ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 29, 1, 99—114, 1981
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

A MODE SOLUTION FOR THE FINITE DEFLECTIONS OF A CIRCULAR PLATE LOADED IMPULSIVELY

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The mode approximation technique as originally presented by MARTIN and SYMONDS [7] is applicable to rigid-plastic structures undergoing infinitesimal deflections. The extension of the mode approach to the finite-deflection range can be done by considering a series of instantaneous modes [5-7], or by assuming a permanent mode shape [9, 16]. The method proposed in [9] is further developed here and applied to the study of a circular plate loaded impulsively. The final deflection is obtained using a method developed by Jones for beams and non-axisymmetric plates [10]. Comparison with experiments and other theoretical treatments show good correlation.

1. INTRODUCTION

The solution of problems of dynamic response of structures to intense loading is rather complicated when parts of it enter the plastic range. However, in some cases the plastic strains associated with the response are so large that one can neglect the elastic strains without significant loss in accuracy. This rigid-plastic idealization of material behaviour is therefore a useful simplification that has allowed some of these problems to become tractable.

Within this theory several exact solutions have been obtained [1, 2]. However, even with this idealization problems were often difficult to solve and consequently some approximate methods were developed. Among these, special relevance has been gained by the bounding theorems [3] and by the mode approximate solutions introduced by MARTIN and SYMONDS [4].

These approximate methods have been developed for the case of small deflections but, sometimes, when considering the effects of intense loading, the nonlinearities due to large deflections must be taken into account. In these cases an exact solution is often out of the question and even approximate ones are sometimes difficult. SYMONDS and CHON [5-7] proposed an extension of the original mode approximation solutions so as to account for finite deflections but, in doing so, they had to resort to a series of instantaneous mode shapes throughout the response, what implied the use of numerical methods. As a consequence, the simplicity and the analytical character of the mode solutions would be lost.

When applying the same procedure to the case of a beam, it was noticed [8] that the shape of the modes was not changing significantly during the response. The same could be observed in the results of CHON and SYMONDS [6] on a circular plate.

This gave the indication [9] that the use of a permanent mode shape might be a reasonable approximation with an unquestionable simplification of the analysis. Indeed, it is believed that when exact solutions cannot be obtained, the approximate methods which are developed should be a compromise between the difficulty of treatment and the accuracy of results.

The procedure proposed in Ref. [9] is applied here to study the behaviour of a circular rigid-plastic plate under impulsive loading, and it is shown how the mode approximation technique could be extended to the case of finite deflections, retaining its original simplicity.

The solution is obtained using the methodology developed by JONES, which has already been applied to beams [10–12], non-axisymmetric plates [10] and shells of revolution [13].

The results are compared with experiments and with other theoretical studies, including a discussion of the relevant differences between the approaches.

2. BASIC APPROACH

The solution of a problem of a rigid-plastic structure subjected to a given prescribed dynamic load involves the determination of a velocity and a stress field such that: 1) the stress field is in dynamic equilibrium with the applied load; 2) the velocity field is zero whenever the yield condition is not satisfied; 3) when the yield condition is verified, the velocity and stress fields are related by the associated flow law; 4) the yield condition is never violated.

However, a complete solution to such a problem can also be obtained using the concepts of limit analysis [14]. Indeed, when the upper and lower bounds are coincident, they are the exact solution.

Therefore, one can depart from a kinematically admissible velocity field, and obtain the equations of motion from the satisfaction of equilibrium, initial conditions, yield condition and associated flow law. Then, if the yield condition is not violated everywhere, the stress field is dynamically admissible and the solution exact.

These principles are also applicable to the mode solutions. The original formulation of MARTIN and SYMONDS [4], which is appropriated to infinitesimal deflections, is based on the choice of a kinematically admissible mode shape which has a constant shape but changes its amplitude with time:

$$\dot{w}(r,t) = \alpha T(t)\,\varphi(r). \tag{2.1}$$

When the mode shape can be associated with an equilibrium stress field such that the yield condition is satisfied everywhere and when the initial conditions are also satisfied, then the mode solution is an exact one. However, in general, the mode solution will not satisfy the initial conditions being therefore only an approximation, even in the small deflections range.

It has been shown that in the case of small deflections, the shape of the deformed structure will converge to a unique mode shape which will remain constant during the latter stages of motion [15].

