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MAGNETO-ELASTIC STABILITY OF AN UNCONSTRAINED ASSEMBLY OF COILS

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The mechanical stability of a solenoid in the form of a toroidal helix in its own magnetic field is investigated. The solenoid is mechanically unconstrained, i.e. the coils are not supported by a rigid or elastic base. In the first part of the paper, the magnetic field, the forces and the moments acting on one single coil are calculated. The calculations are performed for the undeformed and the deformed states of the torus. The second part contains a stability analysis for the model of a circular elastic spring, which accounts for the extension and the shear deformation.

1. Introduction

This paper deals with the magnetoelastic stability of a set of coils in the form of a toroidal helix. The number of coils in the set is sufficiently large so that coils are nearly planar and approximately normal to a common base plane. They interact with each other through the forces and moments that result from the magnetic field which originates from the currents in each coil.

This type of construction is used for the design of fusion reactors. In general, these structures are constrained by a relatively rigid base and in the study of their mechanical stability these constraints are taken into account. It appears that the mechanical instability of these structures is due to the mutual magnetic forces by the currents in each coil. These forces depend on the mutual distances of planes of the coils and as soon as one coil moves out of the mid-plane between its neighbours, the equilibrium of the forces is disturbed and instability may result. Thus the physical origin of this instability is well understood, and there is the same kind of instability observed at the magnetized beams and plates in a strong magnetic field, cf. e.g. [1].

In contrast with this type of instability, another type of instability is investigated in the present paper. We consider the mechanical behaviour of an unconstrained set of coils in which instability may occur as a consequence of the interior pressure. It appears that there exists a resultant radial force affecting every single coil and directed to the center of every coil. If the force is large enough, the structure will buckle, just as a ring will do.

A number of investigators have studied the stability of the constrained assembly of coils, both theoretically and experimentally. F. C. Moon and S. Chattopadhyay give a detailed analysis of the stability and the vibration modes of one single isolated coil in the field of the others, [2]. They find a critical current for superconductive material of $I_{cr}=3.7 \cdot 10^6$ Amps, which has to be compared with the design current $5.6 \cdot 10^6$ Amps, as given by FILE at al [3]. In reference [2] are given some other contributions, mainly in the field of the reactor magnet design. In his recent paper [4] MOON reviews the problem anew. He also proposes a theory for the vibrations and stability of the constrained toroidal set, based upon the work by MOON and SWANSON [5]. In this work the forces on the coils are taken proportional to the differences in displacements of the planes of the coils and thus a system of differential—difference equations is obtained. The magnetic force constants are calculated from the mutual inductanes between the separate coils [5].

It is obvious that the radial force found in the unconstrained set is counterbalanced in a set constrained by a rigid base. Thus in the references [2] to [5] this force does not appear. Since it may be expected that the constraint has a stabilizing influence on the structure, it seems worthwhile to study the unconstrained case separately. The important question here is also whether a reasonably stiff structure may become instable only in the superconductive state, or already in the normal state.

An exact calculation of the instability of a toroidal helix seems to be impossible. Therefore we have to consider a simplified model for which the calculations may be carried out and for which it is also possible to evaluate errors introduced by the model. This model consists of two parts. In the first one we calculate the forces and moments on every single coil, both for the undeformed and the deformed states. We note that even for the model under discussion the force is only approximately determined, but the error is small if the spacing between the coils is small. If the magnet consists of a few number of coils, the force can only be found by numerical integration.

Having determined the force per unit of length of the center-line of the torus, we investigate the mechanical stability of a spring in the form of a tours. The spring here is approximated by a continuous body, but the extension of the spring, the shear deformation, the torsional moment which acts along the central line of the spring, and the bending moment are taken into consideration. The torsional moment results not only from the rotation of a cross-section, but also from an extension. Correspondingly, a bending moment results from an extension. In the stable position the magnetic force lines are nearly circular. As soon as the spring bends out, the magnetic force lines are deformed. There are two extreme cases: In the first one the force is a follower-force, which maintains its normal position with respect to the plane of the cross-section, while in the second case the force is conservative and retains its original direction. We shall assume that the force is partly a follower-force for which an angle over which it rotates is a fraction of an angle of rotation of a cross-section. It is further assumed that the ratio of the two angles is independent of the rotation itself. This ratio has to be determined by an experiment.

The importance of the present study is that it gives some insight in the physical and mechanical aspects of the deformation patterns that may occur in a helix that carries sufficiently large currents. For practical applications the study may be extended by taking into account the constraints that can be present in an actual design. In a paper to follow, we shall discuss the influence of some constraints on the stability of this structure.

2. The magnetic field

We consider a set of coils in the form of a toroidal helix which may be represented as follows:

$$\begin{aligned} x &= (a+b\cos n\varphi)\cos\varphi, \\ y &= (a+b\cos n\varphi)\sin\varphi, \\ z &= -b\sin n\varphi, \end{aligned} \tag{2.1}$$

where x, y, z are Cartesian coordinates with the z-axis perpendicular to the middle plane of the torus and the x- and y-axes in this plane. The origin of the coordinate systems is placed in the middle of the considered set. In Eqs. (2.1) a is the radius of the center line of the torus, while b is the radius of one coil. The angle φ is measured counterclockwise from the x-axis, while n is the total number of coils in the set.

The magnetic field of the solenoid is given by

$$\mathbf{B} = \frac{\mu I}{4\pi}\oint \frac{\mathbf{ds}\times\mathbf{r}}{r^3}. \tag{2.2}$$

In Eq. (2.2) μ is the magnetic permeability of the surrounding medium and I is the current carried by the coils. If $\mathbf{B}$ is measured at a point P, $\mathbf{r}$ denotes the vector from a point Q on the helix to P and r is equal to $|\mathbf{r}|$. The line-element in Q is denoted by $\mathbf{ds}$.

