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MIXED FINITE ELEMENT SOLUTION OF QUASI-STATIC PROBLEMS OF VISCOPLASTIC PLATES (*)

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The paper is concerned with the numerical analysis of rigid, viscoplastic and elastic viscoplastic plates subjected to static loading. The small deflection theory of thin plates is employed. The constitutive equations of viscoplasticity are taken in the form proposed by Perzyna. A numerical solution scheme is formulated by using the mixed element method in which the nodal values of bending moments and of deflection are the unknown discrete parameters to be determined. Both the triangular elements presented by Hellan and Herrmann and the rectangular element proposed by Bäcklund have been used. Two methods have been considered for solving the resulting system of ordinary first-order differential equations with nonlinear coefficients. Sample problems which have been solved include simplysupported circular plates subjected to uniform load and to concentrated load at the the center and simply-supported rectangular plate under uniform load. The viscoplastic algorithm has also been used for the determination of limit loads of circular and rectangular rigid plastic plates.

1. Introduction

In certain loading and thermal conditions the time dependent properties of structural materials have a practical significance in the design procedure. The strain rate sensitivity of the material is important in impact or impulsive loading and, in quasi-static loading, creep can be dominant at normal or elevated temperatures, depending on the material considered.

The viscoplastic constitutive model, introduced by Bingham [1] and Hohenemser and Prager [2] and developed later by Perzyna [3], is capable of taking into account the time dependent behaviour of materials. Some problems of rigid viscoplastic plates based on Perzyna's model were solved by Appleby and Prager [4] and by Wierzbicki [5]. Afterwards, several studies on viscoplastic plates including also dynamic effects were published; here only [6] and [7] are mentioned. Perzyna's viscoplastic model has been widely applied, particularly in numerical computations with the finite element method, e.g. [8–11].

In this paper a numerical solution for rigid viscoplastic and elastic viscoplastic plates is presented. The mixed formulation of the finite element method is used in the application to the small deflection theory of thin plates. The mixed method for elastic plates was introduced by Herrmann [12] and Hellan [13]. Hellan extended

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his solution also to stationary and transient creep analyses of plates using Norton's creep model [14, 15]. BACKLUND applied the mixed formulation to the solution of elastic plastic plates [16, 17]. Here the finite elements developed by Herrmann, Hellan, and Backlund are made use of in the analysis of viscoplastic plates. The resulting system of first-order differential equations with nonlinear coefficients has been solved by using two methods, the Euler forward integration and a secant flexibility method. The numerical stability of both methods has been studied experimentally. Some problems of rigid viscoplastic and elastic viscoplastic plates have been solved and the results have been compared with those obtained by previous investigators. The viscoplastic algorithm has also been used for the determination of the limit loads of rigid plastic plates.

2. CONSTITUTIVE EQUATIONS

The constitutive equations of an elastic viscoplastic solid, proposed by PERZYNA [3], are presented in the following form:

$$\begin{aligned} \dot{\varepsilon} &= \dot{\varepsilon}^{e} + \dot{\varepsilon}^{vp}, \\ \dot{\varepsilon}^{e} &= D^{-1}\,\dot{\sigma}, \\ \dot{\varepsilon}^{vp} &= \gamma < \Phi(F) > \frac{\partial F}{\partial \sigma}, \end{aligned} \tag{2.1}$$

where D is the elasticity matrix, γ the fluidity coefficient, and F the yield function in static yielding. The function Φ is chosen to meet the actual material properties. The symbol $<\Phi>$ is defined as follows:

$$<\Phi> = \begin{cases} 0 & \text{if} \quad \Phi \leqslant 0, \\ \Phi & \text{if} \quad \Phi > 0. \end{cases} \tag{2.2}$$

The actual yield function used in this study is the Huber-Mises yield function

$$F = \sqrt{J_2}/k - 1, \tag{2.3}$$

where $J_2 = s_{ij}\, s_{ij}/2$ denotes the second invariant of the stress deviator and k is the yield stress in simple shear. For Φ the power function

$$\Phi(F) = \operatorname{sign} F\, |F|^{m} \tag{2.4}$$

is used.