However, as in general, this mode shape will not coincide with the shape of the loading it becomes necessary to match the mode solution with the actual response. MARTIN and SYMONDS [4] originally proposed to obtain the initial value of the amplitude of the mode solution from the minimization of the functional Δ which represents a measure of the difference in kinetic energy between the mode solution and the actual response:

$$\Delta = \int_A \frac{1}{2} \mu (\dot{w}_i - \alpha\varphi)(\dot{w}_i - \alpha\varphi)\, dA \,. \tag{2.2}$$

Minimizing Δ by setting $\partial\Delta/\partial\alpha$ equal to zero will yield the value of α:

$$\alpha = \frac{\int_A \mu\varphi\dot{w}_i \, dA}{\int_A \mu\varphi^2 \, dA}, \tag{2.3}$$

It happens that for small deflections the mode solutions are, in general, exact except for the initial conditions. However, for large deflections no exact solutions can be obtained since the stress fields which are originated during the motion will include yield violations.

To extend the mode approximation technique to the finite deflection range, two main procedures have been adopted.

The first one, due to SYMONDS and CHON [5–7], consists in avoiding the yield violations by considering the response to be made of a sequence of instantaneous mode shapes such that, at each moment, the resulting stress field will not violate yield anywhere.

The second one, due to the author [9] and also adopted by SYMONDS and WIERZBICKI [16], is based on the postulation of a permanent mode shape throughout the structural response. However, while the author advocates the use of the same mode shape as in the small deflection range, i.e. the mode in which the structure collapses under the impulsive loading; Symonds and Wierzbicki neglect the effect of bending moments and choose the shape of the final stage of motion when there is only membrane action.

One of the main features of the mode approximation technique is the choice of a mode shape that satisfies all the field equations and constraint conditions. The other one is the matching procedure resulting from having the initial conditions not satisfied. Matching the mode solution to the actual response ensures that two configurations approach each other during the early phases of motion.

These two considerations are maintained in the procedure proposed in Ref. [9] and applied now to solve a specific problem. The difference between small [4] and large deflection will be reflected only in the field equations which include one more term accounting for the additional effect of the membrane forces that are developed because of axial restraint.

These mode solutions maintain the basic features of the original mode approach, they are easy to apply since they do not need the resort to numerical methods and, finally, comparisons with experiments have shown that they give quite satisfactory results [9].

Although not explicitly associating his approximate method with this general formulation, JONES [10] has developed a procedure to estimate the finite deflections of structures, which consists in choosing the collapse velocity field of the structure and assuming its shape to remain constant during the response.

The method has been originally developed for beams and nonaxisymmetric plates. Here we will use it in the case of axisymmetric plates and, within the formalism, of the mode approximation technique, we will apply it to the case of a rigid--plastic circular plate subjected to an impulsive load.

3. Application to axi-symmetric plates

This formulation, as developed by JONES [10], departs from the principle of virtual velocities:

$$\dot{D}_i = \dot{D}_e , \tag{3.1}$$

which equates the total internal and external energy dissipation rates.

The rate of external energy dissipation is given by

$$\dot{D}_e = \int_A \{(P_i - \mu\ddot{u}_i)\,\dot{u}_i + (P_3 - \mu\ddot{w})\,\dot{w}\}\,dA, \qquad i = 1, 2, \tag{3.2}$$

where the summation convention is implied. The area A extends over the entire deformed mid-plane of the plate and may be taken as the original area for moderate deflections.

For a circular plate the consistent dynamic equilibrium equations for finite deflections are given by [17]

$$\begin{gathered} N_\theta - (rN_r)' - rP_i + r\mu\ddot{u}_i = 0, \qquad i = 1, 2, \\ -(rM_r)'' + M_\theta' + (rN_r\, w_r')' + rP_3 - r\mu\ddot{w} = 0, \\ rQ = -(rM_r)' + M_\theta . \end{gathered} \tag{3.3}$$

Substituting Eqs. $(3.3)_{1,2}$ in Eq. (3.2) one obtains

$$\dot{D}_e = \int_A \left\{[N_\theta - (rN_r)']\frac{\dot{u}_i}{r} + [(rM_r)'' - M_\theta' - (rN_r\, w')']\frac{\dot{w}}{r}\right\} dA . \tag{3.4}$$

