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# MAGNETOHYDRODYNAMIC FLOW PAST A SPHERE

S. SWARUP and P. C. SINHA (NEW DELHI)

This paper deals with the slow steady flow of an incompressible, viscous, electrically-conducting fluid past a sphere in the presence of a uniform magnetic field. The magnetic field is weak and is aligned along the free-stream direction. The solution is sought by the method of matched asymptotic expansions under the assumptions that the magnetic pressure number $\beta = O(1)$ and $R_m \ll 1$, $R \ll 1$.

## 1. Introduction

The MHD flow past a sphere at small Reynolds number has been the subject of study for many past years. The first attempt in this direction was made by Chester [3] who obtained a solution for the Stokes flow neglecting the disturbances of the magnetic field and the inertia of the fluid. Assuming that the Reynolds number $R$ and the magnetic Reynolds number $R_m$ are small as compared to the Hartmann number $M$, the author found the drag on a sphere correct to the first order to be

$$D = D_S\left(l + \frac{3}{8} M\right),$$

$D_S$ being the Stokes drag. Chang [2] generalized this to include any body of revolution aligned with its axis parallel to the uniform stream at infinity.

Ludford [10] and Gotoh [6, 7] took into account the effects neglected by Chester by using an Oseen type approximation in which the quadratic terms in the disturbance quantities are neglected in the equation of motion. Chester [4] restudied the problem for a three-dimensional body and gave a general criterion for the use of Oseen's equation as an approximate representation of the full Navier-Stokes equation. Further contribution on the subject have been made by Bois [1], Levy [9] and Cabannes [13].

In the present paper the slow, steady flow of an incompressible, viscous, electrically-conducting fluid past a sphere in the presence of a uniform magnetic field is studied by using the method of matched asymptotic expansions. The applied magnetic field is weak and is aligned along the free stream direction. The basic assumptions made here are that the magnetic pressure number $\beta = (M^2/RR_m)$ is of $O(1)$ and $R_m \ll 1$, $R \ll 1$. The solution requires the matching of two asymptotic expansions for each quantity, one of them valid near the body and the other far from it.

## 2. Mathematical formulation

Consider the steady motion of an incompressible, viscous, electrically-conducting fluid past a sphere of radius $a$ in the presence of a uniform magnetic field parallel to the undisturbed flow. Let the free stream velocity be $U\hat{\imath}$ so that $H_\infty \hat{\imath}$ is the magnetic field at infinity. The governing equations for an axi-symmetric flow, in dimensionless form, are determined as follows:

$$
\begin{aligned}
&R(\mathbf{q}\cdot\nabla)\mathbf{q} = -\nabla p + \nabla^2\mathbf{q} + \beta R(\nabla\times\mathbf{H})\times\mathbf{H},\\
&\nabla\cdot\mathbf{q} = 0,\\
&\nabla\times\mathbf{H} = R_m(\mathbf{q}\times\mathbf{H}),\\
&\nabla\cdot\mathbf{H} = 0,
\end{aligned}
\tag{2.1}
$$

where $R\,(=Ua/\nu)$ is the Reynolds number, $R_m(=Ua\mu\sigma)$ is the magnetic Reynolds number, $M(=\mu H_\infty a\sqrt{\sigma/\rho\nu})$ is the Hartmann number, nad $\beta\,(=M^2/RR_m)$ is the magnetic pressure number.

The pertinent boundary conditions are

$$
\begin{aligned}
&\mathbf{q}=0 \quad \text{at } r=1,\\
&\mathbf{H} \quad \text{is continuous at} \quad r=1.
\end{aligned}
\tag{2.2}
$$

## 3. Method of solution

The above system can be solved by introducing two expansions: an inner expansion (Stockes region) which holds near the surface of the sphere and an outer expansion (Oseen region) which holds in the region for away from the sphere. Now we shall discuss the construction of these expansions and the corresponding equations.

As in the case of a classical problem the inner expansion, which holds in the Stokes region, is of the form

$$
\begin{aligned}
&\mathbf{q} \approx \mathbf{q}_0 + R\mathbf{q}_1 + \dots,\\
&p \approx p_0 + Rp_1 + \dots,\\
&\mathbf{H} \approx \mathbf{H}_0 + R_{m-1}H + \dots .
\end{aligned}
\tag{3.1}
$$

Further terms in the expansions of $\mathbf{q}$ and $p$ beyond those shown involve the product of the powers of $R$ and $\log R$, as have been obtained by Proudman and Pearson [11] and Chester and Breach [5] for the non-conducting flow.

