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THEORETICAL ANALYSIS OF REACTIONS IN THE KNEE JOINT CAUSED BY IMPACT

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A theoretical model is described, which makes it possible to estimate the impulsive reactions in the knee joint, when an impact is given to the lower leg, e.g. by kicking a ball. The leg is represented as a double pendulum and its impulsive motion is treated by means of Lagrange's equations. It is shown how the impact impulses parallel and perpendicular to the contact tangent of the joint are determined from the angular velocities before and after the impact. As an example a few film sequences are numerically treated.

Injuries of the knee joint are great problems in industrial and sport medicine. Notwithstanding the fact that repeated overloads during a long time often create insufficiencies, many injuries are undoubtedly caused at a single occasion, when the violence is great enough, for example, at an impact.

For natural reasons an impact in the knee giving acute injury cannot be studied experimentally. However, in order to get some insight into the character of an impact process, especially as regards the impulsive reaction in the knee, we have studied an experimental series of increasing strength of impact to the lower leg, but still within physiologically permissible limits. A number of subjects were thus instructed to kick (a) against a normal soccer football (0.5 kg), (b) against a heavy ball (3.0 kg) and (c) against a ball fixed against a wall. The motion was recorded by film pictures taken at a frequency of 64 frames per second.

The experimental records and the result of the theoretical analysis of the impulsive reaction will be reported elsewhere [1, 2]. In the present paper we will give a full description of the method of the theoretical analysis, showing how the impulsive reaction in the knee can be calculated from the experimentally recorded angular velocities before and after the impact.

As a measure of the impulsive reaction in the knee joint we may use the impulse, i.e. the time integral of the impact force over the comparatively short time of impact. The aim of our analysis is thus to calculate this impulse from the registered changes in velocity over the time of impact. Moreover, it is of course of great physiological interest to know to what extent the impulsive reaction is taken up by the joint surfaces at their point of contact and to which extent this is made by the ligaments of the knee. To this end the analysis was made so as to make possible a separate calculation of the two impulse components perpendicular and parallel to the knee joint surface, i.e. approximately parallel and perpendicular to the lower leg, respectively. The result is given below in Eqs. (48), (49) with Eqs. (37), (38). Assuming that the muscles do not produce impulsive moments in the joints, it seemed reasonable to use the double pendulum, represented in Fig. 1, as a mathematical model of the motion of the thigh OA and the lower leg AB. When for this purpose we use a coordinate system fixed to the hip joint O, we may assume that there is no change of velocity

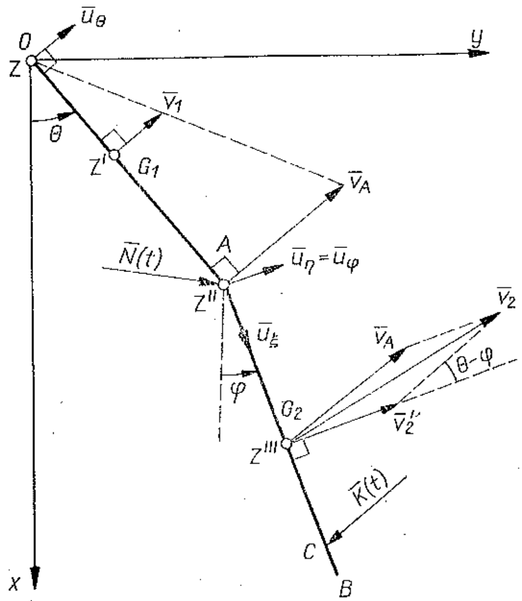

FIG. 1. Model of the leg as a double pendulum

of this point during the time of impact. This assumption is quite reasonable in virtue of the great mass ratio between the body and the leg, and it was also verified by our film pictures. In fact, at the time of impact the translation of the hip joint had solwed down and almost stopped, ($v = 0.5 - 1.5$ m/s), and no sudden change of this translation was observed during the time of impact. Consequently, as regards the the analysis of the impact, the hip joint O can safely be regarded as a fixed point.