Assuming the state of plane stress, as is customary in the treatment of plates the stress and strain vectors and the elasticity matrix are

$$\begin{aligned} \sigma &= (\sigma_x\ \sigma_y\ \tau_{xy})^T, \\ \varepsilon &= (\varepsilon_x\ \varepsilon_y\ \gamma_{xy})^T, \end{aligned} \tag{2.5}$$

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{2.6}$$

The rate of viscoplastic strain is

(2.7) $$\dot{\varepsilon}^{vp}=\frac{\gamma}{6k\sqrt{J_2}}<\Phi(F)>P_0\,\sigma,$$

where

(2.8) $$P_0=\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & 0\\ 0 & 0 & 6\end{bmatrix}.$$

According to Eq. (2.1) the stress rate can be expressed in the form

$$\dot{\sigma}=D(\dot{\varepsilon}-\dot{\varepsilon}^{vp}).$$

The rate of moment is defined by integration through the plate thickness

$$\dot{M}=\int\dot{\sigma}zdz=\int D(\dot{\varepsilon}-\dot{\varepsilon}^{vp})zdz=\int Dz^2\,dz\dot{\kappa}-\int D\dot{\varepsilon}^{vp}\,zdz$$

assuming that the Kirchhoff hypothesis $\varepsilon=z\kappa$ holds for the plate. The solution for $\dot{\kappa}$ yields

(2.9) $$\dot{\kappa}=C\dot{M}+\dot{\kappa}^{vp},$$

where

(2.10) $$C=\frac{12}{Eh^3}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix},$$

(2.11) $$\dot{\kappa}^{vp}=C\int D\dot{\varepsilon}^{vp}\,zdz=P_0\int\frac{\gamma}{6k}\,\frac{<\Phi(F)>}{\sqrt{J_2}}\,\sigma zdz.$$

For the special case of a rigid viscoplastic plate obeying the Huber-Mises yield condition

$$F=\sqrt{\sigma_{11}^2-\sigma_{11}\,\sigma_{22}+\sigma_{22}^2+3\sigma_{12}^2}/\sqrt{3}k-1=0$$

and Φ according to Eq. (2.4), the following constitutive equation can be derived:

(2.12) $$\dot{\kappa}^{vp}=\frac{B\,(M_{\mathrm{eff}}/M_0-1)^m}{M_{\mathrm{eff}}}P_0\,M,\quad \text{when}\quad M_{\mathrm{eff}}>M_0,$$

where

(2.13) $$\begin{aligned}&B=(\gamma/\sqrt{3}kh)\,[(2m+1)/2m]^m,\\ &M_{\mathrm{eff}}=\sqrt{M_x^2-M_x\,M_y+M_y^2+3M_{xy}^2},\\ &M_0=\sqrt{3}kh^2/4,\\ &\dot{\kappa}^{vp}=(\dot{\kappa}_x^{vp}\,\dot{\kappa}_y^{vp}\,\dot{\kappa}_{xy}^{vp})^T=(-\dot{w}_{,xx}^{vp}-\dot{w}_{,yy}^{vp}-2\dot{w}_{,xy}^{vp})^T.\end{aligned}$$

3. Hellinger-Reissner principle

The Hellinger-Reissner functional foı elastic plates was presented by HERRMANN [12] in a form suitable for mixed finite element formulation

$$\pi_R=\iint_A\left[-\frac{1}{2}M^T CM+(\Delta M)^T \nabla w-pw\right]dx\,dy-\int_S M_{ns}\,w_{,s}\,ds- \tag{3.1}$$
$$-\int_{S_\sigma}\bar{V}_n w ds-\int_{S_u} M_n\,\bar{w}_{,n}\,ds,$$

where

$$M=(M_x\,M_y\,M_{xy})^T,$$
$$\Delta=\begin{bmatrix}\frac{\partial}{\partial x}&0&\frac{\partial}{\partial y}\\ 0&\frac{\partial}{\partial y}&\frac{\partial}{\partial x}\end{bmatrix},$$
$$\nabla=\left(\frac{\partial}{\partial x}\;\frac{\partial}{\partial y}\right)^T, \tag{3.2}$$
$$w_{,s}=\frac{\partial w}{\partial s},$$