Integrating by parts all but the first term of this expression results in

$$\begin{aligned} \dot{D}_e = {} & \int_A \left(\frac{N_\theta}{r}\dot{u}_i + N_r\,\dot{u}_i'\right) dA - \int_l N_r\,\dot{u}_i\, dl - \int_l Q\dot{w}dl - \int_l N_r\,\dot{w}\dot{w}'\, dl + \\ & + \int_l (M_r - N_r\, w)\,\dot{w}'\, dl + \int_A (M_r - N_r\, w)\,\dot{w}''\, dA + \int_A \left(\frac{M_\theta}{r} - N_r' w - \frac{N_r}{r} w\right)\dot{w}'\, dA , \end{aligned} \tag{3.5}$$

where use of Eq. $(3.3)_1$ has been made to obtain the third term of Eq. (3.5) and the line integrals are performed along circumferential lines.

When the load acts only in the transverse direction ($P_i=0$), it is reasonable to neglect u_i and $\dot{u}_i$, compared to w and $\dot{w}$. Introducing these simplifications in Eqs. (3.2) and (3.5) and equating them, one obtains

$$(3.6) \qquad \int_A (P_3-\mu\ddot{w})\,\dot{w}dA = -\int_l \{(Q+N_r\,\dot{w}')\,\dot{w}+(M_r-N_r\,w)\,\dot{w}'\}\,dl + \\ +\int_A \left\{(M_r-N_r\,w)\,\dot{w}''+(M_\theta-rN_r'\,w-N_r\,w)\frac{\dot{w}'}{r}\right\}dA\,.$$

The velocity fields to be postulated will hold throughout the structural response not allowing therefore the existence of travelling hinge circles. In this case w and $\dot{w}$ must be continuous throughout, but $\dot{w}'$ may be discontinuous across a stationary hinge circle.

Therefore the first term of the line integral of Eq. (3.6) vanishes and the equation is finally reduced to

$$(3.7) \qquad \int_A (P_3-\mu\ddot{w})\,\dot{w}dA = -\int_l \overline{M_r\,\dot{w}'\,dl} + \int_l \left\{(M_r-N_r\,w)\,\dot{w}''+ \right. \\ \left. +(M_\theta-rN_r'\,w-N_r\,w)\frac{\dot{w}'}{r}\right\}dA\,,$$

where now the line integral is performed along a hinge circle.

Incidentally, it can be shown that this equation could also be obtained from the equation of a shell of revolution [13] by using the general coordinates appropriate to a circular plate.

4. Yield condition

In a circular plate undergoing finite deflections the state of stress is defined by four stress resultants: N_r, N_θ, M_r and M_θ. Similarly the state of strain has four components. These quantities are related by the equilibrium equations and by the yield condition and associated flow law.

The plastic yield condition may be represented as a surface in a Cartesian space whose coordinates are the generalized stresses. The associated flow rule defines then the generalized strain rate vector as the normal to the surface. Therefore, if the yield surface consists of plane facets, as long as the stress point remains on a given plane, the yield vector remains in the same direction and the yield mechanism does not change. This is why in analytical work preference is often given to the Tresca yield condition although the Huber-Mises condition might give a slightly better description of the behaviour of metal structures.

The use of the exact yield surface is very complicated when considering finite deflections since it becomes a four-dimensional surface in the stress space [18]. Therefore approximate yield conditions must developed. Moreover, it has been shown that by suitably linearizing the yield condition not only the equations are made linear but also the stress and velocity equations are uncoupled [19].

In the linearization process two main approaches have been used. One consists in approximating the exact yield surface by a polyhedron [20, 21] and the other consists in using an approximate description of the structure itself [19, 6].

Hodge suggested that the exact yield condition for an ideal sandwich shell could be used as an approximation to the uniform shell case [19].

The rational behind his proposal derives from the theory of limit analysis and states that if a yield condition A is wholly interior to another condition B, then the collapse load computed according to A is not greater than the collapse load computed relative to B. Indeed this would be applicable since the sandwich surface is interior to the exact one.

Drucker and Shield [20], on the other hand, proposed a simple approximation to the yield condition of shells of revolution based on the argument that in most rotationally symmetric shell problems the moments are generally small, compared with direct stresses.