### 3.1. *Outer expansion*

Since the inertial terms and the viscous terms are comparable in the Oseen region, the outer expansion can formally be obtained from the exact solution by the limit process $R\to 0$ with the outer variables $(\tilde{x}, \tilde{y})$ which are defined as

$$
\tilde{x} = Rx, \quad \tilde{y} = Ry
$$

fixed. In terms of the outer variables the surface of the sphere becomes $\tilde{r}=R$ and hence, in the limit $R\to 0$, the sphere cannot cause any disturbance. Therefore, the outer limit is $\tilde{\mathbf{q}}_0=\hat{\imath}$. The governing equations in the outer region are

$$(3.2)\qquad \begin{aligned} (\tilde{\mathbf{q}}\cdot\tilde{\nabla})\,\tilde{\mathbf{q}} &= -\tilde{\nabla}\tilde{p}+\tilde{\nabla}^2\,\tilde{\mathbf{q}}+\beta(\tilde{\nabla}\times\tilde{\mathbf{H}})\,,\\ \tilde{\nabla}\cdot\tilde{\mathbf{q}} &= 0\,,\\ \tilde{\nabla}\times\tilde{\mathbf{H}} &= \frac{R_m}{R}\,(\tilde{\mathbf{q}}\times\tilde{\mathbf{H}})\,,\\ \tilde{\nabla}\cdot\tilde{\mathbf{H}} &= 0\,, \end{aligned}$$

where

$$(3.3)\qquad \tilde{p}=\frac{1}{R}\,p\,,\qquad \tilde{\mathbf{q}}=\mathbf{q}\,.$$

The appropriate boundary conditions are

$$(3.4)\qquad \tilde{\mathbf{q}}\to\hat{\imath}\,,\quad \tilde{\mathbf{H}}\to\hat{\imath}\quad\text{and}\quad \tilde{p}\to 0\quad\text{as}\quad \tilde{r}\to\infty\,.$$

Clearly, the outer expansion is of the form

$$(3.5)\qquad \begin{aligned} \tilde{\mathbf{q}} &\approx \hat{\imath}+R\tilde{\mathbf{q}}_1+\dots\,,\\ \tilde{p} &\approx \tilde{p}_0+R\tilde{p}_1+\dots\,,\\ \tilde{\mathbf{H}} &\approx \hat{\imath}+R_m\tilde{\mathbf{H}}_1+\dots\,. \end{aligned}$$

It is required that the outer solution, besides satisfying the boundary condition at infinity, matches asymptotically with the inner solution in some overlapping domain.

### 3.2. *Zeroth-order inner solution*

Substituting the relations (3.1) into Eqs. $(2.1)_1$ to $(2.1)_4$, the zeroth-order inner equations reduce to

$$(3.6)\qquad \begin{aligned} \nabla^2\,\mathbf{q}_0 &= \nabla p_0\,, & \nabla\cdot\mathbf{q}_0 &= 0\,,\\ \nabla\times\mathbf{H}_0 &= 0\,, & \nabla\cdot\mathbf{H}_0 &= 0\,, \end{aligned}$$

whose solutions subject to appropriate boundary and matching conditions are [12]

$$(3.7)\qquad \begin{aligned} q_0 &= \left(1-\frac{3}{4r}-\frac{1}{4r^3}\right)\hat{\imath}-\left(\frac{3x}{4}\left(\frac{1}{r^3}-\frac{1}{r^5}\right)\right)\mathbf{r}\,,\\ p_0 &= -\frac{3x}{2r^3}\,,\\ \mathbf{H}_0 &= \hat{\imath}\,. \end{aligned}$$

### 3.3. *First-order outer solution*

The equations to be satisfied by $\tilde{\mathbf{q}}_1$, $\tilde{p}_1$ and $\tilde{\mathbf{H}}_1$ are

$$(3.8)\qquad \begin{aligned} \left(\nabla^2-\frac{\partial}{\partial\tilde{x}}\right)\mathbf{q}_1 &= \nabla\tilde{p}_1\,, & \tilde{\nabla}\cdot\tilde{\mathbf{q}}_1 &= 0\,,\\ \tilde{\nabla}\times\tilde{\mathbf{H}}_1 &= \tilde{\mathbf{q}}_1\times\hat{\imath}\,, & \tilde{\nabla}\cdot\tilde{\mathbf{H}}_1 &= 0\,. \end{aligned}$$