M_n, M_{ns} are the internal bending and twisting moments at the boundary. $\bar{M}_n$, $\bar{V}_n$, $\bar{w}$, $\bar{w}_{,n}$ are the prescribed external forces and displacements on the parts S_σ and S_u of the boundary, respectively. In the expression (3.1) the bending moment and the deflection satisfy the boundary conditions $M_n=\bar{M}_n$ on S_σ and $w=\bar{w}$ on S_u. The stationary condition of π_R is

$$\delta\pi_R=\iint_A[-\delta M^T\kappa+(\Delta\delta M)^T\nabla w+(\Delta M)^T\nabla\delta w-p\delta w]\,dx\,dy+ \tag{3.3}$$
$$-\int_S(\delta M_{ns}\,w_{,s}+M_{ns}\,\delta w_{,s})\,ds-\int_{S_\sigma}\bar{V}_n\,\delta w\,ds+\int_{S_u}\delta M_n\,\bar{w}_{,n}ds=0,$$

where

$$\kappa=(\kappa_x\,\kappa_y\,\kappa_{xy})^T=(-w_{,xx}-w_{,yy}-2w_{,xy})^T \tag{3.4}$$

equals $\kappa=CM$ for elastic plates. The condition (3.3) provides the equation for finding the solution of the plate problem. The stationary condition holds also in rate form where the quantities are replaced by their respective rates: $w\leftrightarrow\dot{w}$, $\kappa\leftrightarrow\dot{\kappa}$, $M\leftrightarrow\dot{M}$, etc.

4. The mixed finite element formulation

In the mixed method the deflection and the moments are approximated by polynomials inside the elements

$$w=Nq, \quad M=Lr. \tag{4.1}$$

N and L are matrices of the shape functions, and q and r vectors of nodal deflections and moments, respectively. The values of internal moments on element boundaries or on the boundary of the plate can be presented in the form

$$M_n = L_n r, \qquad M_{ns} = L_{ns} r, \tag{4.2}$$

where L_n and L_{ns} depend on the orientation of the boundary and on the shape functions. Inserting the expressions (4.1), (4.2) and (2.9) into the condition (3.3) in rate form, the following system of first-order differential equations results:

$$\begin{bmatrix} H & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{\bar{R}} + \dot{R}^{vp} \\ \dot{Q} \end{bmatrix}, \tag{4.3}$$

where

$$\begin{aligned} H &= -\iint_A L^T C L \, dx\, dy, \\ G &= \iint_A (\Delta L)^T (\nabla N)\, dx\, dy - \int_S L_{ns}^T N_{,s} ds, \\ \dot{R} &= \int_{S_u} L_n^T \dot{\bar{w}}_{,n}\, ds, \\ \dot{Q} &= \iint_A N^T \dot{p}\, dx\, dy + \int_{S_\sigma} N^T \dot{\bar{V}}_n ds. \end{aligned} \tag{4.4}$$

The quantity $\dot{R}^{vp}$

$$\dot{R}^{vp} = \iint L^T \dot{\kappa}^{vp}\, dx\, dy \tag{4.5}$$

is interpreted as the rate of a pseudo-load. The system of equations (4.3), including all elements of the plate, provides the solution of the plate problem.

The elements which were used in computations are the triangular element (Fig. 1a) by HELLAN [13] and HERRMANN [12] and the rectangular element (Fig. 1b) by Backlund [17]. In the Hellan-Herrmann element the state of stress is uniform and the moment parameters are the values of bending moment M_1, M_2, M_3 on the sides of the triangle. The deflection is a linear function of position with nodal values as parameters. In the Backlund element the bending moment M_x varies linearly with respect to x but does not depend on y, and the bending moment M_y linearly

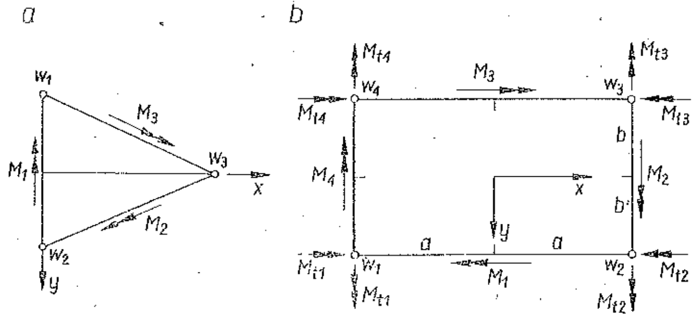

FIG. 1. a) Triangular element by Hellan and Herrmann. b) Rectangular element by Backlund.

with respect to y but does not depend on x. The shape functions for the twisting moment contain the term xy in addition to the linear terms. The nodal values $M_{t1}, \ldots, M_{t4}$ are the parameters. The deflection is approximated in the same way as the twisting moment. For details the reference is made to the papers mentioned above.