Therefore they ignored the hoop moment M_θ and neglected the interaction between the radial moment and the direct stresses. This is referred to as the one-moment limited interaction surface.

However, Hodge [21] noted that in dynamic problems it might not be justified to disregard M_θ because it may become important in regions near the axis of symmetry. Therefore he proposed a two-moment limited interaction surface in which all interaction between moments and between membrane forces are maintained. It recognizes though, that in most problems the moments and the membrane forces will not be of simultaneous importance so that the relations between moment and force are neglected (Fig. 1).

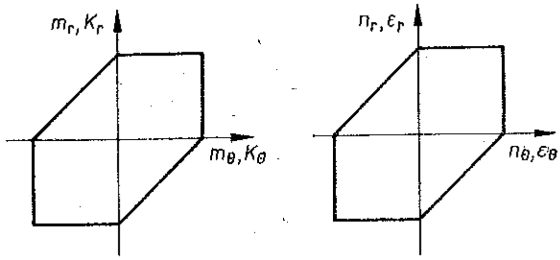

Fig. 1. Two-moment limited interaction yield surface.

Hodge studied then the problem of a spherical cap, using the exact yield surface [18] and the different approximations. He obtained comparable results but in the case studied the two-moment limited interaction gave better predictions than the sandwich approximation and, furthermore, it simplified the analysis.

This concept of limited interaction yield surfaces has also been adopted by Jones and co-workers in problems of beams [10–12], non-axisymmetric plates [10] and shells of revolution [13]. Also, as already stated in [9], extensive comparisons with experimental work have shown the results to be satisfactory.

Therefore the two-moment limited interaction surface will be adopted in the present study.

5. Mode solution

We will now seek a solution for the case of a simply supported circular plate subjected to a uniform impulsive axi-symmetric load over a central region of radius a (Fig. 2) defined by:

$$\dot{w}_i = V_0, \quad 0 \leqslant r \leqslant a; \qquad \dot{w}_i = 0, \quad r > a, \tag{5.1}$$

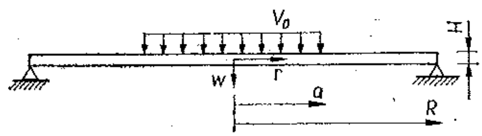

FIG. 2. Impulsively loaded, simply supported circular plate.

We will assume a modal velocity field of the form of Eq. (2.1) where, as defined by MARTIN and SYMONDS [4], the parameter α is given by Eq. (2.3).

The mode shape chosen will be equal to the final mode shape of the corresponding infinitesimal deflections dynamic problem [22]. It has been shown [22] that for moderate loads the displacement field will have the following permanent mode shape:

$$\varphi(r) = \left(1 - \frac{r}{R}\right). \tag{5.2}$$

For large loads the initial displacement field will have a shape of a truncated cone:

$$\varphi(r) = 1, \quad 0 \leqslant r \leqslant \xi; \qquad \varphi(r) = \frac{R-r}{R-\xi}, \quad \xi \leqslant r \leqslant R. \tag{5.3}$$

This response shape is not a permanent one since the hinge located at $r=\xi$ will travel towards the center of the plate until, at later stages of the response, the plate will take the conical mode shape [Eq. (5.2)] and maintain it until it comes to rest. Indeed, MARTIN [15] showed that the response of a structure in the small deflection range will tend towards a mode-form solution.

Following an earlier proposal [9], in the present analysis, which is valid for finite deflections, the corresponding mode shape will therefore be given by Eq. (5.2). This mode shape is common for displacement, velocity and acceleration fields.

Substituting Eqs. (5.1) and (5.2) in Eq. (2.3) will result in

$$\alpha = V_0 \frac{a^2}{R^2}\left(6 - 4\frac{a}{R}\right) \equiv \eta V_0 \tag{5.4}$$

and therefore,

(5.5) $$\dot{w}(r,t) = \dot{W}(t)\left(1 - \frac{r}{R}\right),$$

where $\dot{W}(t) = \eta V_0 T(t)$.

The consistent strain rates [17] take the form

(5.6) $$\dot{K}_r = 0, \qquad \dot{K}_\theta = -\frac{\dot{W}}{rR};$$ $$\dot{\varepsilon}_r = \frac{W\dot{W}}{R^2}, \quad \dot{\varepsilon}_\theta = 0$$

for the velocity field of Eq. (5.5).