According to Fig. 1 we introduce the following notations:

O	hip joint (regarded as fix),
A	knee joint,
l_1	OA, length of thigh,
l_2	AB, length of lower leg,
θ	angle of deflection of OA,
φ	angle of deflection of AB,
G_1	centre of mass of thigh OA,
$k_1 l_1$	OG_1,
G_2	centre of mass of lower leg AB,
$k_2 l_2$	AG_2,
x, y, z	coordinate system with origin in O,
x	vertical axis, directed downwards,

y horizontal axis in the sagittal plane OAB,
z horizontal axis perpendicular to the sagittal plane,
z' axis$\|z$, and with origin in G_1,
z'' axis$\|z$, and with origin in A,
z''' axis$\|z$, and with origin in G_2,
m_1 mass of thigh OA,
m_2 mass of lower leg,
$J_z = J_0$ moment of inertia with respect to z of thigh OA,
$J_{z'} = J_1$ moment of inertia with respect to z' of thigh OA,
$J_{z''} = J_A$ moment of inertia with respect to z'' of lower leg AB,
$J_{z'''} = J_2$ moment of inertia with respect to z''' of lower leg AB,
$\bar{v}_1$ velocity of G_1, being $\perp OA$,
$\bar{v}_A$ velocity of A, being $\perp OA$,
$\bar{v}_2'$ velocity of G_2 relative to A, being $\perp AB$,
$\bar{v}_2$ $\bar{v}_A + \bar{v}_2'$ absolute velocity of G_2,
τ time of imapct,
C point of impact on lower leg,
b AC,
$\bar{K}(t)$ impulsive force of impact on lower leg in $C \times \bar{S} = \int_0^\tau \bar{K} dt$,
$\bar{N}(t)$ impulsive force of impact on thigh from lower leg in $A \times \bar{R} = \int_0^\tau \bar{N} dt$,
$\bar{u}_\theta$ $(-\sin\theta, \cos\theta)$ = unit vector $\perp OA$,
$\bar{u}_\xi = \bar{u}_{AB}$ $(\cos\varphi, \sin\varphi)$ = unit vector$\|AB$,
$\bar{u}_\eta = \bar{u}_\varphi$ $(-\sin\varphi, \cos\varphi)$ = unit vector $\perp AB$.

The position of the system at any time t is defined by the angles θ and φ, which will be used as generalized Lagrangian coordinates. Lagrange's equations for a system specified by the generalized coordinates q_α $(\alpha = 1, 2)$ are

$$\frac{\delta T}{\delta q_\alpha} \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha, \quad (\alpha = 1, 2), \tag{1}$$

where T is the kinetic energy of the system, $q_1 = q_\theta = \theta$ and $q_2 = q_\varphi = \varphi$ are the generalized coordinates, and $Q_1 = Q_\theta$ and $Q_2 = Q_\varphi$ are the corresponding generalized forces.

As seen from Fig. 1, we have

$$\bar{v}_1 = k_1 l_1 \dot{\theta} \bar{u}_\theta, \tag{2}$$

$$\bar{v}_A = l_1 \dot{\theta} \bar{u}_\theta, \tag{3}$$

$$\bar{v}_2' = k_2 l_2 \dot{\varphi} \bar{u}_\varphi, \tag{4}$$

$$\bar{v}_2 = \bar{v}_A + \bar{v}_2' = l_1 \dot{\theta} \bar{u}_\theta + k_2 l_2 \dot{\varphi} \bar{u}_\varphi, \tag{5}$$

$$\bar{v}_1^2 = (k_1 l_1 \dot{\theta})^2, \tag{6}$$

and since $\bar{u}_\theta \cdot \bar{u}_\varphi = \cos(\theta - \varphi)$:

(7) $$v_2^2 = (l_1 \dot{\theta})^2 + (k_2 l_2 \dot{\varphi})^2 + 2l_1 k_2 l_2 \theta\varphi \cos(\theta - \varphi).$$

Consequently, the kinetic energy T of the system is given by

(8) $$T = T_1 + T_2 .$$

where

(9) $$T_1 = \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} m_1 (k_1 l_1 \dot{\theta})^2 ,$$

(10) $$T_2 = \frac{1}{2} J_2 \dot{\varphi}^2 + \frac{1}{2} m_2 [(l_1 \dot{\theta})^2 + (k_2 l_2 \dot{\varphi})^2 + 2l_1 k_2 l_2 \dot{\theta}\dot{\varphi} \cos(\dot{\theta} - \varphi)].$$

This expression for T gives

(11) $$\frac{\partial T}{\partial \dot{\theta}} = (J_1 + m_1 k_1^2 l_1^2 + m_2 l_1^2) \dot{\theta} + m_2 l_1 k_2 l_2 \dot{\varphi} \cos(\theta - \varphi),$$

(12) $$\frac{\partial T}{\partial \dot{\varphi}} = (J_2 + m_2 k_2^2 l_2^2) \dot{\varphi} + m_2 l_1 k_2 l_2 \dot{\theta} \cos(\theta - \varphi),$$

(13) $$\frac{\partial T}{\partial \theta} = -m_2 l_1 k_2 l_2 \dot{\theta}\dot{\varphi} \sin(\theta - \varphi),$$

(14) $$\frac{\partial T}{\partial \varphi} = m_2 l_1 k_2 l_2 \dot{\theta}\dot{\varphi} \sin(\theta - \varphi).$$

Immediately before the impact, which is supposed to start at the time $t=0$, we have

(15) $$\theta = \theta_0 , \quad \varphi = \varphi_0 , \quad \dot{\theta} = \dot{\theta}_0 , \quad \dot{\varphi} = \dot{\varphi}_0 .$$

Then the lower leg receives a blow of impulse over a small time interval $0 < t < \tau$.