5. Solution methods

For elastic viscoplastic plates the plate is divided into layers through thickness, and the state of stress is evaluated at the middle surface of each layer. Division to 10 layers was used in the computation of numerical examples. Division is not needed for rigid viscoplastic plates.

5.1. *Euler procedure*

The Euler procedure for the solution is as follows:

1) Assume that the solution w, M, σ is known at time t.
2) Evaluate $\Delta\varepsilon^{vp}=\dot{\varepsilon}^{vp}\,\Delta t$ in each layer using Eq. (2.7).
3) Evaluate $\Delta\kappa^{vp}$ and ΔR^{vp} using Eq. (2.11) and (4.5), respectively.
4) Evaluate $\Delta R=\dot{R}\Delta t$ and $\Delta Q=\dot{Q}\Delta t$.
5) Solve Δr and Δq from Eq. (4.3).
6) Evaluate $\Delta w=N\Delta q$ and $\Delta M=L\Delta r$.
7) Evaluate $\Delta\kappa=C\Delta M+\Delta\kappa^{vp}$, $\Delta\varepsilon=z\Delta\kappa$, $\Delta\varepsilon^{e}=\Delta\varepsilon-\Delta\varepsilon^{vp}$, $\Delta\sigma=D\Delta\varepsilon^{e}$.
8) Evaluate the solution at time $t+\Delta t$: $w+\Delta w$, $M+\Delta M$, $\sigma+\Delta\sigma$.
9) Go to 2).

In the case of rigid viscoplastic plate the steps 2) and 7) are dropped and $\Delta\kappa^{vp}$ in step 3) is computed using Eq. (2.12).

5.2. *Secant flexibility method*

Assume that for an elastic viscoplastic plate the relationship between curvature and moment vectors can be expressed in the form

$$\kappa=CM+PM, \tag{5.1}$$

where PM represents the viscoplastic part of the curvature and P is a matrix depending on the current configuration. The variational principle (3.3) then results in the system of equations

$$\begin{bmatrix} H_s & G \\ G^T & 0 \end{bmatrix}\begin{bmatrix} r \\ q \end{bmatrix}=\begin{bmatrix} R \\ Q \end{bmatrix}, \tag{5.2}$$

where H_s is a secant flexibility matrix

$$H_s=H+H^{vp} \tag{5.3}$$

with H, according to Eq. $(4.4)_1$, and

$$H^{vp} = -\int\int_A L^T PL\, dx\, dy. \tag{5.4}$$

The way of constructing the relationship $\kappa^{vp} = PM$ is somewhat artificial. The formula (2.11) can be written in the form

$$\dot{\kappa}^{vp} = \bar{P}M, \tag{5.5}$$

where

$$\bar{P} = \begin{bmatrix} 2M_x^{vp}/M_x & -M_y^{vp}/M_y & 0 \\ -M_x^{vp}/M_x & 2M_y^{vp}/M_y & 0 \\ 0 & 0 & 6M_{xy}^{vp}/M_{xy} \end{bmatrix} \tag{5.5}$$

with

$$M^{vp} = \frac{\gamma}{6k} \int \frac{<\Phi(F)>}{\sqrt{J_2}} \sigma z\, dz. \tag{5.7}$$

Integration with respect to time yields

$$\kappa^{vp}(t_k) = \int_0^{t_k} \dot{\kappa}^{vp}\, dt \approx \kappa_k^{vp} = \sum_{i=1}^{k} \dot{\kappa}_i\, \Delta t_i = \sum_{i=1}^{k} \bar{P}_i\, M_i\, \Delta t_i. \tag{5.8}$$

The quantities $\bar{P}_i$ and M_i are determined at the onset of the time interval t_i. The approximation of the matrix P is found as follows:

$$P(t_k) \approx P_k = \begin{bmatrix} \kappa_{xk}^{vp}/M_{xk} & 0 & 0 \\ 0 & \kappa_{yk}^{vp}/M_{yk} & 0 \\ 0 & 0 & \kappa_{xyk}^{vp}/M_{xyk} \end{bmatrix}. \tag{5.9}$$

For rigid viscoplastic plate the relationship follows directly from the formula (2.12)

The solution of the Eq. (5.2) provides the values of w and M at time t_k, but the construction of the matrix P requires the use of an incremental procedure.