If n is very large, we may consider the solenoid to be a continous torus and the field inside the torus may then be approximated by

$$B_\varphi = \frac{\mu n I}{2\pi R}, \qquad B_R = B_z = 0, \tag{2.3}$$

where B_R and B_φ are the radial and circumferential components of $\mathbf{B}$. In Eq. (2.3) R is the distance of the point P from the center axis of the torus ($x=y=0$), and the current is positively oriented in the clockwise direction. We have

$$R = a + b\cos\psi, \tag{2.4}$$

where ψ is the angle in a cross-section of the torus perpendicular to its center-line, measured in a clockwise direction.

As has been stated above, the formulae (2.3) hold only approximately. In the Appendix we prove that they are asymptotic approximations of the exact formulae.

3. The forces and the moments

For the continuous model we have for the force per one coil

$$\mathbf{F}=I\oint \mathbf{ds}\times\mathbf{B}, \tag{3.1}$$

while the moment $\mathbf{N}$ per one coil with respect to the center of cross-section is given by

$$\mathbf{N}=I\oint \boldsymbol{\rho}\times(\mathbf{ds}\times\mathbf{B}), \tag{3.2}$$

where $\boldsymbol{\rho}$ is the vector from the center to a point of the coil. The formula (3.2) may be rewritten as

$$\mathbf{N}=I\oint [(\boldsymbol{\rho},\mathbf{B})\,\mathbf{ds}-(\boldsymbol{\rho},\mathbf{ds})\,\mathbf{B}]. \tag{3.3}$$

Making use of Eqs. (2.3), (2.4) and (3.1), where the line-element $\mathbf{ds}$ is taken for the continuous torus (i.e. we put ψ instead of $n\varphi$ into the formulae (2.1) and later we express ds in term of $d\psi$), we find

$$\mathbf{F}=-\frac{\mu n I^2 b^2}{\sqrt{a^2-b^2}\,(a+\sqrt{a^2-b^2}}\,\mathbf{e}_R, \tag{3.4}$$

where $\mathbf{e}_R$ is a unit vector.

To determine the moment $\mathbf{N}$ we have to admit a small inconsistency in our model. We have assumed above that the set of coils is approximated by a continuous torus because of n being large. With such an assumption all coils are planar, an event which does not happen with a real toroidal or straight helix. If the coils were planar, no moments would have existed. This statement may be easily verified by making use of Eqs. (2.3), (2.4) and (3.3) and by replacing the real $\mathbf{ds}$ by one similar to that used for the calculation of Eq. (3.4). Therefore, when looking for the moment $\mathbf{N}$, we have to take into consideration the varying slope of each coil with respect to the proper, perpendicular cross-section of the torus. By replacing $n\varphi$ by $\psi\in(0,2\pi)$ and by applying the line-element $\mathbf{ds}$ exactly in accordance with the formulae (2.1), we obtain the non-vanishing relations that describe $\mathbf{N}$. A careful calculation allows us to find, for n being large, the following results:

$$\begin{aligned} N_R &= -\mu I^2 b, \\ N_\varphi &= \frac{2b\mu I^2}{\pi n}\simeq 0, \\ N_z &= 0. \end{aligned} \tag{3.5}$$

It is clear that the moment N_R acts so as to place the coil exactly in the plane perpendicular to the center-line of the solenoid.

4. Forces and moments in a disturbed pattern

It is well known from the experiment that the assembly of coils may be unstable. A very small change in the position of one coil gives rise to the creation of some additional forces and moments that can even destroy the constiuction. Such a change also gives rise to changes in the magnetic field **B**. The exact analysis is highly involved and leads to very complicated formulae. Therefore we consider the simplified model which, however, describes the phenomenon studied with sufficient accuracy. The model takes into consideration a disturbance of the magnetic field as well as of the position of the coil.

Let one coil be displaced out of its original position. If the displacement of the center point of the coil is negligible (as we shall assume), such a change can be described by a vector of small revolution $\boldsymbol{\omega}_1$. Additionally, a change in the position of the coil causes a distance in the magnetic field **B**. Therefore we introduce another vector $\boldsymbol{\omega}_2$ responsible for this. Of interest to us is the relative change of the coil and of the field **B**.

We shall assume that

$$\boldsymbol{\omega}_2=(1-\alpha)\,\boldsymbol{\omega}_1 \tag{4.1}$$

with $0<\alpha<1$, where an exact value of the parameter α is unknown and should be determined by an experiment.

Hence the relative position of the disturbed field and displaced coil can be described by a difference between $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. To avoid any additional calculation concerning **ds** and $\boldsymbol{\rho}$ it is better to keep the coil fixed and to rotate the field only. Thus the vector describing the small relative revolution of the field is

$$\boldsymbol{\omega}=\boldsymbol{\omega}_2-\boldsymbol{\omega}_1=-\alpha\boldsymbol{\omega}_1\,. \tag{4.2}$$

Notice, that Eq. (4.2) includes both cases of disturbance considered above.

For $\alpha=0$ we have $\boldsymbol{\omega}_1=\boldsymbol{\omega}_2$, i.e. $\boldsymbol{\omega}=0$ and then we deal with a kind of follower-force. As may be expected no extra forces and moments occur. For the contrary case of $\alpha=1$ the relative revolution of the field **B** is equal to $-\boldsymbol{\omega}_1$ because $\boldsymbol{\omega}_2=0$ and in fact we find the field **B** to be conservative. The intermediate case, defined by the parameter α, seems to be the closest to the real situation.