We integrate Lagrange's equations (1) from $t=0$ to $t=\tau$, noting the following rules:

1) θ and φ are considered constant since the system will not be displaced considerably during the time interval τ

$$(\theta_0 = \theta = \theta_\tau , \quad \varphi_0 = \varphi = \varphi_\tau).$$

2) $\dot{\theta}$ and $\dot{\varphi}$ are bounded to finite values, so that their integrals over the small interval τ can be neglected.

3) $\ddot{\theta}$ and $\ddot{\varphi}$ may be "infinite", i.e. very large, so that their integrals over the small interval τ assume finite values.

4) $\bar{K}$, the impulsive force, may also be "infinite", making its integral over the small interval τ finite.

We then obtain for the left hand sides of Eq. (1)

$$\text{(16)}\qquad \int_0^\tau \frac{\delta T}{\delta\theta}\,dt \equiv \int_0^\tau \left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta}\right]dt = \left(\frac{\partial T}{\partial\dot\theta}\right)_\tau - \left(\frac{\partial T}{\partial\dot\theta}\right)_0,$$

$$\text{(17)}\qquad \int_0^\tau \frac{\delta T}{\delta\varphi}\,dt \equiv \int_0^\tau \left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\varphi}\right) - \frac{\partial T}{\partial\varphi}\right]dt = \left(\frac{\partial T}{\partial\dot\varphi}\right)_\tau - \left(\frac{\partial T}{\partial\dot\varphi}\right)_0,$$

the integrals of $\partial T/\partial\theta$ and $\partial T/\partial\varphi$ being neglected since their integrands are finite in virtue of Eqs. (13) and (14).

Inserting the expressions (11) and (12) and noting that $\theta_0=\theta_\tau$ and $\varphi_0=\varphi_\tau$, we obtain

$$\text{(18)}\qquad \int_0^\tau \frac{\delta T}{\delta\theta}\,dt \equiv (J_1+m_1k_1^2l_1^2+m_2l_1^2)(\dot\theta_\tau-\dot\theta_0)+m_2l_1k_2l_2(\dot\varphi_\tau-\dot\varphi_0)\cos(\theta_0-\varphi_0),$$

$$\text{(19)}\qquad \int_0^\tau \frac{\delta T}{\partial\varphi}\,dt \equiv (J_2+m_2k_2^2l_2^2)(\dot\varphi_\tau-\dot\varphi_0)+m_2l_1k_2l_2(\dot\theta_\tau-\dot\theta_0)\cos(\theta_0-\varphi_0).$$

For the integration of the right hand side of Eq. (1) we have to determine the generalized impulsive forces Q_θ and Q_φ, which is made by means of the expression for the virtual work of the impact force $\bar K$ of Fig. 1:

$$\text{(20)}\qquad \delta W = \bar K\cdot\delta\bar r = K_x\delta x + K_y\delta_y.$$

Denoting the coordinates for the point of impact, C, by (x, y), we get

$$\text{(21)}\qquad x = l_1\cos\theta + b\cos\varphi,$$

$$\text{(22)}\qquad y = l_1\sin\theta + b\sin\varphi,$$

giving

$$\text{(23)}\qquad \delta x = -l_1\sin\theta\delta\theta - b\sin\varphi\delta\varphi,$$

$$\text{(24)}\qquad \delta y = l_1\cos\theta\delta\theta + b\cos\varphi\delta\varphi.$$

Insertion of Eqs. (23) and (24) in Eq. (20) gives

$$\text{(25)}\qquad \delta W = -K_x(l_1\sin\theta\delta\theta + b\sin\varphi\delta\varphi) + K_y(l_1\cos\theta\delta\theta + b\cos\varphi\delta\varphi).$$