5.3. *Starting procedure*

In the case of an elastic viscoplastic plate the elastic solution is determined at time $t=0$, and the time integration is started with this initial condition. In the case of a rigid viscoplastic plate an elastic solution is first determined assuming certain elastic properties. Using the solution procedure of an elastic viscoplastic plate, a steady state of stress is sought, i.e. a state in which the moments do not change. In fact, this is the state of plastic yield. This state is used as the initial condition at $t=0$ for the time integration of a rigid viscoplastic plate.

6. Time step selection and numerical stability

Zienkiewicz and Cormeau [8] performed numerical experiments in order to find the largest feasible time interval in the displacement method. Cormeau [18] derived theoretically explicit stability criteria for certain viscoplastic flow rules. More recently Hughes and Taylor [19] have considered the stability of some algorithms for elasto/viscoplastic finite element analysis.

In this study the stability limit is presented in the form:

for the Euler procedure

$$\|\dot{\kappa}\|\, \Delta t < \tau_1 \, \|\kappa^e\|;$$

for the secant flexibility procedure

$$\|\dot{\kappa}\|\, \Delta t < \tau_2 \, \|\kappa\|,$$

where the norms are

$$\|\dot{\kappa}\| = \sqrt{\dot{\kappa}_x^2 + \dot{\kappa}_y^2 + \dot{\kappa}_{xy}^2},$$

$$\|\kappa^e\| = \sqrt{(\kappa_x^e)^2 + (\kappa_y^e)^2 + (\kappa_{xy}^e)^2},$$

$$\|\kappa\| = \sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_{xy}^2}.$$

The norms were evaluated in the element in which max $\|\kappa^e\|$ occurred. τ_1 and τ_2 are experimental coefficients of which the values varied $\tau_1 = 0.05, \ldots, 0.5$ and $\tau_2 = 0.05, \ldots, 0.3$. For rapid changes of stress a small value of τ_1 and τ_2 was necessary while for nearly stationary state a larger value could be used. Since $\|\kappa\|$ is larger than $\|\kappa^e\|$, larger time steps could be used in the secant flexibility method. However, the computer time spent at each time step of the secant flexibility method was about four times as compared to the Euler method. The experimental time step limit cannot be considered to be satisfactory, but the subject should be studied more thoroughly.

7. Numerical examples

7.1. *Rigid viscoplastic circular plate*

A rigid viscoplastic circular plate, simply supported at the edge, subjected to uniform step load is considered first. The bending moments and the rate of deflection are presented in Fig. 2 and compared with those obtained by Wierzbicki [5] numerically from a nonlinear system of differential equations. Good agreement between Wierzbicki's results and the present ones can be observed. The finite element solution was calculated using both the Euler procedure and the secant flexibility method which gave identical results.

The second example is a similar plate subjected to a concentrated load at the center. The distributions of bending moments and rate of deflection are shown in Fig. 3.

In these two examples the triangular element was used.

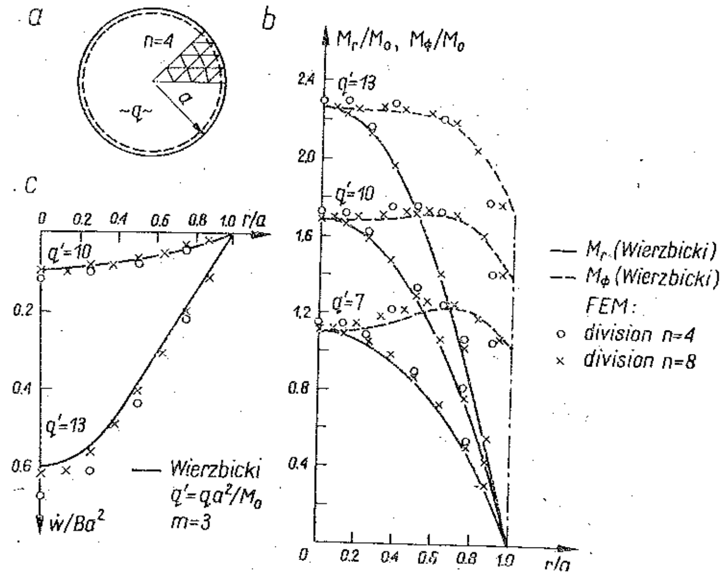

FIG. 2. Simply supported circular plate under uniform load. a) Element division. b) Bending moments. c) Rate of deflection