By using Hodge's two-moment limited interaction yield surface (Fig. 1) and the associated flow rule, we then have for $\dot{K}_\theta = 0$ and $\dot{K}_\theta < 0$

(5.7) $$M_\theta = -M_0 \quad \text{and} \quad -M_0 \leqslant M_r \leqslant 0,$$

where $M_0 = \sigma_0\, H/4$ is the fully plastic moment.

For $\dot{\varepsilon}_\theta = 0$ and $\dot{\varepsilon}_r > 0$ we must have

(5.8) $$N_r = N_0 \quad \text{and} \quad 0 \leqslant N_\theta \leqslant N_0,$$

where

$$N_0 = \sigma_0\, H = 4M_0/H.$$

Following Jones' approach [10] we will substitute Eqs. (5.5) and (5.7) in Eq. (3.7). As no hinges are allowed in this mode shape (5.2), the line integral of Eq. (3.7) vanishes. Noting also that for impulsive loading $P_3 = 0$, the result will be

(5.9) $$-\int_A \mu \ddot{W}\left(1 - \frac{r}{R}\right)^2 \dot{W} dA = \int_A \left\{-M_0 - N_0\, W\left(1 - \frac{r}{R}\right)\right\}\left(-\frac{\dot{W}}{rR}\right) dA.$$

Performing the integrations, one can obtain the equation of motion:

(5.10) $$\ddot{W} + \frac{24M_0}{\mu R^2 H} W = -\frac{12M_0}{\mu R^2}.$$

The solution of this equation will give the time history of $\dot{w}$ and consequently the velocity fields as defined in Eq. (5.5).

In terms of the following nondimensional variables,

(5.11) $$\ddot{w}_* = \frac{\ddot{W}H}{V_0^2}, \quad w_* = \frac{W}{H}, \quad \lambda = \frac{\mu V_0^2 R^2}{M_0 H}.$$

The equation of motion becomes

(5.12) $$\ddot{w}_* + \frac{24}{\lambda} w_* = -\frac{12}{\lambda}.$$

and its solution is

$$w_* = \eta \sqrt{\frac{\lambda}{24}} \sin \sqrt{\frac{24}{\lambda}}\, t + \frac{1}{2} \cos \sqrt{\frac{24}{\lambda}}\, t - \frac{1}{2}, \tag{5.13}$$

where the constants have been obtained from the initial conditions

$$\text{at } t=0 \quad w_*=0, \quad \dot{w}_*=\eta. \tag{5.14}$$

The motion stops at $t=t_f$ when $\dot{w}_*=0$. Therefore, differentiating Eq. (5.13) with respect to time and equating to zero at $t=t_f$ will yield the expression for the duration of the response:

$$\tan \sqrt{\frac{24}{\lambda}}\, t_f = \eta \sqrt{\frac{\lambda}{6}} \tag{5.15}$$

from what the following results:

$$\sin \sqrt{\frac{24}{\lambda}}\, t_f = \sqrt{\frac{\lambda\eta^2}{6+\lambda\eta^2}}, \qquad \cos \sqrt{\frac{24}{\lambda}}\, t_f = \sqrt{\frac{6}{6+\lambda\eta^2}}. \tag{5.16}$$

The final deflection which occurs at $t=t_f$ is obtained by substituting Eq. (5.16) in Eq. (5.13):

$$w_{*f} = \frac{1}{2}\left(\sqrt{1+\frac{\lambda\eta^2}{6}} - 1\right). \tag{5.17}$$

In the special case of having the impulsive load uniformly distributed over the entire surface of the plate, a will be equal to R and, from Eq. (5.4), it results that $\eta=2$. Therefore the duration of the response and the final deflection amplitude will be given by

$$\begin{aligned} \tan \sqrt{\frac{24}{\lambda}}\, t_f &= \sqrt{\frac{2\lambda}{3}}, \\ w_{*f} &= \frac{1}{2}\left(\sqrt{1+\frac{2\lambda}{3}} - 1\right). \end{aligned} \tag{5.18}$$

In this case a plot of final deflection amplitude versus the nondimensional measure of kinetic energy is shown in Fig. 3.