The solution of Eq. $(3.8)_1$ is well-known [12]

$$(3.9)\qquad \begin{aligned} \mathbf{q}_1 &= \frac{3}{2\tilde{r}^3}\,\tilde{\mathbf{r}} - \frac{3}{4r}\,e^{1/2(\tilde{x}-\tilde{r})}\,\hat{\imath} - \frac{3}{4}\,e^{1/2(\tilde{x}-\tilde{r})}\left(\frac{1}{\tilde{r}^2}+\frac{2}{\tilde{r}^3}\right)\tilde{\mathbf{r}},\\ \tilde{p}_1 &= -\frac{3\tilde{x}}{2\tilde{r}^3},\end{aligned}$$

while the solution of Eq. $(3.8)_2$ is obtained as

$$(3.10)\qquad \tilde{\mathbf{H}}_1 = \left[-\frac{3}{2\tilde{r}} + \frac{3}{2\tilde{r}}\,e^{1/2(\tilde{x}-\tilde{r})}\right]\hat{\imath} + \tilde{\nabla}\left[\frac{3}{4}\,P_1(\mu) - \frac{3}{2\tilde{r}}\,e^{1/2(\tilde{x}-\tilde{r})} + \sum_{n=0}^{\infty}\frac{\tilde{A}_n P_n(\mu)}{\tilde{r}^{n+1}}\right],$$

where $\mu=\cos\theta$ and $\tilde{A}_n$ are constants to be determined from the matching condition.

### *3.4. First order inner solution*

The corresponding first-order equations in the Stokes region are

$$(3.11)\qquad \begin{aligned} &\nabla^2\mathbf{q}_1 = \nabla p_1 + (\mathbf{q}_0\cdot\nabla)\mathbf{q}_0, \qquad \nabla\cdot\mathbf{q}_1 = 0,\\ &\nabla\times\mathbf{H}_1 = \mathbf{q}_0\times\hat{\imath}, \qquad \nabla\cdot\mathbf{H}_1 = 0,\end{aligned}$$

with the solutions [12]

$$(3.12)\qquad \begin{aligned} \mathbf{q}_1 = {}& \frac{3}{32}\left(-\frac{3}{r}-\frac{1}{r^3}+\frac{2x}{r^5}\right)\hat{\imath} + \frac{3}{32}\left[\frac{1}{r^3}+\frac{1}{r^5}-3x\left(\frac{1}{r^3}-\frac{1}{r^5}\right)-\right.\\ &\left.-x^2\left(\frac{7}{r^5}+\frac{5}{r^7}\right)\right]\mathbf{r} + \frac{3}{32}\left[4-x\left(\frac{4}{r}-\frac{3}{r^2}+\frac{1}{r^4}\right)\right]\hat{\imath}+\\ &+\frac{3}{32}\left[\frac{2}{r}-\frac{3}{r^2}-\frac{1}{r^4}+x^2\left(-\frac{2}{r^3}+\frac{6}{r^4}+\frac{4}{r^5}+\frac{4}{r^6}\right)\right]\mathbf{r},\\ p_1 = {}& \frac{3}{32}\left[-\frac{6}{r^2}-\frac{2}{r^4}-\frac{1}{3r^6}+x^2\left(\frac{12}{r^4}+\frac{12}{r^6}-\frac{1}{r^8}\right)\right]-\frac{9}{16}\frac{x}{r^3}+\frac{7}{16}\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right),\end{aligned}$$

while $\mathbf{H}_1$ has been found to be

$$(3.13)\qquad \begin{aligned} \mathbf{H}_1 = {}& \frac{3x}{4}\left(\frac{1}{r}-\frac{1}{3r^3}\right)\hat{\imath} + \nabla\left[\left(\frac{1}{12r}-\frac{r}{8}\right)P_2(\mu) - \frac{r}{4} + \right.\\ &\left. + \sum_{n=0}^{\infty}\left[\frac{A_n}{r^{n+1}}P_n(\mu) + B_n r^n P_n(\mu)\right]\right],\end{aligned}$$

where the constants $A_n$, $B_n$ are to be determined from the boundary and matching conditions.

### *3.5. Solutions inside the sphere*

Inside the sphere, $\mathbf{q}=0$, $\mathbf{H}_0=\hat{\imath}$. Also, the equations satisfied by $\mathbf{H}_1^*$ are given by

$$(3.14)\qquad \nabla\times\mathbf{H}_1^* = 0, \qquad \nabla\cdot\mathbf{H}_1^* = 0$$

having the solution

$$\mathbf{H}_1^* = \sum_{n=0}^{\infty} B_n^* \nabla [r^n P_n(\mu)], \tag{3.15}$$

where $B_n^*$ are determined from proper boundary conditions.

