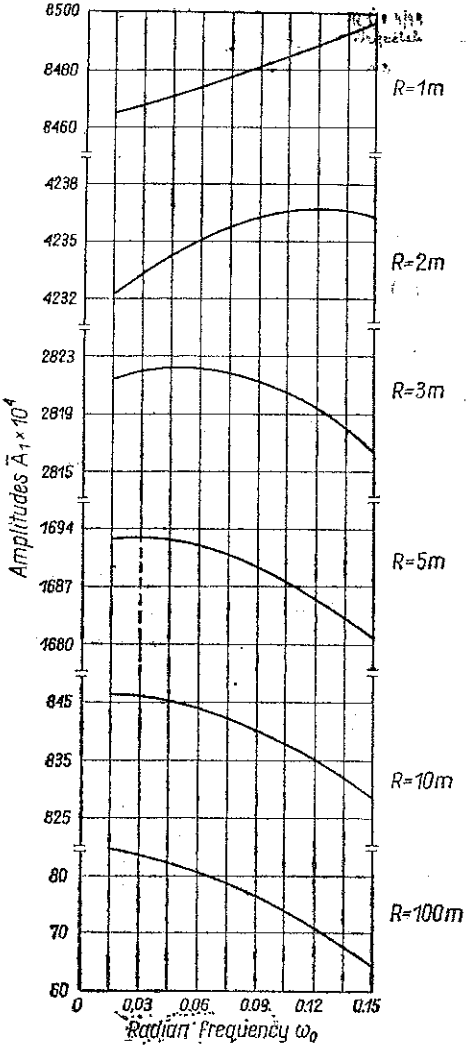

FIG. 6. Amplitudes of the fluid dilatations of the first kind waves as functions of the frequency.

Figures 3 and 4 show the amplitudes A_1 and A_2 as functions of the distance from the fluid source for various frequencies. The amplitude of the slow wave decreases rapidly with the distance (Fig. 4) and this is caused by high attenuation of this wave.

The curves in figs. 5 and 6 show the change of amplitude of the first kind of waves with the radian frequency ω_0 at various distances from the source. The monotonic increase of the amplitude of these waves is observed for small distances from sources. The curves have the maximum for the larger distances and this maxi-

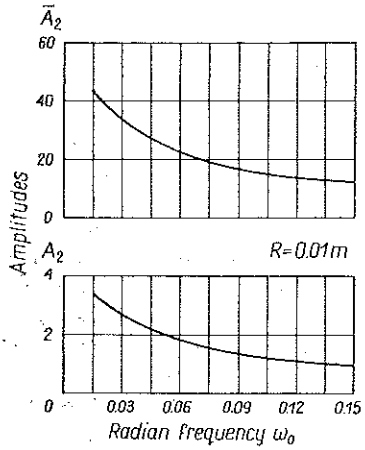

FIG. 7. Amplitudes of the fluid and solid skeleton dilatations of the second kind waves as function of the frequency.

mum moves in the direction of lower frequencies if the distance R increases. For large distances the amplitude decreases with an increase of frequency. The amplitudes of the second kind wave in the solid skeleton and in the fluid decrease rapidly with an increase of frequency (Fig. 7). This is caused by high attenuation of this wave.

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Streszczenie

PROPAGACJA FAL SPRĘŻYSTYCH WYWOŁANYCH DZIAŁANIEM ŹRÓDŁA CIECZY W OŚRODKU KONSOLIDUJĄCYM

W pracy rozpatrzono propagację fal sprężystych wywołanych działaniem źródła cieczy w ośrodku porowatym o sprężystym szkielecie nasyconym cieczą. Podstawę rozważań stanowią równania ruchu sformułowane przez M. A. Biota. W pierwszej części pracy podano równania propagacji fal sprężystych dla przypadku działania źródła cieczy w tego rodzaju ośrodku. Następnie rozpatrzono działanie źródła punktowego, harmonicznie zmiennego w czasie. Rozważania są poprawne dla niższych częstości, przy których pozostaje prawidłowe założenie o przepływie Poiseuille'a. Źródło cieczy powoduje propagację dwóch fal dylatacyjnych. Szczegółową dyskusję ich własności przedstawiono w ostatniej części pracy.

Резюме

РАСПРОСТРАНЕНИЕ УПРУГИХ ВОЛН ВЫЗВАННЫХ ДЕЙСТВИЕМ ИСТОЧНИКА ЖИДКОСТИ В КОНСОЛИДИРУЮЩЕЙ СРЕДЕ

В работе рассмотрено распространение упругих волн, вызванных действием источника жидкости в пористой среде с упругим скелетом, насыщенной жидкостью. Основу рассуждений составляют уравнения движения, сфорумлированные М. А. Биотом. В первой части работы приведены уравнения распространения упругих волн для случая действия источника жидкости в этого рода среде. Затем рассмотрено действие точечного источника, гармонически переменного во времени. Рассуждения справедливы для низших частот, при которых остается справедливым предположение о течении Пуазейля. Источник жидкости вызывает распространение двух дилатоционных волн. Подробное обсуждение их свойств представлено в последней части работы.

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