Thus the disturbances of forces and moments are determined as follows:

$$\begin{aligned}\Delta\mathbf{F}&=I\oint \mathbf{ds}\times(\boldsymbol{\omega}\times\mathbf{B})=I\oint(\boldsymbol{\omega},\mathbf{ds})\,\mathbf{B}\,,\\ \Delta\mathbf{N}&=I\oint \boldsymbol{\rho}\times[\mathbf{ds}\times(\boldsymbol{\omega}\times\mathbf{B})]=I\oint(\boldsymbol{\rho}\times\mathbf{B})(\boldsymbol{\omega},\mathbf{ds})\,,\end{aligned} \tag{4.3}$$

where $\boldsymbol{\rho}$, **ds** and **B** are the same as in the previous analysis and $\boldsymbol{\omega}$ is defined by Eq. (4.2). Making use of Eq. (4.3), we find

$$\begin{aligned}&\Delta F_R=0\,,\qquad \Delta F_z=0\,,\\ &\Delta F_\varphi=-\alpha\omega_{1z}\,2\pi nI^2 K\,,\qquad \Delta N_R=\alpha\omega_{1R}\,2\pi nI^2 Km^* a\,,\\ &\Delta N_\varphi=0,\ \text{and}\qquad \Delta N_z=\alpha\omega_{1z}\,2\pi nI^2 Ka\,,\end{aligned} \tag{4.4}$$

where

$$K=\frac{\mu m^2}{2\pi m^*(1+m^*)}\,,\qquad m=\frac{b}{a}\,,\qquad m^*=\sqrt{1-m^2}\,,$$

and ω_{1R} and ω_{1z} are the components of the vector of a small revolution of the coil $\boldsymbol{\omega}_1$. In our further analysis these quantities will be expressed in terms of the displacements and of the angle of rotation describing a kinematics of the toroidal solenoid being considered.

5. Kinematics of the toroidal helix

To consider the mechanical properties of the toroidal helix we have to introduce certain geometrical and kinematical quantities such as displacements and curvatures. We shall treat the helix as a toroidal spring.

Let u, v and w be the coordinates of a displacement vector in radial, circumferential and z-directions, respecively. Let β stand for a small angular displacement with respect to the center-line of the toroidal helix. The curvatures and the twist of the disturbed toroidal helix ,when treated as a ring, are given in the general case by the following equations:

$$\kappa_R = -\frac{\beta}{a} + \frac{d^2 w}{ds^2}, \quad \kappa_z = \frac{1}{a} - \frac{d^2 u}{ds^2} + \frac{1}{a}\frac{dv}{ds}, \quad \tau = \frac{d\beta}{ds} + \frac{1}{a}\frac{dw}{ds}. \tag{5.1}$$

In Eq. (5.1), s denotes the line parameter in the circumferential, counterclockwise direction, and a is the radius of the center line of the spring, as formerly.

Thus defined curvatures and twist are in agreement with the system of coordinates introduced in Chapter 1. For an undisturbed toroidal spring we have, of course, $\kappa_R=0$, $\kappa_z=1/a_0$ and $\tau=0$, where a_0 stands for the initial radius of the toroidal helix. Hence, in fact, for the deformed spring $\kappa_R=\Delta\kappa_R$, $\kappa_z=1/a+\Delta\kappa_z$ and $\tau=\Delta\tau$.

The elongation of the spring in the circumferential direction is defined by

$$\varepsilon = \frac{dv}{ds} + \frac{u}{a}. \tag{5.2}$$

If β, v and w are equal to zero, i.e. if the ring is compresed in its plane with a constant pressure, we get

$$\kappa_R = 0, \quad \tau = 0 \quad \text{and} \quad \Delta\kappa_z = \frac{1}{a} - \frac{1}{a_0} = -\frac{u}{aa_0} = -\frac{\varepsilon}{a}. \tag{5.3}$$

These formulae will be useful in our further analysis.

Now we will determine the relations between the geometrical quantities that describe the spring. The following notations will be used:

a and b are radii defined formerly, and we assume $a \gg b$; l is the length of the wire, so that $0 \leqslant l \leqslant 2\pi bn$, $L=2\pi bn$ is the total length of the wire; s is the length of the center line, so that $0 \leqslant s \leqslant 2\pi a$, $S=2\pi a$ is the total length; n is the total number of the coils; ε is the average angle between the coils and a plane perpendicular to the center line of the spring, so that the total length of the spring can be expressed as $2\pi bn \tan \varepsilon$. We may define ε by virtue of the assumption $a \gg b$.

Now the formulae (2.1) can be rewritten in the following approximate form that holds locally:

$$x(\psi)\simeq a+b\cos\psi,\quad y(\psi)\simeq\frac{a}{n}\psi,\quad z(\psi)=-b\sin\psi, \tag{5.4}$$

where ψ stands for $n\varphi$; $\psi\in(0,2\pi n)$. Hence we obtain

$$l=\frac{s}{\sin\varepsilon}=\frac{b\psi}{\cos\varepsilon},\quad dl=\frac{ds}{\sin\varepsilon}=\frac{bd\psi}{\cos\varepsilon},\quad \text{and}\quad \frac{b}{a}=m=\frac{1}{n\tan\varepsilon}. \tag{5.5}$$

Thus we have found that for the toroidal spring with $a\gg b$ the geometrical relations that concern a straight spring can be approximately adapted.

6. Constitutive equations

We derive constitutive equations by means of the energy method. To this end we have to calculate the forces and moments acting in an arbitrary cross-section of the wire. Let us represent a straight helix by (cf. (5.4))

$$x=b\cos\psi,\quad y=b\psi\tan\varepsilon,\quad z=-b\sin\psi. \tag{6.1}$$

A natural coordinate system consists of three unit vectors in the directions of the tangent **t**, the principal normal **n**, and the binormal **b**. The direction cosines of the **t, n, b**-system with respect to the original xyz-system may be obtained with the aid of the Serret–Frenet formulae. In Table 1 we give the results.