Comparing Eq. (25) with the alternative expression

$$\text{(26)}\qquad \delta W = Q_\theta\delta\theta + Q_\varphi\delta\varphi$$

and noting that $\theta=\theta_0=\theta_\tau$, $\varphi=\varphi_0=\varphi_\tau$, we obtain

$$\text{(27)}\qquad Q_\theta = -l_1(K_x\sin\theta_0 - K_y\cos\theta_0),$$

$$\text{(28)}\qquad Q_\varphi = -b(K_x\sin\varphi_0 - K_y\cos\varphi_0).$$

Integrating now the right hand sides of Eq. (1) from $t=0$ to $t=\tau$ and introducing the orthogonal components of the impulse of the impact force,

(29) $$S_x=\int_0^\tau K_x(t)\,dt\,,$$

(30) $$S_y=\int_0^\tau K_y(t)\,dt\,,$$

we obtain the generalized components of the impulse:

(31) $$S_\theta=\int_0^\tau Q_\theta\,dt=-l_1\sin\theta_0\,S_x+l_1\cos\theta_0\,S_y\,,$$

(32) $$S_\varphi=\int_0^\tau Q_\varphi\,dt=-b\sin\varphi_0\,S_x+b\cos\varphi_0\,S_y\,.$$

The use of Eqs. (18), (19), (31) and (32) gives us Lagrange's equations (1) integrated over τ:

(33) $$(J_1+m_1k_1^2l_1^2+m_2l_1^2)(\dot\theta_\tau-\dot\theta_0)+m_2l_1k_2l_2(\dot\varphi_\tau-\dot\varphi_0)\cos(\theta_0-\varphi_0)=S_\theta\,,$$

(34) $$(J_2+m_2k_2^2l_2^2)(\dot\varphi_\tau-\dot\varphi_0)+m_2l_1k_2l_2(\dot\theta_\tau-\dot\theta_0)\cos(\theta_0-\varphi_0)=S_\varphi\,.$$

By using the moment of inertia of the thigh with respect to an axis through the hip joint 0,

(35) $$J_0=J_1+m_1k_1^2l_1^2$$

and of the lower leg with respect to the knee joint A,

(36) $$J_A=J_2+m_2k_2^2l_2^2\,,$$

Eqs. (33) and (34) become

(37) $$(J_0+m_2l_1^2)(\dot\theta_\tau-\dot\theta_0)+m_2l_1k_2l_2(\dot\varphi_\tau-\dot\varphi_0)\cos(\theta_0-\varphi_0)=S_\theta\,,$$

(38) $$J_A(\dot\varphi_\tau-\dot\varphi_0)+m_2l_1k_2l_2(\dot\theta_\tau-\dot\theta_0)\cos(\theta_0-\varphi_0)=S_\varphi\,.$$

In order to calculate the impulse

(39) $$\bar R=\int_0^\tau \bar N\,dt$$

of the impact force $\bar N$ from the lower leg on the thigh in the knee joint A, we apply the law of linear momentum on the lower leg on which the forces $\bar K$, $-\bar N$ and $m_2\,\bar g$ act:

(40) $$m_2\frac{d}{dt}\bar v_2=\bar K-\bar N+m_2\bar g\,.$$

Integrating Eq. (40) over τ and noting that the integral over $m_2 \bar{g}$ is negligible, since $m_2 \bar{g}$ is finite, we obtain

$$m_2 [(\bar{v}_2)_\tau - (\bar{v}_2)_0] = \bar{S} - \bar{R}. \tag{41}$$

We project these equations on the directions

$$-\bar{u}_\xi = (\cos\varphi, \sin\varphi), \tag{42}$$

along the lower leg, and

$$\bar{u}_\eta = \bar{u}_\varphi = (-\sin\varphi, \cos\varphi), \tag{43}$$

perpendicular to the lower leg.

By introducing Eq. (5) in Eq. (41), multiplying by $\bar{u}_\xi$ and $\bar{u}_\eta$, respectively, and noting that

$$\bar{u}_\theta = (-\sin\theta, \cos\theta),$$

we obtain

$$R_\xi = S_x \cos\varphi_0 + S_y \sin\varphi_0 - m_2 l_1 (\dot{\theta}_\tau - \dot{\theta}_0) \sin(\varphi_0 - \theta_0), \tag{44}$$

$$R_\eta = -S_x \sin\varphi_0 + S_y \cos\varphi_0 - m_2 [l_1 (\dot{\theta}_\tau - \dot{\theta}_0) \cos(\varphi_0 - \theta_0) + k_2 l_2 (\dot{\varphi}_\tau - \dot{\varphi}_0)]. \tag{45}$$