It should be noted that although this solution (5.17) has been developed for the case of uniformly distributed impulsive load (5.1), it remains valid for any other kind of impulsive load. The variation of the load distribution will only imply a new value of the parameter η which was obtained from Eq. (2.3) by using Eq. (5.2) and the shape of the load distribution.

6. Comparison with other studies

There is a vast literature on the dynamic plastic behaviour of circular plates and therefore we will not attempt to look at all the solution methods that have

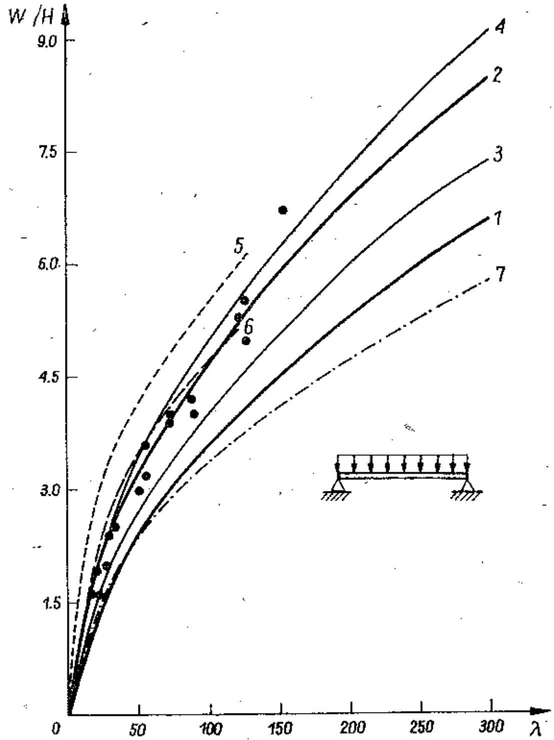

FIG. 3. Estimates of final deflection for a simply supported plate: 1—present solution, 2—present solution (inscribing yield surface), 3—JONES [26], 4—JONES (inscribing yield surface), [26], 5—WIERZBICKI [28], 6—KALISZKY [29], 7—SYMONDS and WIERZBICKI [16]; ● —FLORENCE experiments [27].

been adopted. Only some treatments which are considered to be related with the present work will be discussed.

The mode approximation technique has been originally developed for rigid--plastic structures which undergo small deflections when subjected to impulsive loading [4]. It has been applied mostly in infinitesimal deflection situations and a review of its capabilities has been given by MARTIN [23] and by SYMONDS and CHON [24]. In the latter, various mode solutions for circular plates are presented and it is shown that they are the same as the ones developed in the early works such as for example [2].

WANG [22] considers a simply supported circular plate under impulsive loading, whereas HOPKINS and PRAGER [2] had considered pressure loading. As mentioned previously, we adopted here a mode shape equal to the one used by Wang.

The first studies of the finite deflection problems are by FLORENCE [25] who looked at an annular plate and by JONES [26] who considered a simply supported circular plate under impulsive load, i.e. exactly the same case that has been considered here.

Florence used an interaction between M_θ and N_θ only, while Jones adopted the limited interaction concept of HODGE [21] (Fig. 1). Jones used the same deformation mechanism as WANG [22] and a more accurate theory to obtain the final deflection. He considered two stages of motion, having in the first one a hinge circle travelling towards the centre of the plate, and retaining during this stage the effect of both moment and membrane forces.

The second stage occurs after the travelling hinge circle has reached the center. Since Jones considered only the effect of membrane forces, he obtained a deformation shape in terms of a Bessel function of the first kind of zero order. Comparison of his results with the experiments of FLORENCE [27] showed an excellent agreement.

The attempts to extend the mode approximation technique to the finite deflection range began with SYMONDS and CHON [5, 6]. The mode technique, although being an approximate procedure, cannot be applied to the finite deflection case without further generalizations or assumptions.

When studying the circular plate problem CHON and SYMONDS [6] decided to consider the response to be made of a series of instantaneous modes. Each mode was found by an iterative procedure and therefore their approach implied the use of numerical methods with the consequent loss of one of the basic features of the mode technique: its simplicity. In a further study [7] they obtained better correlation with experiments but the numerical procedure became even more complicated since finite elements were used.