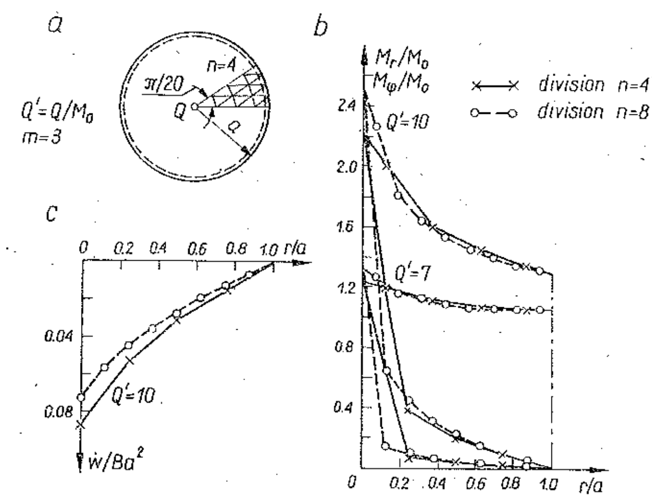

FIG. 3. Simply supported circular plate under concentrated load. a) Element division. b) Bending moments. c) Rate of deflection

7.2. *Rigid viscoplastic square plate*

A rigid viscoplastic square plate, simply supported at the edges and subjected to uniform step load, was analysed using both triangular and rectangular elements. The bending moments and the rate of deflection are plotted in Fig. 4. The agreement of bending moments is satisfactory but the rate of deflection based on triangular elements exceeds that based on rectangular elements by about 25%. For elastic plates the rectangular elements give more accurate deflection than the triangular ones, so there is reason to consider it more accurate for viscoplastic plates, too.

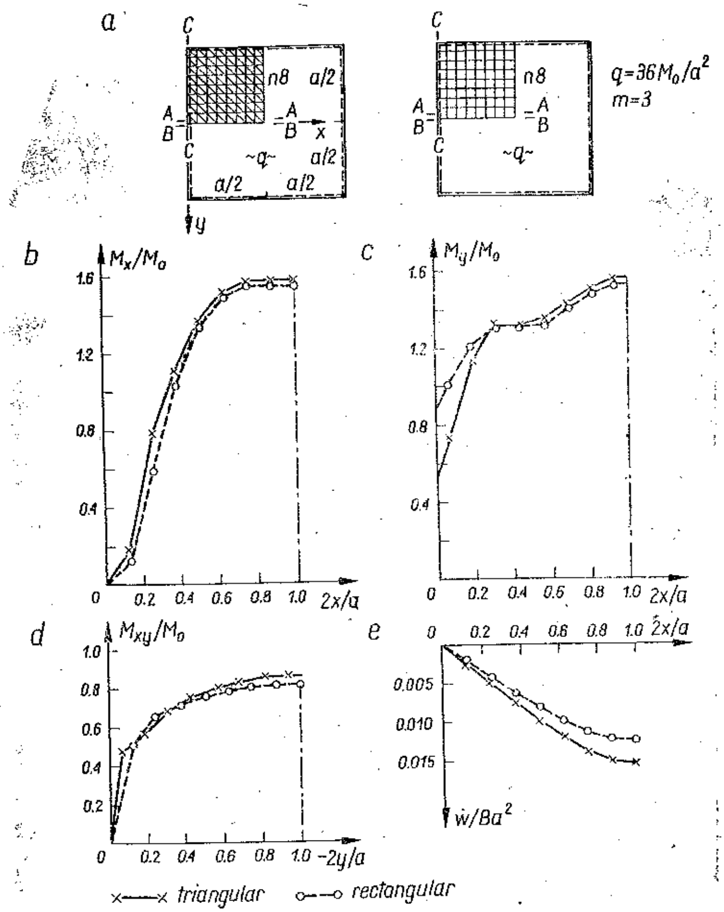

FIG. 4. Square plate under uniform load. a) Element divisions. b) Bending moment M_x on line A–A. c) Bending moment M_y on line A–A. d) Twisting moment M_{xy} on support line C–C. e) Rate of deflection on center line B–B.

