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CHOICE OF COLLOCATION POINTS FOR AXISYMMETRIC NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS IN STATICS OF SHALLOW SPHERICAL SHELLS

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The present work investigates the optimum choice of collocation points which gives for a given accuracy the minimum number of collocation points. A covergence study has been conducted for the axisymmetric nonlinear analysis of a shallow spherical shell under a uniformly distributed load with four different choices of collocation points, viz. equidistant collocation points; collocation at maxima of a Chebyshev polynomial; collocation at zeros of a Chebyshev polynomial and zeros of a Legendre polynomial (Gaussian points) as collocation points. It has been found that the Gaussian collocation method has the fastest rate of convergence and it yields accurate results even with small number of collocation points. The results for the nonlinear static analysis of elastic circular plates and shallow spherical shells obtained by the Gaussian collocation method have been presented and are found to be in good agreement with the results available.

NOTATIONS

a, h, K^*, q base radius, thickness, curvature and uniformly distributed load,
ω^*, ψ^* transverse deflection and stress function,
$\sigma_r^{m*}, \sigma_r^{b*}, \sigma_r^{t*}$ membrane, bending and total radial stress,
E, ν Young's modulus and Poisson's ratio,
$()'$ differentiation w.r.t. space variable ρ.
Subscripts J, p, i load step, predicted value and value at i-th collocation point,
ΔP load step increment,
$P_{CL}=2Eh^2 K^{*2}/[3(1-\nu^2)]^{1/2}$ — classical buckling pressure for complete spherical shell,

$$\omega^*_{avg}/h=[48(1-\nu^2)]^{\frac{1}{2}} \int_0^1 \frac{\omega^*}{h} \rho d\rho,$$

CE, SE — clamped edge, simply supported edge.

1. INTRODUCTION

Nonlinear problems are encountered in modern engineering structures because the materials are being utilised to their fullest potential and the economy in design necessitates subjecting the structures to large deformations under extreme loads. Plates and shells form essential structural elements in the aerospace industry where light weight construction is of utmost importance. In this work the problem of static large deflection nonlinear analysis of circular plates and shallow spherical shells has been investigated using collocation methods.

Several methods for the nonlinear analysis are well established such as: perturbation, finite difference, finite element, Rayleigh–Ritz, Galerkin, Kontorovich, collocation. There is a need to examine, evaluate and improve the existing methods, as well as to develop new approaches for solving nonlinear problems. The present study examines the interior point collocation method and evaluates the efficacy of the different choices of location of collocation points. The orthogonal point collocation method using the zeros of an orthogonal polynomial as the collocation points has been extensively employed for nonlinear problems in chemical engineering [1, 2] and has been mathematically investigated [3, 4]. LANCZOS [5] introduced this method using zeros of a Chebyshev polynomial as collocation points. VILLADSEN *et al.* [2] employed zeros of other orthogonal polynomials, e.g. Legendre polynomials.

The interior global point collocation method with the following set of collocation points, viz., (i) equidistant, (ii) maxima of a Chebyshev polynomial, (iii) zeros of a Chebyshev polynomial (Chebyshev collocation method) and (iv) zeros of a Legendre polynomial (Gaussian collocation method) have been employed for the axisymmetric nonlinear static analysis of shallow spherical shells. Donnell type nonlinear coupled differential equations in terms of transverse displacement ω and stress function ψ have been employed. The functions ω and ψ have been expanded as a polynomial in the space variable ρ. The coefficients are evaluated by using appropriate boundary conditions and the collocation equations corresponding to the differential equations. The load is incremented in steps ΔP and an iterative scheme is used to solve nonlinear equations. At each iteration one of the product terms constituting the nonlinearity has been predicted as the mean of its values at the two previous iterations. The predictions at the first iteration are taken as the quadratically extrapolated values from the previous three steps. A convergence study using the four collocation schemes has been carried out for a clamped shallow spherical shell with $K=3$. The Gaussian collocation method not only shows the fastest rate of convergence but also yields quite accurate results with a smaller number of collocation points. The equidistant collocation method has the slowest rate of convergence. The collocation methods based on maxima and zeros of a Chebyshev polynomial are quite efficient but have somewhat a slower rate of convergence as compared to the Gaussian collocation method. Keeping this in view, another convergence study for a larger shell parameter ($K=21$) using Chebyshev and Gaussian collocation methods is conducted, which also confirms that the Gaussian method is marginally superior. The results for circular plate and shallow shells with $K=1, 2, 3$ for clamped and simply supported edges have been obtained using the Gaussian collocation method and have been found to agree quite well with the results available.