A different and simpler approach suggested in [9] and used here consists in maintaining the basic procedure used in the small deflections range [4]. Therefore the permanent mode shape appropriate for the infinitesimal deflection case is maintained on the large deflection range. This is very similar to what has been done by JONES [10], WIERZBICKI [28] and KALISZKY [29] whoadopted one collapse mechanism for the whole structural response.

It should be noted that the method developed by JONES [10] can also be derived from variational formulations [30]. It had not been applied previously to circular plates; it has been adopted here to obtain the mode solution in the finite deflection range. It is a solution which includes both bending and membrane effects, while SYMONDS and WIERZBICKI [16] consider only the membrane behaviour.

They followed the same idea of estimating the response with a permanent mode shape. However, instead of choosing the shape of the initial phases of motion they chose the shape appropriate to the stage where only membrane effects exist. Therefore they obtained a mode shape in terms of Bessel functions as JONES [26] did for the second phase of his solution. FLORENCE [25] also used a membrane solution which gave worse results than his limited interaction approximation. However, he used an approximate linear profile as the velocity field.

As Symonds and Wierzbicki have given no account for bending, their solution is valid for both simply supported and clamped conditions. Their equation of motion is equivalent to Eqs. (5.10) and (5.18) without the right hand side which is the term accounting for the flexural effects.

As can be seen in Fig. 3, their predictions underestimate the final deflection of the plates. It is expected, however, that their results will improve for much larger deflections, when the energy dissipated in membrane effects becomes more important.

In Fig. 3, besides the different theoretical formulations, the experimental results of FLORENCE [27] on 6061-T6 aluminium plates are also shown.

For the present approach and for JONES' solution [26] two curves are shown. They both use the limited interaction yield surface of HODGE [21]. As originally suggested, another inscribing yield surface 0.618 times smaller can also be used, giving a sort of "upper bound" to the maximum deflection.

With the present approach an approximate solution could also be obtained very easily for the case of clamped boundary conditions. If the same mode shape is used, it becomes necessary to account only for the extra work that is dissipated in the plastic hinge circle that develops at the support. This implies minor changes in the formulation as can be seen in the Appendix. Results of this procedure are plotted in Fig. 4 and compared with other studies.

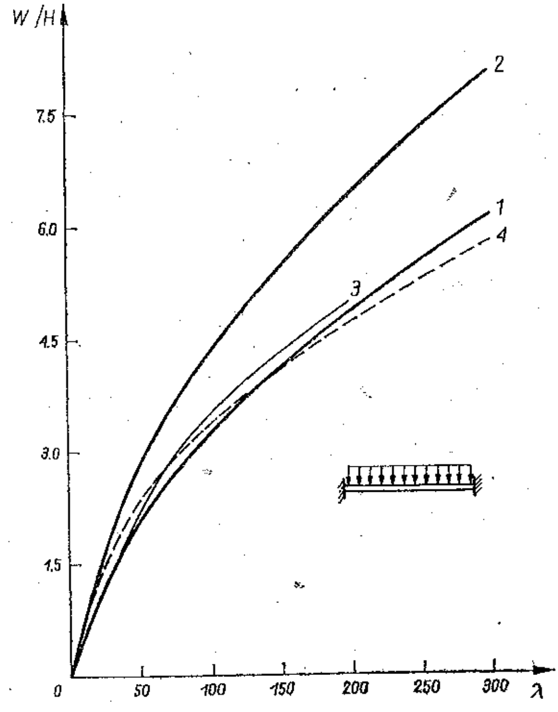

FIG. 4. Estimate of final deflection for a clamped plate: 1—present solution, 2—present solution (inscribing yield surface), 3—CHON and SYMONDS [6], 4—SYMONDS and WIERZBICKI [16].

However, it should be kept in mind that this is not the solution that would be obtained by following what is proposed in [9] and here. If that were done, then a logarithmic mode shape would have to be used, since this would be the appropriate one for a clamped circular plate subjected to impulsive loading [31].

7. Concluding remarks

One possible extension of the mode approximation technique to the finite deflections range has been presented. It follows a previous suggestion [9] and it allows for very simple solutions; this is therefore in the spirit of the original infinitesimal solutions of MARTIN and SYMONDS [4].

It provides an alternative to the approach of SYMONDS and CHON [5–7], being simpler and of comparable accuracy. It is of a complexity similar to that of the recently proposed method of SYMONDS and WIERZBICKI [16], providing, however, better estimates for modelately large deflections.