7.3. *Elastic viscoplastic circular plate*

The circular, simply-supported plate subjected to uniform step load was also analysed assuming elastic viscoplastic material. The distribution of bending moments in stationary state is presented in Fig. 5a. It can be seen that the bending moments of elastic viscoplastic and rigid viscoplastic plates are practically identical. The rate of deflection as function of time is plotted in Fig. 5b. It approaches very rapidly the rate of deflection of the rigid viscoplastic plate.

The same plate was studied under uniform load decreasing as a function of time (Fig. 6a). In Fig. 6b we can see the bending moments at the instant of zero load.

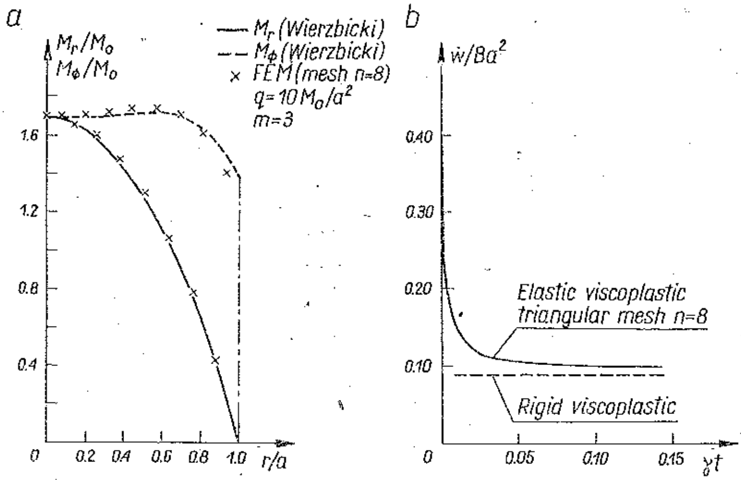

FIG. 5. Elastic viscoplastic circular plate under uniform load. a) Bending moments in stationary state b) Rate of deflection

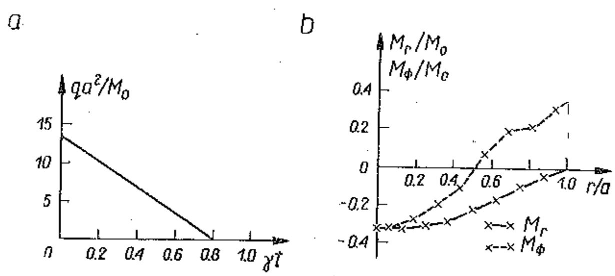

FIG. 6. Elastic viscoplastic circular plate under uniform time dependent load. a) Load intensity as function of time b) Bending moments at instant $\gamma t=0.8$

7.4. Limit loads of rigid plastic plates

An estimate of the limit load of a rigid plastic plate can be determined considering the plate as viscoplastic. The limit load is the largest load under which the viscoplastic plate has a zero rate of deflection. The estimate of the limit load was calculated in the following steps:

1) The plate is first solved as elastic under arbitrary primary load q_0.

2) The smallest value of the ratio $\nu=M_0/M_{\text{eff}}$ is sought. The primary load is multiplied by a factor which slightly exceeds ν and the corresponding elastic solution is determined.

3) The viscoplastic solution of the plate subjected to the above load is determined.

4) If the rate of deflection is smaller than a prescribed small value, an increment of load is added and a new cycle of computation is performed. If the rate of deflection exceeds the prescribed value, the limit load has been exceeded and the computation is terminated.

The load increment was 2% of the load νq_0. The Huber-Mises yield condition was employed. Simply-supported circular, square and rectangular plates were analysed using triangular elements. The element divisions are shown in Fig. 7.

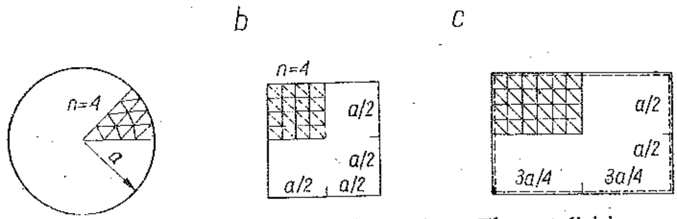

FIG. 7. Limit loads of rigid plastic plates. Element divisions

The limit load found for a circular plate was 6.59 M_0/a^2. According to SAWCZUK and JAEGER [20] the limit load is 6.52 M_0/a^2 for the Huber-Mises yield condition. The yield line theory with the square yield condition gives 6 M_0/a^2.