Table 1.

	x	y	z
t	$-\sin\psi\cos\varepsilon$	$\sin\varepsilon$	$-\cos\psi\cos\varepsilon$
n	$-\cos\psi$	0	$\sin\psi$
b	$\sin\psi\sin\varepsilon$	$\cos\varepsilon$	$\cos\psi\sin\varepsilon$

Having determined the direction cosines, we may resolve the forces and moments in these directions. We find

$$\begin{aligned} M_t&=W\sin\varepsilon+Nb\cos\varepsilon-M_z\cos\psi\cos\varepsilon+M_x\sin\psi\cos\varepsilon,\\ M_n&=M_z\sin\psi-M_x\cos\psi,\\ M_b&=W\cos\varepsilon-Nb\sin\varepsilon+M_z\cos\psi\sin\varepsilon+M_x\sin\psi\sin\varepsilon. \end{aligned} \tag{6.2}$$

From Eq. (6.2) we obtain a formula for the strain energy from the moments,

$$U_1=\frac{s}{2\sin\varepsilon}\left\{\frac{M_x^2+M_z^2}{2}\left[\frac{1}{EI_1}+\frac{\sin^2\varepsilon}{EI_2}+\frac{\cos^2\varepsilon}{GI_0}\right]+\frac{1}{EI_2}(Nb\sin\varepsilon-W\cos\varepsilon)^2+\right.$$
$$\left.+\frac{1}{GI_0}(Nb\cos\varepsilon+W\sin\varepsilon)^2\right\}, \tag{6.3}$$

where EI_1 and EI_2 are the bending stiffnesses in the normal and binormal directions, respectively, while GI_0 is the torsional rigidity.

There is also a contribution to the strain energy from the shear forces, cf. [6]. We find

$$U_2=\frac{Q_x^2+Q_z^2}{2EI_1}\pi nb^3=\frac{s}{2\sin\varepsilon}\frac{Q_x^2+Q_z^2}{2EI_1}b^2\cos\varepsilon. \tag{6.4}$$

The strain energy $\bar{U}$ per unit of length of the coil, expressed in the local forces and moments, is obtained by combining Eqs. (6.3) and (6.4), with the result that

$$\bar{U}=\frac{dU}{ds}=\frac{dU_1}{ds}+\frac{dU_2}{ds}=\frac{1}{2\sin\varepsilon}\left[\frac{M_R^2+M_z^2}{2}\left(\frac{1}{EI_1}+\frac{\sin^2\varepsilon}{EI_2}+\frac{\cos^2\varepsilon}{GI_0}\right)+\right.$$
$$\left.+\frac{1}{EI_2}(Nb\sin\varepsilon-W\cos\varepsilon)^2+\frac{1}{GI_0}\left[(Nb\cos\varepsilon+W\sin\varepsilon)^2+\frac{Q_R^2+Q_z^2}{2EI_1}b^2\cos\varepsilon\right]\right]. \tag{6.5}$$

We differentiate $\bar{U}$ with respect to N and W and we obtain the following formulae:

$$\begin{aligned} N&=\varepsilon\gamma_{11}-\varDelta\tau\gamma_{12},\\ W&=-\varepsilon\gamma_{12}+\varDelta\tau\gamma_{22}, \end{aligned} \tag{6.6}$$

where

$$\begin{aligned} \gamma_{11}&=\frac{\sin\varepsilon}{b^2}(GI_0\cos^2\varepsilon+EI_2\sin^2\varepsilon),\\ \gamma_{22}&=\sin\varepsilon(GI_0\sin^2\varepsilon+EI_2\cos^2\varepsilon),\\ \gamma_{12}&=\frac{\sin^2\varepsilon\cos\varepsilon}{b}(EI_2-GI_0). \end{aligned} \tag{6.7}$$

In Eq. (6.7), γ_{11} stands for the compressive rigidity and γ_{22} is the torsional rigidity of the spring. Note that if the angle ε is small, we can assume with sufficient accuracy that $\sin^2\varepsilon=0$ and $\cos^2\varepsilon=1$. Then, since $\sin\varepsilon=S/L$, we may express the rigidities γ_{11}, γ_{22} and γ_{12} as follows:

$$\gamma_{11}\equiv A\simeq\frac{aGI_0}{nb^3},\qquad \gamma_{22}\equiv C\simeq\frac{aEI_2}{nb},\qquad \gamma_{12}=0. \tag{6.8}$$

To derive the other constitutive equation we consider only that $M_R\neq0$ and $Q_z\neq0$. Then we have

$$\bar{U}=\frac{dU}{ds}=\frac{M_R^2}{2B}+\frac{Q_z^2}{2D}, \tag{6.9}$$

where

$$B=\frac{2\sin\varepsilon}{\dfrac{1}{EI_1}+\dfrac{\sin^2\varepsilon}{EI_2}+\dfrac{\cos^2\varepsilon}{GI_0}}\simeq\frac{2EI_1a}{nb\left(1+\dfrac{EI_1}{GI_0}\right)},\qquad D=\frac{2EI_1a}{nb^3}. \tag{6.10}$$

In Eq. (6.10), B denotes the flexural rigidity of the spring and D stands for the shear rigidity.

The relation (6.9) may be rewritten in the following form:

$$\delta \bar{U} = M_R \, \delta \frac{d\theta}{ds} + Q_z \, \delta\gamma, \tag{6.11}$$

where θ is a rotation of a cross-section and $\gamma = -\theta + \int^{s} \Delta\kappa_R \, ds$. It follows from Eqs. (6.9) and (6.11) that $M_R = B \frac{d\theta}{ds}$ and $Q_z = D\gamma$. Eliminating θ we arrive at the following constitutive equation:

$$\Delta\kappa_R = \frac{M_R}{B} + \frac{1}{D} \frac{dQ_z}{ds}. \tag{6.12}$$

In the same way we obtain

$$\Delta\kappa_z = \frac{M_z}{B} - \frac{1}{d} \frac{dQ_R}{ds}. \tag{6.13}$$

The rigidities A, B, C and D are inversely proportional to the number of coils per unit of length of the spring, hence:

$$A = A_0 \frac{a}{a_0}, \quad B = B_0 \frac{a}{a_0}, \quad C = C_0 \frac{a}{a_0}, \quad \text{and} \quad D = D_0 \frac{a}{a_0}, \tag{6.14}$$

where the subscript o denotes the values of the rigidities and of the radius a for the spring when unloaded. With regard to the radius b we assume $b = b_0$.