The final deflection of the plate was obtained using a method developed by JONES [10] which has previously been applied to beams [10–12], non-axisymmetric plates [10] and shells of revolution [13].

This method has the great advantage of its analytical character and also of its applicability to non-axisymmetric plates which are relevant engineering structures that might be difficult to study by means of other methods.

8. Appendix. Approximate solution for a clamped plate

To obtain a mode solution to a clamped plate impulsively loaded, the same general procedure should be followed using, however, as a mode shape instead of Eq. (5.2), a logarithmic profile [31].

We will not seek here the mode solution but just an approximation based on the use of the same mode shape.

As this mode shape does not satisfy the clamped boundary condition $\varphi'=0$, we must have a hinge circle developing at the supports. This will be the difference relative to the simply supported solution.

Therefore, when introducing Eqs. (5.5) and (5.7) in Eq. (3.7), one must account also for the contribution of the line integral which should be performed around the hinge circle.

If that is done, the resulting equation of motion will be, in nondimensional form,

$$\ddot{w}_* + \frac{24}{\lambda} w_* = -\frac{24}{\lambda}$$

and its solution is

$$w_* = \eta \sqrt{\frac{\lambda}{24}} \sin \sqrt{\frac{24}{\lambda}}\, t + \cos \sqrt{\frac{24}{\lambda}}\, t - 1$$

for the following initial conditions:

$$\text{at } t=0, \quad w_*=0, \quad \dot{w}_*=\eta,$$

where η is given by Eq. (5.4).

Following exactly the same procedure as for the simply supported case, we will obtain the final response time and deflection amplitude

$$\tan\sqrt{\frac{24}{\lambda}}\,t_f=\eta\sqrt{\frac{\lambda}{24}},$$

$$w_{*f}=\sqrt{1+\frac{\lambda\eta^2}{24}}-1.$$

This can be used in the case of loading over the entire surface of the plate giving

$$\tan\sqrt{\frac{24}{\lambda}}\,t_f=\sqrt{\frac{\lambda}{6}},$$

$$w_{*f}=\sqrt{1+\frac{\lambda}{6}}-1,$$

what can be seen in Fig. 4.

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STRESZCZENIE

ROZWIĄZANIE MODALNE DLA DUŻYCH UGIĘĆ IMPULSOWO OBCIĄŻONEJ PŁYTY KOŁOWEJ

Metodę rozdzielenia zmiennych w formie zaproponowanej przez MARTINA i SYMONDSA [4] stosuje się do konstrukcji sztywno-plastycznych w zakresie nieskończenie małych ugięć. Uogólnienie tej metody do problemów dużych ugięć dokonywane było bądź poprzez rozpatrzenie sekwencji chwilowych postaci ruchu [5–7], bądź też przyjmując stały kształt pola prędkości [9, 16]. Metoda zaproponowana w [9] jest dalej rozwinięta w obecnej pracy i zastosowana do zagadnienia impulsowo obciążonej płyty kołowej. Końcowe ugięcia obliczone są na podstawie metody opracowanej przez Jonesa dla belek i dowolnych płyt [10]. Porównanie uzyskanych wyników z doświadczeniami oraz innymi rozwiązaniami teoretycznymi wskazuje na dobrą ich zgodność.

Резюме

МОДАЛЬНОЕ РЕШЕНИЕ ДЛЯ БОЛЬШИХ ПРОГИБОВ КРУГОВОЙ ПЛИТЫ НАГРУЖЕННОЙ ИМПУЛЬСНЫМ ОБРАЗОМ

Метод разделения переменных,в форме предложенной Мартином и Симондсом [4], применяется для жестко-пластических конструкций в области бесконечно малых прогибов. Обобщение этого метода к проблемам больших прогибов проводилось или путем рассмотрения последователсностей мгновенных видов движения [5–7], или же принимая постоянную форму поля скоростей [9, 16]. Метод предложенный в [9] дальше развит в настоящей работе и применен к проблеме круговой плиты нагруженной импульсным образом. Остаточные прогибы рассчитаны на основе метода разработанного Джонсом для балоки произвольных плит [10]. Сравнение товполученных результа с экспериментами, а также с другими теоретическими решениями указывает на хорошее совпадение.

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Received February 15, 1980