For a square plate the value 24.5 M_0/a^2 for limit load was obtained. BACKLUND [17] found a lower bound 24.9 M_0/a^2 and an upper bound 25.1 M_0/a^2. The yield line theory gives the value 24 M_0/a^2.

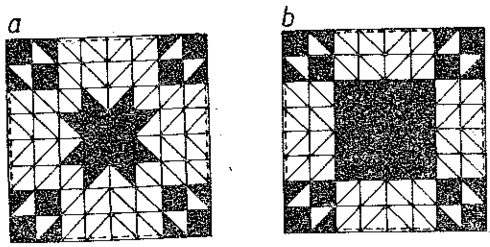

FIG. 8. Spread of plastic zones of square plate. a) Just before collapse. b) At collapse

For a rectangular plate with a span ratio $b/a=3/2$ the the value or limit load was 17.5 M_0/a^2, while the yield lines theory gives 17.0 M_0/a^2.

In Fig. 8 the patterns of plastic yield just before and after the collapse are shown for a square plate.

8. DISCUSSION

The computed examples indicate that rigid or elastic viscoplastic circular plates can be analysed with fair accuracy using the mixed finite element method with a triangular Hellan-Herrmann element.

For rectangular plates, differences have been noticed between triangular and rectangular elements, in particular in the rate of deflection. Comparison with elastic plates suggests that the rectangular element is more accurate.

An experimental stability criterion can be used in the time integration, but it cannot be considered as satisfactory. A theoretical derivation on the lines presented by CORMEAU [18] could be possible and preferred.

The limit loads of rigid plastic plates can be satisfactorily determined using the viscoplastic method.

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STRESZCZENIE

ROZWIĄZANIA QUASI-STATYCZNYCH ZAGADNIEŃ PŁYT LEPKOSPRĘŻYSTYCH ZA POMOCĄ MIESZANYCH ELEMENTÓW SKOŃCZONYCH

W pracy zajęto się analizą numeryczną płyt sztywno-plastycznych, sprężystych i sprężysto-lepkoplastycznych poddanych obciążeniom statycznym. Zastosowano teorię małych ugięć płyt cienkich. Konstytutywne równania lepkosprężystości przyjęto w postaci zaproponowanej przez Perzynę. Schemat obliczeniowy oparto na metodzie mieszanych elementów skończonych, w których dyskretny układ niewiadomych stanowią wartości momentów zginających i ugięć w punktach węzłowych. Zastosowano zarówno elementy trójkątne zaproponowane przez Hellana i Herrmanna jak i elementy prostokątne Bäcklunda. Zastosowano również dwie metody rozwiązania otrzymanych równań różniczkowych zwyczajnych pierwszego rzędu o nieliniowych współczynnikach. Spośród rozwiązanych przykładów wymienić można swobodnie podparte płyty kołowe pod działaniem obciążenia równomiernego oraz siły skupionej działającej w środku a także równomiernie obciążoną płytę prostokątną. Zastosowano również algorytm lepkosprężysty do wyznaczenia obciążeń granicznych sztywno-plastycznych płyt kołowych i prostokątnych.

Резюме

РЕШЕНИЯ КВАЗИСТАТИЧЕСКИХ ЗАДАЧ ВЯЗКОУПРУГИХ ПЛИТ ПРИ ПОМОЩИ СМЕШАННЫХ КОНЕЧНЫХ ЭЛЕМЕНТОВ

В работе занимаются численным анализом жестко-пластических, вязкоупругих и упруго-вязкопластических плит подвергнутых статическим нагрузкам. Применена теория малых прогибов тонких плит. Определяющие уравнения вязкоупругости приняты в виде предложенном П. Пежина. Расчетная схема опирается на метод смешанных конечных элементов, в котором дискретную систему неизвестных составляют значения изгибных моментов и прогибов в узловых точках. Применены так треугольные элементы, предложенные Хелланом и Херрманом, как прямоугольные элементы Беклунда. Применены тоже два метода решения полученных обыкновенных дифференциальных уравнений первого порядка с нелинейными коэффициентами. Среди решенных примеров можно указать свободно подпретые круговые плиты под действием равномерной нагрузки и сосредоточенной силы действующей в центре, а также равномерно нагруженную прямоугольную плиту. Применен тоже вязкоупругий алгоритм для определения предельных нагрузок жестко-пластических круговых и прямоугольных плит.

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