Thus, making use of Eqs. (6.6), (6.8), (6.12), (6.13) and (6.14) we obtain the following constitutive equations:

$$\begin{gathered} N = \varepsilon A_0 \frac{a}{a_0}, \quad W = \Delta\tau C_0 \frac{a}{a_0}, \\ \Delta\kappa_R = \frac{M_R a_0}{B_0 a} + \frac{dQ_z}{ds} \frac{a_0}{D_0 a}, \quad \text{and} \quad \Delta\kappa_z = \frac{M_z a_0}{B_0 a} - \frac{dQ_R}{ds} \frac{a_0}{D_0 a}. \end{gathered} \tag{6.15}$$

7. The general equations of equilibrium

In the general three-dimensional theory, the set of equilibrium equations for a curved beam has the following form:

$$\begin{aligned} &\frac{dQ_R}{ds} + \tau Q_z - \kappa_z N + P_R = 0, && \frac{dM_R}{ds} + \tau M_z - \kappa_z W + Q_z + L_R = 0, \\ &\frac{dQ_z}{ds} + \kappa_R N - \tau Q_R + P_z = 0, && \frac{dM_z}{ds} + \kappa_R W - \tau M_R - Q_R + L_z = 0, \\ &\frac{dN}{ds} + \kappa_z Q_R - \kappa_R Q_z + P_\varphi = 0, && \frac{dW}{ds} + \kappa_z M_R - \kappa_R M_z + L_\varphi = 0. \end{aligned} \tag{7.1}$$

In Eq. (7.1), κ_R, κ_z and τ are the curvatures and twist defined by Eq. (5.1); Q_y and Q_z are the shearing forces; N is the normal force; M_R and M_z stand for the bending moments; W denotes the torque; P_R, P_z and P_φ are the external forces; and L_R, L_z and L_φ denote the external moments.

By virtue of the considerations given in Chapters 3, 4 and 5, we have

$$P_R=\frac{nF_R}{2\pi a}=-\frac{n^2}{a}I^2 K, \quad P_\varphi=\frac{n\Delta F_\varphi}{2\pi a}=-\alpha\omega_{1z}\frac{n^2}{a}I^2 K, \quad P_z=0,$$

(7.2) $$L_\varphi=\frac{n(N_R+\Delta N_R)}{2\pi a}\equiv L_{R0}+\Delta L_R=-\frac{\mu n I^2 b}{2\pi a}+\alpha\omega_{1R}\, n^2 I^2 Km^*,$$

$$L_\varphi=0, \quad L_z=\frac{n\Delta N_z}{2\pi a}=\alpha\omega_{1z}\, n^2 I^2 K,$$

where

(7.3) $$\omega_{1z}=\frac{1}{a}\left(v-a\frac{du}{ds}\right) \quad \text{and} \quad \frac{d\omega_{1R}}{ds}=\kappa_R.$$

The full set of equations describing an equilibrium of the assembly of coils under consideration contains also the constitutive equations (6.15).

We shall split up the set (7.1) into two sets. The first one will describe the so-called intermediate state, when ΔF and ΔN are equal to zero and curvatures and twist are given by Eq. (5.3). The second set will deal with the disturbances themselves. Then we shall determine the equation describing a dependence between the critical current I_{cr} and the mechanical properties of the set of coils under consideration.

8. The intermediate state

The set of equilibrium equations now consists of the equations (7.1) with the curvatures and twist being given by Eq. (5.3) and with the external force P_R and moment L_{R0} given by Eqs. $(7.2)_1$ and $(7.2)_4$, respectively. In addition, the constitutive equations have to be taken into account. For the intermediate state only Eqs. $(6.15)_1$ and $(6.15)_4$ are essential. Thus this state is described by the following set of equilibrium and constitutive equations:

$$\frac{dQ_{R0}}{ds}-\frac{N_0}{a}+P_{R0}=0, \quad \frac{dQ_{z0}}{ds}=0, \quad \frac{dN_0}{ds}+\frac{Q_{R0}}{a}=0,$$

(8.1) $$\frac{dM_{R0}}{ds}-\frac{W_0}{a}+Q_{z0}+L_{R0}=0, \quad \frac{dM_{z0}}{ds}-Q_{R0}=0, \quad \frac{dW_0}{ds}+\frac{M_{R0}}{a}=0,$$

$$N_0=\varepsilon A_0\frac{a}{a_0}, \quad \text{and} \quad M_{z0}=-\frac{B_0}{a}\varepsilon.$$

In Eq. (8.1), we have marked the internal and external moments and forces with the subscript o to distinguish them from the disturbances. Hence we obtain

$$Q_0=0, \quad Q_{z0}=0, \quad N_0=-ap, \quad M_{R0}=0, \quad M_{z0}=\frac{B_0}{A_0}p, \quad W_0=-\frac{\mu abp}{2\pi Kn}, \tag{8.2}$$

where we have denoted

$$p=\frac{n^3}{a}KI^2. \tag{8.3}$$

In the system of equations (8.1) as well as in the further analysis, the radius a of the deformed spring is unknown. To determine it we can use the equations (8.1) and $(8.2)_3$. Hence, by virtue of Eq. $(5.3)_3$ we find the following equation:

$$a=a_0\left(1-\frac{a_0}{A_0}p\right). \tag{8.4}$$

Of course, p itself also depends on a (cf. (8 3)). However, the formula (8.3) will allow us to replace a by a value p that is to be determined in the further analysis.