In order to determine the constants involved in Equations (3.10), (3.13) and (3.15), we first match the two solutions given by Eqs. (3.10) and (3.13). The remaining undetermined constants in Eq. (3.13) and those in Eq. (3.15) are then obtained by applying the boundary condition $(2.2)_2$.

Thus, the first-order and inner solutions in magnetic fields reduce to

$$\tilde{\mathbf{H}}_1 = \frac{3}{2\tilde{r}}[-1+e^{1/2(\tilde{x}-\tilde{r})}]\hat{\imath} + \frac{3}{2}\tilde{\nabla}\left[\frac{1}{2}P_1(\mu) - \frac{1}{\tilde{r}}e^{1/2(\tilde{x}-\tilde{r})} + \frac{1}{\tilde{r}}\right],$$

$$\mathbf{H}_1 = \frac{3x}{4}\left(\frac{1}{r} - \frac{1}{3r^3}\right)\hat{\imath} + \nabla\left[\left(\frac{1}{12r} - \frac{r}{8} - \frac{1}{40r^3}\right)P_2(\mu) - \frac{3r}{8}P_1(\mu) - \left(\frac{r}{4} + \frac{1}{12r}\right)\right]. \tag{3.16}$$

Also, the solution inside the sphere is

$$\mathbf{H}_{11}^* = \nabla\left[-\frac{3}{8}rP_1(\mu) + \frac{1}{10}r^2P_2(\mu)\right]. \tag{3.17}$$

*3.6. Second-order outer solutions*

The second-order outer solution $\tilde{\mathbf{q}}_2$ is of $O(R_m)$ and can be obtained by solving the equations

$$\left(\tilde{\nabla}^2 - \frac{\partial}{\partial \tilde{x}}\right)\tilde{\mathbf{q}}_2 = \nabla\tilde{p}_2 + \beta(q\tilde{q}_{1y}\hat{\jmath} + \tilde{q}_{1z}\hat{k}), \qquad \tilde{\nabla}\cdot\mathbf{q}_2 = 0, \tag{3.18}$$

with the boundary conditions that it vanishes at infinity. From Eqs. (3.18), it follows that $\tilde{p}_2$ satisfies the equation

$$\tilde{\nabla}^2\tilde{p}_2 = \beta\frac{\partial\tilde{q}_{1x}}{\partial\tilde{x}}, \tag{3.19}$$

the solution of which is given by

$$\tilde{p}_2 = \sum_{n=0}^{\infty}\frac{C_nP_n(\mu)}{\tilde{r}^{n+1}} + \beta\left[\frac{1}{2\tilde{r}}P_2(\mu) - \frac{3}{4}e^{1/2(\tilde{x}-\tilde{r})}\left(\frac{1}{r} + \frac{\tilde{x}}{\tilde{r}^2} + \frac{2\tilde{x}}{\tilde{r}^3}\right)\right]. \tag{3.20}$$

Using the relations $(3.9)_1$ and (3.20), Eq. $(3.18)_1$ reduces to

$$\left(\tilde{\nabla}^2 - \frac{\partial}{\partial\tilde{x}}\right)\tilde{\mathbf{q}}_2 = \tilde{\nabla}\sum_{n=0}^{\infty}\frac{C_nP_n(\mu)}{\tilde{r}^{n+1}} + \frac{\beta}{2}\tilde{\nabla}\left[\frac{1}{\tilde{r}}P_2(\mu)\right] - \frac{3\beta}{4}\tilde{\nabla}\left[\left(\frac{1}{\tilde{r}} + \frac{\tilde{x}}{\tilde{r}^2} + \frac{2\tilde{x}}{\tilde{r}^3}\right)e^{1/2(\tilde{x}-\tilde{r})} - \frac{3\beta}{2}\left(\hat{\jmath}\frac{\partial}{\partial\tilde{y}} + \hat{k}\frac{\partial}{\partial\tilde{z}}\right)\left(\frac{1}{\tilde{r}}\right) + \frac{3\beta}{2}\left[\left(\hat{\jmath}\frac{\partial}{\partial\tilde{y}} + \hat{k}\frac{\partial}{\partial\tilde{z}}\right)\frac{1}{\tilde{r}}e^{1/2(\tilde{x}-\tilde{r})}\right]. \tag{3.21}$$