Solving Eqs. (31) and (32) for S_x and S_y gives us

$$S_x = (l_1^{-1} S_\theta \cos\varphi_0 - b^{-1} S_\varphi \cos\theta_\theta)/\sin(\varphi_0 - \theta_0), \tag{46}$$

$$S_y = (l_1^{-1} S_\theta \sin\varphi_0 - b^{-1} S_\varphi \sin\theta_0)/\sin(\varphi_0 - \theta_0). \tag{47}$$

Inserting Eqs. (46) and (47) in Eq. (44) and Eq. (32) in Eq. (45) we obtain

$$R_\xi = [l_1^{-1} S_\theta - b^{-1} S_\varphi \cos(\varphi_0 - \theta_0)]/\sin(\varphi_0 - \theta_0) - m_2 l_1 (\dot{\theta}_\tau - \dot{\theta}_0) \sin(\varphi_0 - \theta_0), \tag{48}$$

$$R_\eta = b^{-1} S_\varphi - m_2 [l_1 (\dot{\theta}_\tau - \dot{\theta}_0) \cos(\varphi_0 - \theta_0) + k_2 l_2 (\dot{\varphi}_\tau - \dot{\varphi}_0)]. \tag{49}$$

Equations (48) and (49) together with Eqs. (37) and (38) make it possible to determine the impulses of impact in the knee from the measured values of the angular velocities $\dot{\theta}_0$, $\dot{\varphi}_0$ and $\dot{\theta}_\tau$, $\dot{\varphi}_\tau$ before and after the impact at the given angles θ_0, φ_0.

As an example of the application of the method of analysis we give the values related to one of the experimental instances reported in [1, 2]. The values of the

Table 1. Anthropometric data

J_0 [kgm²]	J_A [kgm²]	m_2 [kg]	k_2 [1]	l_1 [m]	l_2 [m]	b [m]
0.405	0.479	4.6	0.49	0.45	0.53	0.53

lengths, masses, moments of inertia etc. have been obtained from anthropometric measurements as described in [2]. The velocities before and after the impact were determined from the photographic records by means of graphical differentation. Table 1 gives the anthropometric data used in this example and in Table 2 the kinematic data and the calculated impulses of impact are given for the three different types of kicking.

Table 2. Kinematic data and calculated impulses of impact

	Soccer football	Heavy ball	Wall
θ_0 [°]	40	39	40
φ_0 [°]	−27	−5	−5
$\dot{\theta}_0$ [s^{-1}]	5.6	3.4	1.0
$\dot{\theta}_\tau$ [s^{-1}]	3.4	0.0	−1.0
$\dot{\varphi}_0$ [s^{-1}]	28.0	28.0	21.2
$\dot{\varphi}_\tau$ [s^{-1}]	23.5	10.2	−17.4
R_ξ [Ns]	3.1	7.7	14.6
R_η [Ns]	2.2	8.4	12.8

Concluding remarks

The validity of the assumption that the muscles do not produce impulsive moments in the joints cannot be verified only by means of the experimental mechanical data available at present. However, this assumption is to some extent supported by simultaneous electromyographic recordings which show that important muscles were not active during the time of impact. The EMG-recordings, though, did not include all the muscle groups which are involved in hip and knee joint movements. For this reason further work is being carried out in order to decide upon this more or less open question by means of more complete measurements.

Acknowledgement

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Streszczenie

ANALIZA TEORETYCZNA REAKCJI W PRZEGUBIE KOLANOWYM — SPOWODOWANYCH UDERZENIEM

Przedstawiono teoretyczny model umożliwiający ocenę reakcji impulsu w przegubie kolanowym, występujących przy uderzeniu w dolną część nogi (goleń), na przykład przy kopaniu piłki. Noga jest modelowana za pomocą podwójnego wahadła, a jej ruch impulsywny opisany jest równaniami Lagrange'a. Pokazano, jak impulsy uderzenia równoległe i prostopadłe do stycznej kontaktu przegubu są określone z prędkością kątowych przed i po uderzeniu. Jako przykład policzono numerycznie kilka sekwencji z filmu.

Резюме

ТЕОРЕТИЧЕСКИЙ АНАЛИЗ РЕАКЦИИ В КОЛЕННОМ СУСТАВЕ ВЫЗВАННЫХ УДАРОМ

Представлена теоретическая кмодель дающая возможность оценки реакций импульса в коленном суставе, выступающий при ударе в нижнюю часть ноги (голень), например при ударе мяча. Нога моделируется при помощи двойного маятника, а его импульсное движение описывается уравнениями Лагранжа. Показано как импульсы удара, параллельные и перпендикулярные к касательной контакта сустава, определены из угловых скоростей до и после удара. Как пример численно расчитано несколько кадров из фильма.

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