9. The general case of the three-dimensional disturbed state

Now we deal with the deformed toroidal helix. Let the disturbances be denoted as Q_R, Q_z, ..., and κ_R, $\Delta\kappa_z$, τ. Making use of the formulae (7.1) to (8.2) we arrive at the following set of linearized equilibrium equations:

$$Q_R'-\frac{N}{a}+ap\Delta\kappa_z=0, \quad Q_z'-ap\kappa_R=0, \quad N''+\frac{Q_R'}{a}-\alpha p\Delta\kappa_z=0,$$

$$M_R''+\frac{B_0}{A_0}p\tau'-\frac{W'}{a}+\frac{\mu ab}{2\pi Kn}p\Delta\kappa_z'+Q_z'+a_0\,\alpha pm^*\,\kappa_R=0, \tag{9.1}$$

$$M_R''-\frac{\mu ab}{2\pi Kn}p\kappa_R'-Q_R'+\alpha ap\Delta\kappa_z=0, \qquad W'+\frac{M_R}{a}-\frac{B_0}{A_0}p\kappa_R=0.$$

In Eq. (9.1), $(\cdot)'\equiv\frac{d}{ds}(\cdot)$. The set of equations (9.1) has to be completed by the constitutive equations (6.15). The latter can be rewritten in the form

$$\tau=\frac{a_0\,W}{C_0\,a}, \qquad \kappa_R=\frac{a_0\,M_R}{aB_0}+\frac{a_0\,Q_z'}{aD_0}, \qquad \Delta\kappa_z=\frac{a_0\,M_z}{aB_0}-\frac{a_0\,Q_R'}{aD_0}. \tag{9.2}$$

Equation $(6.15)_1$ is, in fact, the definition of ε and therefore we do not use it.

The complete system of equations consists of Eqs. (9.1) and (8.2) and, in addition, Eq. (8.4). This set contains 11 unknown functions: Q_R, Q_z, N, M_R, M_z, W, κ_R, $\Delta\kappa_z$, τ, a and p. The equations are nonlinear, but the products of the type $p\kappa_R$ or $p\tau$ are not troublesome because p is assumed to be independent of s. Thus, in fact, p is a kind of parameter in the system (9.1) and (9.2), and so is a.

In the following analysis we determine the value p that leads to instability. Later on, making use of Eqs. (8.3) and (8.4), we shall be able to determine the critical current I_{cr} and the radius of the deformed spring a.

Resolving the system (9.1) and (9.2), we arrive at the following two equations for M_R and M_z:

$$a^2 q^2 M_z'''' + q\left[1+\frac{a^2 a_0 p}{B_0} + \alpha a_0 p\left(\frac{1}{D_0}+\frac{a^2}{B_0}\right)\right] M_z'' = \nu a^2 q \frac{a_0 p}{B_0} M_R''' + $$

$$(9.3) \qquad + \frac{\nu a_0 p}{B_0}\left(1+\frac{\alpha a_0 p}{D_0}\right) M_R',$$

$$\frac{a_0 pq}{D_0} M_z''' - \frac{a_0 pq}{B_0} M_z' = \frac{M_R''}{\nu}\left[\frac{a_0^2 \nu^2 p^2}{B_0 D_0} + q\left(1+\frac{\alpha a_0 p}{D_0}\right)\right] +$$

$$+ \frac{M_R}{\nu a^2}\left(1+\frac{\alpha a_0 p}{D_0}\right)\left[q\left(1-\frac{p a_0 B_0}{C_0 A_0}\right) - \frac{a_0 p}{A_0} + \frac{a_0^2 p^2 B_0}{C_0 A_0^2} + \frac{a_0 a^2 p}{B_0} + \alpha m^* \frac{a a_0^2 p}{B_0}\right],$$

where

$$(9.4) \qquad \nu = \frac{\mu b}{2\pi K n}, \qquad q = 1 - \frac{a_0 p}{D_0}.$$

Now, substituting into Eq. (9.3) the moments M_R and M_z in the form

$$(9.5) \qquad M_R = \mathcal{M}_R \cos\left(\frac{k}{a} s\right), \qquad M_z = \mathcal{M}_z \sin\left(\frac{k}{a} s\right),$$

and making use of Eq. $(9.4)_2$, we obtain the equation for $\bar{p}$:

$$(9.6) \qquad -\bar{p}^2 \nu^2 L_1 L_3 [k^2 - 1 - (k^2+\alpha)\bar{p} L_1]\left[k^2 + \frac{L_3}{L_1}(1-\bar{p})^2\right] =$$

$$= a_0^2 [k^2 - 1 - \bar{p} L_1 (k^2+\alpha) - \bar{p} l_3 (1-\bar{p})^2 (1+\alpha)] [(1-k^2)(1+\bar{p}L_1)(1-\bar{p}L_1) -$$

$$- k^2 \frac{\nu^2}{a_0^2} \bar{p}^2 L_1 L_2 + \bar{p} W_1 + \bar{p}^2 W_2 + \bar{p}^3 W_3 + \alpha \bar{p}^4 L_1 L_3],$$

where

$$W_1 = -1 - L_2 + L_3(1+\alpha m^*), \qquad W_3 = \alpha L_1 L_2 (1+L_1) + L_3 - \alpha L_1 L_3 (2+\alpha m^*),$$

$$W_2 = L_1 L_2 (1-\alpha) + L_2 - L_1 + L_1 L_3 (1+\alpha m^*) - L_3 (2+\alpha m^*),$$

$$L_1 = \frac{A_0}{D_0}, \qquad L_2 = \frac{B_0}{C_0}, \qquad L_3 = \frac{a_0^2 A_0}{B_0} \quad \text{and} \quad \bar{p} = \frac{a_0 p}{A_0}.$$

As Eq. (9.6) is rather complicated, we shall try to find the critical pressure p_{cr} for the simplified case, when the out-of-plane displacements are neglected.

10. The plane case

The force N_0 and moment M_0 given by Eqs. $(8.2)_3$ and $(8.2)_5$ hold for the intermediate plane state. Then we consider the deformed state. The complete system of equations for the disturbances reads

$$
(10.1)\qquad \begin{aligned}
&Q_R' - N_0\,\Delta\kappa_z - \frac{N}{a} = 0, \qquad N' + \frac{Q_R}{a} + P_\varphi = 0,\\
&M_z' - Q_R + L_z = 0, \qquad \Delta\kappa_z = \frac{a_0 M_z}{aB_0} - \frac{a_0 Q_R'}{aD_0},
\end{aligned}
$$

where P_φ and L_z are given by Eqs. $(7.3)_2$ and $(7.3)_6$, respectively.