Let

$$\tilde{\mathbf{q}}_2 = \tilde{\mathbf{q}}_{1p} + \tilde{\mathbf{q}}_{2c}\,. \tag{3.22}$$

The particular solution $\tilde{\mathbf{q}}_{2p}$ of (3.21) is

$$\begin{aligned}\tilde{\mathbf{q}}_{2p} = &-\tilde{\nabla}\sum_{n=0}^{\infty} C_n \frac{d^n}{d\tilde{x}^n}\left(\frac{1}{\tilde{r}}\right) + \frac{\beta}{2}\left[\hat{\imath}\left\{\frac{1}{2}\left(\frac{1}{\tilde{r}} - \frac{3\tilde{x}^2}{\tilde{r}^3}\right) - \frac{3\tilde{x}}{\tilde{r}^3}\right\} - \right. \\ &\left. - \hat{\jmath}\left\{\frac{\tilde{y}}{\tilde{r}(\tilde{x}+\tilde{r})} + \frac{3\tilde{x}\tilde{y}}{2\tilde{r}^3} + \frac{3\tilde{y}}{\tilde{r}^3}\right\} - \hat{k}\left\{\frac{\tilde{z}}{\tilde{r}(\tilde{x}+\tilde{r})} + \frac{3\tilde{x}\tilde{z}}{2\tilde{r}^3} + \frac{3\tilde{z}}{\tilde{r}^3}\right\}\right] - \\ &- \frac{3\beta}{4}\left[\hat{\imath}\left\{\tilde{x}^2\left(\frac{1}{2\tilde{r}^2} + \frac{1}{\tilde{r}^3}\right) - 2\tilde{x}\left(\frac{1}{\tilde{r}^2} + \frac{2}{\tilde{r}^3}\right) - \left(\frac{1}{2} + \frac{1}{\tilde{r}}\right)\right\} + \right. \\ &\left. + \frac{1}{2}(\tilde{y}\hat{\jmath} + \tilde{z}\hat{k})\left(\frac{1}{\tilde{r}} + \frac{\tilde{x}}{\tilde{r}^2} + \frac{2\tilde{x}}{\tilde{r}^3}\right)\right] e^{1/2(\tilde{x}-\tilde{r})} + \frac{3\beta}{2}\left[\frac{1}{\tilde{r}(\tilde{x}+\tilde{r})}(\tilde{y}\hat{\jmath} + \tilde{z}\hat{k})\right] + \\ &+ \frac{3\beta}{4}\left[(\tilde{y}\hat{\jmath} + \tilde{z}\hat{k})\frac{1}{\tilde{r}}\, e^{1/2(\tilde{x}-\tilde{r})}\right],\end{aligned} \tag{3.23}$$

and the complementary function $\tilde{\mathbf{q}}_{2c}$ of Eq. (3.21) is given by

$$\tilde{\mathbf{q}}_{2c} = -3\beta\hat{\imath}\, e^{1/2(\tilde{x}-\tilde{r})}\left(\frac{1}{2\tilde{r}} + \frac{\tilde{x}}{2\tilde{r}^2} + \frac{\tilde{x}}{\tilde{r}^3}\right) + \tilde{\mathbf{q}}_{21}\,, \tag{3.24}$$

where $\mathbf{q}_{21}$ singular the equations

$$\left(\tilde{\nabla}^2 - \frac{\partial}{\partial\tilde{x}}\right)\tilde{\mathbf{q}}_{21} = 0\,, \qquad \tilde{\nabla}\cdot\tilde{\mathbf{q}}_{21} = 0\,, \tag{3.25}$$

having the solution

$$\tilde{\mathbf{q}}_{21} = -\tilde{\nabla}\varphi + \tilde{\nabla}\chi + \left(-\chi,\ -\frac{\partial\psi}{\partial\tilde{z}},\ \frac{\partial\psi}{\partial\tilde{y}}\right), \tag{3.26}$$

with

$$\varphi = \sum_{n=0}^{\infty} D_n \frac{d^n}{d\tilde{x}^n}\left(\frac{1}{\tilde{r}}\right) + R\log\tilde{r} + \tilde{x})\,,$$

$$\chi = \sum_{n=0}^{\infty} E_n \frac{d^n}{d\tilde{x}_n}\left[\frac{1}{\tilde{r}}\, e^{1/2(\tilde{x}-\tilde{r})}\right] + S e^{\tilde{x}} \int_{\frac{\tilde{r}+\tilde{x}}{2}}^{\infty} \frac{e^S}{S}\, dS\,,$$

$$\psi = \sum_{n=0}^{\infty} F_n \frac{d^n}{d\tilde{x}^n}\left[\frac{1}{\tilde{r}}\, e^{1/2(\tilde{x}-\tilde{r})}\right].$$

In Eq. (3.24), the exponential term satisfies the Eq. (3.25) and this term taken togetherwith Eq. (3.23) satisfies the continuity Eq. $(3.18)_2$.