After finding the solutions for the above equations and then making use of Eq. $(9.5)_2$, we find the relation

$$
(10.2)\qquad a_0\,p = \frac{(k^2-1)\,B_0}{a^2\,(1+\alpha) + (k^2+\alpha)\dfrac{B_0}{D_0}}.
$$

The critical pressure p_{cr} is determined by $k=2$. However, Eq. (10.2) also contains the radius a, an unknown. Therefore, the formula (10.2) has to be considered together with Eq. (8.4). Combining these two formulae we arrive at the following equation:

$$
(10.3)\qquad \bar{p}^3 - 2\bar{p}^2 + \left(1 + \frac{B_0\,(4+\alpha)}{a_0^2\,D_0\,(1+\alpha)}\right)\bar{p} - \frac{3B_0}{a_0^2\,A_0\,(1+\alpha)} = 0,
$$

where $\bar{p} = \dfrac{a_0\,p}{A_0}$.

The determination of $\bar{p}_{cr}$ (i.e. also p_{cr}) allows us to find a from Eq. (8.4) and later on to find the critical current I_{cr}. The latter can be expressed by

$$
(10.4)\qquad I_{cr}^2 = \frac{2\pi A_0}{n^2\,\mu m^2}\,\bar{p}_{cr}\,(1-\bar{p}_{cr})\sqrt{(1-\bar{p}_{cr})^2 - m^2}\,\left[1 - \bar{p}_{cr} + \sqrt{(1-\bar{p}_{cr})^2 - m^2}\right],
$$

where $m = \dfrac{b}{a_0}$.

11. Discussion

This investigation of instability has led us to an equation determining the dimensionless critical pressure $\bar{p}_{cr}$ that causes the loss of stability in the helix under consideration. In the three-dimensional case we have obtained for $\bar{p}_{cr}$ an algebraic equation of degree 7 that has at least one positive root. The equation simplifies considerably when the planer model of instability is investigated. In this case we have obtained an equation of degree 3 that also has at least one positive root. For the planar case we have performed some numerical calculations considering a toroidal helix with $n=120$, $a=6$ m and $b=0.5$ m made of a copper wire. We have listed in Table 2 the more interesting results of the calculations for some sizes of the wire and for two values of α.

Table 2.

Cross-section	Circular, the radius of the wire =0.01 m		Circular, the radius of the wire =0.1 m		Rectangular, 0.1 m×0.2 m	
A_0, N	289		$2.89.10^6$		$3.83.10^5$	
B_0, Nm^2	66.2		$6.62.10^5$		$4.85.10^4$	
D_0, N	490		$4.90.10^6$		$2.60.10^5$	
R, Ω	$1.86.10^{-2}$		$1.86.10^{-4}$		$2.92.10^{-4}$	
a, —	0.3	0.7	0.3	0.7	0.3	0.7
$\bar{p}_{cr}$, —	0.0206	0.0153	0.0206	0,0152	0.0069	0.0050
p_{cr}, Nm^{-1}	0.9954	0.7365	9945	7365	442	321
I_{cr}, Amps	745.8	648.5	74580	64850	16100	13750
$I_{cr}^2 R$, kW	10.35	7,82	1035	782	75.45	55.21
B, gauss	30	26	2980	2594	644	550

In Table 2 the dimensions a and b of the coil are of the order of magnitude of the corresponding dimensions of the magnets designed for fusion reactors. It appears that the magnetic fields are much weaker. To increase these fields the currents have to be increased, a state which may be realised by enlarging the dimensions of the wire. When such action is taken, the Joule heat becomes dangerous, and for this reason superconductive material has to be used in the design of fusion reactors. However, even with this kind of material it appears that the unconstrained set of coils becomes mechanically unstable at a much smaller critical current than the constrained one does, as could be expected. If the dimensions of the magnet are decreased, the phenomenon of magneto-elastic stability may be studied in the laboratory at room temperature. Such a study is important for the understanding of the physical basis on which the technology for the design of fusion reactors rests. The study may be extended by the introduction of constraints.

Appendix

The formula (2.3) that describes approximately the magnetic field **B** in the toroidal helix is derived here.

Let us consider a discrete model of planar coils insteated of the helix. A parameterization of such an assembly may be given by

$$x=(a+b\cos\psi)\cos\varphi_k,\quad (a+b\cos\psi)\sin\varphi_k,\quad z=-b\sin\psi. \tag{A.1}$$

In Eq. (A.1), $\varphi_k=\dfrac{2\pi k}{n}$, where φ_k denotes an angle that determines the place of each coil in the xy-plane, and $k=1, 2, ..., n$. ψ stands for the angle in the plane of the coil. Making use of the formula (2.2) with

$$\mathbf{r}=[R-(a+b\cos\psi)\cos\varphi_k;\ -(a+b\cos\psi)\sin\varphi_k;\ b\sin\psi], \tag{A.2}$$

(since we can choose the coordinates of any point, we will look for the field **B** at $x=R$, $y=0$ and $z=0$, a choice which is allowed because of the symmetry of the assembly as well as because of the assumption $b \ll a$), we obtain the following coordinates of the magnetic field **B**:

$$\text{(A.3)} \qquad B_R=0, \quad B_z=0, \text{ and } B_\varphi=\sum_{k=1}^{n}\frac{\mu I}{4\pi}\int_{-\pi}^{\pi}\frac{d\psi}{r^3}\left[b^2\cos\varphi_k-b(R-a\cos\varphi_k)\cos\psi\right],$$

where $r=|\mathbf{r}|$. Now we are interested in the integral on the right-hand side of Eq. $(\text{A.3})_3$. We introduce the abbreviations

$$\text{(A.4)} \qquad \begin{aligned} &p_1=b^2\cos\varphi_k, \quad p_2=-b(R-a\cos\varphi_k),\\ &p_3=R^2+a^2+b^2-2aR\cos\varphi_k, \quad \text{and} \quad p_4=2b(a-R\cos\varphi_k). \end{aligned}$$