In terms of inner variables ($\tilde{r}=\mathrm{R}r$), Eq. (3.22) is written as

$$\tilde{\mathbf{q}}_2=\beta\left[\frac{1}{\mathrm{R}^2}\left(-\frac{3x}{2r^3}\hat{\imath}-\frac{3y}{2r^3}\hat{\jmath}-\frac{3z}{2r^3}\hat{k}\right)+\frac{1}{\mathrm{R}}\left\{\hat{\imath}\left(-\frac{1}{2r}-\frac{3x^2}{2r^3}\right)+\right.\right.$$
$$+\hat{\jmath}\left(\frac{y}{r(x+r)}-\frac{3xy}{2r^3}\right)+\hat{k}\left(\frac{z}{r(x+r)}-\frac{3xz}{2r^3}\right)\Big\}+\left\{\hat{\imath}\left(\frac{3}{4}-\frac{3x}{8r}-\frac{3x^3}{8r^3}\right)+\right.$$
$$+\hat{\jmath}\left(\frac{3y}{8r}-\frac{3x^2y}{8r^3}\right)+\hat{k}\left(\frac{3z}{8r}-\frac{3x^2z}{8r^3}\right)\Big\}+R\left\{\hat{\imath}\left(\frac{3x}{8}-\frac{9r}{32}-\frac{3x^4}{32r^3}\right)+\right.$$
$$\left.+\hat{\jmath}\left(\frac{9xy}{32r}-\frac{3y}{16}-\frac{3x^3y}{32r^3}\right)+\hat{k}\left(\frac{9xz}{32r}-\frac{3z}{16}-\frac{3x^3z}{3r^3}\right)\Big\}+O(R^2)\right]-$$
$$-\nabla\sum_{n=0}^{\infty}\frac{C_n}{R^{n+2}}\frac{d^n}{dx^n}\left(\frac{1}{r}\right)+\tilde{\mathbf{q}}_{21}, \tag{3.27}$$

and in view of the matching conditions, the constants appearing in Eq. (3.26) have been found to be

$$R=\beta,\quad S=0,$$
$$D_0=-\frac{3\beta}{2},\qquad D_n=0\quad\text{for } n\geqslant 1,$$
$$E_0=-3\beta,\qquad E_n=0\quad\text{for } n\geqslant 1,$$
$$F_n=0\quad\text{for } n\geqslant 0.$$

Thus, by the suitable choice of $\tilde{\mathbf{q}}_{21}$, the terms of $O(1/R^2)$, $O(1/R)$ and $O(1)$ in Eq. (3.27) are matched and the remaining terms of $O(R)$ will be matched with the corresponding solution in the Stokes region.

### 3.7. *Second-order inner solutions*

To determine second-order inner solutions, we have to solve the equations

$$\begin{aligned}\nabla^2\mathbf{q}_2&=\nabla p_2-\beta(\nabla\times\mathbf{H}_1)\times\hat{\imath},\\ \nabla\cdot\mathbf{q}_2&=0.\end{aligned}\tag{3.28}$$

and employ the matching condition. Using Eq. $(3.16)_2$ — Eq. $(3.28)_1$ becomes

$$\nabla^2\mathbf{q}_2=\nabla p_2+\frac{3\beta}{4}x\left(\frac{1}{r^5}-\frac{1}{r^3}\right)(y\hat{\jmath}+z\hat{k}). \tag{3.29}$$

Taking the divergence of Eq. (3.29) and using Eq. $(3.28)_2$, we get

$$\nabla^2 p_2=\frac{3\beta}{4}\left[x^3\left(\frac{3}{r^5}-\frac{5}{r^7}\right)+x\left(\frac{3}{r^5}-\frac{1}{r^3}\right)\right], \tag{3.30}$$

whose particular solution $p_{2c}$, where $p_2=p_{2c}+p_{2p}$, is

$$p_{2c}=\frac{3\beta}{16}\left[x^3\left(\frac{2}{r^5}-\frac{1}{r^3}\right)-x\left(\frac{1}{r}+\frac{6}{5r^3}\right)\right]. \tag{3.31}$$