Introducing Eq. (A.4) into the integral we obtain after some calculations

$$\text{(A.5)} \qquad \int_{-\pi}^{\pi}\frac{p_1+p_2\cos\psi}{(p_3+p_4\cos\psi)^{3/2}}d\psi=\frac{4}{(p_3+p_4)^{1/2}}\left\{\frac{p_2\,p_3}{p_4(p_3-p_4)}[\mathsf{K}(l)-\mathsf{E}(l)]+\right.$$
$$\left.+\frac{p_1\,\mathsf{E}(l)-p_2\,\mathsf{K}(l)}{p_3-p_4}\right\}.$$

where $l^2=\dfrac{2p_4}{p_3+p_4}$ and $\mathsf{E}(l)$ $\mathsf{K}(l)$ and are the complete elliptic integrals of the first and second kind, respectively. By virtue of the assumption $a \ll b$ we have $l^2 \simeq m=b/a$ and $l \to 0$ if $m \to 0$. Then, taking into account the expansions of K and E with respect to l [7] and assuming $l \gg 1$, we can write the approximate form

$$\text{(A.6)} \qquad \mathsf{K}(l)\simeq\frac{\pi}{2}\left(1+\frac{l^2}{4}\right), \quad \mathsf{E}(l)\simeq\frac{\pi}{2}\left(1-\frac{l^2}{4}\right), \quad \text{and} \quad \mathsf{K}(l)-\mathsf{E}(l)\simeq\frac{\pi l^2}{4}.$$

Now we can evaluate the integral on the left-hand side of Eq. (A.5) to get the following approximate form:

$$\text{(A.7)} \qquad \int_{-\pi}^{\pi}\frac{p_1+p_2\cos\psi}{(p_3+p_4\cos\psi)^{3/2}}d\psi=\frac{\pi(2p_1\,p_3+p_1\,p_4-3p_2\,p_4)}{(p_3^2-p_4^2)(p_3+p_4)^{1/2}}.$$

Hence, by virtue of Eq. $(\text{A.3})_3$, we find

$$\text{(A.8)} \qquad B_\varphi\simeq\frac{\mu I}{4}\sum_{k=1}^{n}\frac{2p_1\,p_3+p_1\,p_4-3p_2\,p_4}{(p_3^2-p_4^2)(p_3+p_4)^{1/2}}.$$

Now we take $R\simeq a$. Moreover, since for $n\to\infty$ the sum (A.8) may be treated approximately like an integral, we make the substitution $k=\dfrac{n\varphi}{2\pi}$, $\Delta k=\dfrac{n\Delta\varphi}{2\pi}$. Thus we approximate

$$\text{(A.9)} \qquad B_\varphi\simeq$$
$$\simeq\frac{\mu nI}{4\pi R}m^2\int_0^{\pi}\frac{[4(1-m)\sin^4(\varphi/2)+(4+2m-2m^2)\sin^2(\varphi/2)+m^2]\,d\varphi}{\sqrt{(1+m)\sin^2(\varphi/2)+m^2/4}\,[16(1-m^2)\sin^4(\varphi/2)+8m^2\sin^2(\varphi/2)+m^4]},$$

where use has been made of Eq. (A.4). Now let us replace

$$\sin\frac{\varphi}{2}=\frac{m}{2}\,\alpha. \tag{A.10}$$

Hence we obtain

$$B_\varphi \simeq \frac{\mu n I}{2\pi R}\int\limits_0^{2/m} \frac{\left[1+\left(1+\frac{m}{2}-\frac{m^2}{2}\right)\alpha^2+\frac{m^2-m^3}{4}\alpha^4\right]d\alpha}{[(1+\alpha^2)^2-m^2\alpha^4]\sqrt{[1-(m^2\alpha^2/4)][1+(1+m)\alpha^2]}}. \tag{A.11}$$

For $m \ll 1$ we can write down B_φ in the approximate form:

$$B_\varphi \simeq \frac{\mu n I}{2\pi R}\int\limits_0^{\infty}\frac{d\alpha}{(1+\alpha)^{3/2}}+O(m). \tag{A.12}$$

Since the integral in Eq. (A.12) is equal to 1, the formula (2.3) is obtained as a simple consequence of Eq. (A.12).

Streszczenie

MAGNETO-SPRĘŻYSTA STATECZNOŚĆ UKŁADU CEWEK BEZ WIĘZÓW

W pracy bada się mechaniczną stateczność zwojnicy mającej kształt toroidalnej linii śrubowej, na którą oddziaływuje jej własne pole magnetyczne. Zwojnica ta jest pozbawiona więzów natury mechanicznej, tzn. cewki nie są przymocowane do jakiejkolwiek sztywnej czy sprężystej podstawy. W pierwszej części pracy wyznaczono pole magnetyczne, siły i momenty działające na każdą pojedynczą cewkę, zarówno dla zwojnicy nieodkształconej, jak i odkształconej. W części drugiej przedstawiona jest analiza stateczności dla modelu sprężyny zwojowej, która uwzględnia również rozciąganie oraz ścinanie.

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Резюме

МАГНИТОУПРУГАЯ УСТОЙЧИВОСТЬ СИСТЕМЫ КАТУШЕК БЕЗ СВЯЗЕЙ

В работе исследуется механическая устойчивость обмотки, имеющей форму тороидальной винтовой линии, на которую воздействует ее собственное магнитное поле. Эта обмотка не имеет связей механической природы, т.зн. катушки не прикреплены к какому-нибудь жесткому или упругому основанию. В первой части работы определены магнитное поле, силы и моменты, действующие на каждую единичную катушку, так для недеформирувмой, как и деформируемой обмотки. Во второй части представлен анализ устойчивости для модели спиральной пружины, которая учитывает тоже растяжение и сдвиг.

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