Writing $\mathbf{q}_2=\mathbf{q}_{2c}+\mathbf{q}_{2p}$, the particular solution $\mathbf{q}_{2p}$ of Eq. (3.29) which satisfies the continuity Eq. (3.28), is given by

$$\text{(3.32)}\quad \mathbf{q}_{2p}=\frac{3\beta}{32}\left[\hat{\imath}\left\{x^4\left(\frac{1}{r^5}-\frac{1}{3r^3}\right)-\frac{6x^2}{5r^3}-r-\frac{13}{15r}\right\}+\right.$$
$$\left.+(y\hat{\jmath}+z\hat{k})\left\{x^3\left(\frac{1}{r^5}-\frac{1}{3r^3}\right)+x\left(\frac{1}{r}-\frac{23}{15r^3}\right)\right\}\right],$$

while the complementary function $\mathbf{q}_{2c}$ satisfies the equations

$$\text{(3.33)}\quad \nabla^2\mathbf{q}_{2c}=\nabla p_{2c},\quad \nabla\cdot\mathbf{q}_{2c}=0,$$

the solution of which is [8]

$$\text{(3.34)}\quad \mathbf{q}_{2c}=K_1\left[-\frac{r^2}{6}\nabla\left\{\frac{1}{r^2}P_1(\mu)\right\}+\frac{2}{3r}\nabla\{rP_1(\mu)\}\right]+$$
$$+K_2\left[-\frac{r^2}{10}\nabla\left\{\frac{1}{r^3}P_2(\mu)\right\}+\frac{1}{10r^3}\nabla\{r^2P_2(\mu)\}\right]+K_3\left[\frac{r^2}{14}\nabla\left\{\frac{1}{r^4}P_3(\mu)\right\}+\right.$$
$$\left.+\frac{4}{105r^5}\nabla\{r^3P_3(\mu)\}\right]+\nabla\left[L_2r^2P_2(\mu)+\frac{M_1}{r^2}P_1(\mu)+\frac{M_2}{r^3}P_2(\mu)+\frac{M_3}{r^4}P_3(\mu)\right],$$

in which other harmonics do not appear in view of the matching condition with the outer solution $\tilde{\mathbf{q}}_2$.

Applying the boundary condition $(2.2)_1$ to Eqs. (3.32) and (3.34), we get

$$K_1=\frac{3\beta}{10},\quad K_2=-\frac{5\beta}{8},\quad K_3=\frac{\beta}{8},$$

$$M_1=\frac{3\beta}{80},\quad M_2=-\frac{\beta}{16},\quad M_3=\frac{\beta}{80},$$

while $L_2=\beta/16$, which has been obtained by matching with the remaining terms of $O(R)$ in $\tilde{\mathbf{q}}_2$. Hence the solution of Eqs. (3.28) which satisfies the boundary condition on the body and the matching condition with the corresponding Oseen solution, are

$$\text{(3.35)}\quad \mathbf{q}_2=\frac{3\beta}{32}\left[\hat{\imath}\left\{x^4\left(\frac{1}{r^5}-\frac{1}{3r^3}\right)-\frac{6x^2}{5r^3}-r-\frac{13}{15r}\right\}+\right.$$
$$\left.+(y\hat{\jmath}+z\hat{k})\left\{x^3\left(\frac{1}{r^5}-\frac{1}{3r^3}\right)+x\left(\frac{1}{r}-\frac{23}{15r^3}\right)\right\}\right]-\beta r^2\nabla\left[\frac{5x^3}{224r^7}-\frac{3x^2}{32r^5}+\right.$$
$$\left.+x\left(\frac{1}{20r^3}-\frac{3}{224r^5}\right)+\frac{1}{32r^3}\right]+\frac{\beta}{5r}\hat{\imath}-\frac{\beta}{32r^3}\nabla(3x^2-r^2)+$$
$$+\frac{\beta}{420r^5}\nabla(5x^3-3xr^2)+\frac{\beta}{32}\nabla\left[\frac{x^3}{r^7}+x^2\left(2-\frac{3}{r^5}\right)-(y^2+z^2)+\right.$$
$$\left.+x\left(\frac{6}{5r^3}-\frac{3}{5r^5}\right)+\frac{1}{r^3}\right],$$

$$\text{(3.36)}\quad p_2=\beta\left[x^3\left(\frac{15}{16r^7}+\frac{3}{8r^5}-\frac{3}{16r^3}\right)-\frac{15}{16}\frac{x^2}{r^5}-x\left(\frac{3}{16r^5}-\frac{3}{40r^3}+\frac{3}{16r}\right)+\frac{5}{16r^3}\right].$$

This determines the solution up to $O(R_m)$. The results of PROUDMAN and PEARSON [11] and CHESTER and BREACH [5] imply that other terms in the expansion involve the product of the powers of $R$ and $\log R$. Accordingly, the solution in the Stokes region is of the form

$$\mathbf{q}=q_0+R\mathbf{q}_1+\beta RR_m\mathbf{q}_2+\dots,$$

$$\mathbf{H}=\hat{\imath}+R_m\mathbf{H}_1+\dots,$$

where $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{H}_1$ are given by $(3.7)$, $(3.12)_1$, $(3.35)$ and $(3.16)_2$, while in the Oseen region the solution will be of the form

$$\tilde{\mathbf{q}}=\hat{\imath}+R\tilde{\mathbf{q}}_1+R_m\tilde{\mathbf{q}}_2+\dots,$$

$$\mathbf{H}=\hat{\imath}+R_m\mathbf{H}_1+\dots,$$

where $\tilde{\mathbf{q}}_1$, $\tilde{\mathbf{q}}_2$ and $\tilde{\mathbf{H}}_1$ are given by Eq$_s$s. $(3.9)_1$ and $(3.10)$.

## 4. Drag on the sphere

We shall now use the solution obtained to compute the drag experienced by the sphere. The only force acting on the body is the drag $D$ along the $X$-axis and is given by [4]

$$(4.1)\qquad D=\int\int_S\Big[\rho\nu\Big\{2x\frac{\partial q_x}{\partial x}+y\Big(\frac{\partial q_x}{\partial y}+\frac{\partial q_y}{\partial x}\Big)+z\Big(\frac{\partial q_x}{\partial z}+\frac{\partial q_z}{\partial x}\Big)\Big\}-$$
$$-xp+\mu\Big(xH_x^2+yH_xH_y+zH_xH_z-\frac{1}{2}x\mathbf{H}^2\Big)-\rho\,(xq_x^2+yq_xq_y+zq_xq_z)\Big]\,dS,$$

where $S$ is an arbitrary closed surface surrounding the sphere. The last term given no contribution since $\mathbf{q}=0$ on the body. The drag obtained by using Eq. (4.1) is

$$(4.2)\qquad D=D_S\Big[1+\frac{3}{8}R+\frac{9}{40}R^2\Big(\log R+\gamma+\frac{5}{3}\log 2-\frac{323}{360}\Big)-$$
$$\frac{2}{15}\beta R_mR+\frac{27}{80}R^3\log R+O(R^3)\Big].$$

The contribution from the Maxwell stress is absent because it produces symmetric changes in the drag force. Eq. (4.2) differs from its non-magnetic counterpart [Chester and Breach (1961)] by a term of order $O(R_m R)$. The magnetic effect is that it tends to reduce the drag; this is evident from the term corresponding to the coefficient of Hartmann number $\beta R_m R$.

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## Streszczenie

### MAGNETOHYDRODYNAMICZNY PRZEPŁYW ZA KULĄ

Niniejsza praca dotyczy swobodnego ustalonego przepływu nieściśliwej, lepkiej elektrycznie przewodzącej cieczy za kulą przy istnieniu równomiernego pola magnetycznego. Pole magnetyczne jest słabe i ukierunkowane wzdłuż swobodnego strumienia. Rozwiązania poszukuje się metodą wyrównujących rozwinięć asymptotycznych przy założeniu że liczba ciśnienia magnetycznego $\beta = O(1)$ oraz $R \ll 1$, $R \ll 1$, $R \ll R$.



## Резюме

### МАГНИТОГИДРОДИНАМИЧЕСКОЕ ОБТЕКАНИЕ СФЕРЫ

Работа обсуждает медленное установившееся течение несжимаемой, вязкой и электропроводящей жидкости вокруг сферы в присутствии однородного магнитного поля. Магнитное поле слабо и параллельно направлению поля свободного течения. Решение ищется при помощи метода асимптотических разложений при предположении, что коэффициент магнитного давления $\beta = O(1)$, а также $R_m \ll 1$, $R \ll 1$, $R_m \ll R$.



DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, HAUZ KHAS, NEW DELHI, INDIA


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