2. Mathematical formulation

The nondimensionalized Donnell type equations in terms of the transverse displacement ω and the stress function ψ for the axisymmetric nonlinear analysis of uniform thin elastic shallow spherical shells [6] are:

$$\left(\nabla^2-\frac{1}{\rho^2}\right)\psi+\frac{1-\nu^2}{2}\left[(\omega')^2/\rho+2K\frac{h}{a}\omega'\right]=0, \tag{2.1}$$

$$\nabla^4\omega-12\left(\frac{a}{h}\right)^2\frac{1}{\rho}\left[\psi'\omega'+\psi\omega''+\frac{Kh}{a}(\rho\psi)'\right]=P, \tag{2.2}$$

where

$$\begin{aligned}&\rho=\frac{r}{a},\quad \omega=\frac{\omega^*}{a},\quad K=K^*\frac{a^2}{h},\quad \psi=\frac{(1-\nu^2)\psi^*}{Eha},\\ &P=\frac{qa^3}{D},\quad D=Eh^3/[12(1-\nu^2)].\end{aligned} \tag{2.3}$$

The boundary conditions are:

(2.4)
a) clamped edge (CE) $\rho=1$: $\rho\psi'-\nu\psi=0$, $\omega'=0$, $\omega=0$,
b) simply supported edge (SE) $\rho=1$: $\rho\psi'-\nu\psi=0$, $\omega=0$, $\rho\omega''+\nu\omega'=0$,
c) symmetry conditions at centre $\rho=0$: $\psi=0$, $\omega'''=0$, $\omega'=0$.

The load is incremented in steps ΔP and at each step ω and ψ are expanded as

$$\psi(\rho)=\sum_{n=1}^{N+2}\rho^{n-1}a_n,\quad \omega(\rho)=\sum_{m=1}^{N+4}\rho^{m-1}b_m, \tag{2.5}$$

where N is the number of collocation points. For the four schemes the collocation points ρ_i $(i=1,2,...,N)$ are chosen as:

(i) equidistant: $\rho_i=i/(N+1)$,
(ii) maxima of a Chebyshev polynomial: $\rho_i=\frac{1}{2}\left[1+\cos\frac{i\pi}{N+1},\right]$
(iii) zeros of a Chebyshev polynomial: $\rho_i=\frac{1}{2}\left[1+\cos\frac{(zi-1)\pi}{2N}\right]$, where the n^{th} degree Chebyshev polynomial $T_n^*(\rho)$, $0\leqslant\rho\leqslant1$, is defined as $T_n^*(\rho)=\cos n[\cos^{-1}(2\rho-1)]$,
(iv) zeros of a Legendre polynomial [7],

The nonlinear equations (2.1) and (2.2) at step J have been solved iteratively by linearizing the nonlinear product terms at each iteration as

$$\omega_J'^2=\omega_J',\omega_{J_p}',\quad (\psi'\omega')_J=\psi_J'\omega_{J_p}',\quad (\psi\omega'')_J=\chi_J\omega_{J_p}'' \tag{2.6}$$

where the typical predicted term f_{J_p} is taken as the mean of its values at the two previous iterations. For the first iteration, predicted value f_{J_p} is extrapolated quadratically from the values of f at three previous steps:

$$f_{J_p}=C(f_{J-1})+B(f_{J-2})+A(f_{J-3}), \tag{2.7}$$

where A, B, C are given by:

$J=1$	$A=0$,	$B=0$,	$C=1$,
$J=2$	$A=0$,	$B=-1$,	$C=2$,
$J\geqslant3$	$A=1$,	$B=-3$,	$C=3$.

Substituting Eqs. (2.5) and (2.6) into Eqs. (2.1) and (2.2), the $2N$ collocation equations for an iteration at the J-th load step can be expressed as

$$(2.8)\quad \sum_{n=1}^{N+2} n\,(n-2)\,\rho_i^{n-1}\;a_n+\frac{1-\nu^2}{2}\left[p_i\,(\omega'_{J_p})_i+2K\frac{h}{a}\,\rho_i^2\right]\left[\sum_{m=1}^{N+4}(m-1)\,\rho_i^{m-2}\,b_m\right]=0,$$

$$(2.9)\quad \sum_{m=1}^{N+4}[(m-1)^2\,(m-3)^2\,\rho_i^{m-2}]\,b_m-12\left(\frac{a}{h}\right)^2\times$$

$$\times\left[\sum_{n=1}^{N+2}\{(\omega'_{J_p})_i\,(n-1)\,\rho_i^n+(\omega''_{J_p})_i\,\rho_i^{n+1}\}\,a_n\right]-12\,\frac{Ka}{h}\left[\sum_{n=1}^{N+2}n\rho_i^{n+1}\,a_n\right]=\rho_i^3\,P.$$

The complete set of $2N+6$ discretized equations for a and b are obtained by appending the appropriate six boundary conditions of Eq. (2.4) to Eqs. (2.8) and (2.9). These are solved using Gaussian elimination with pivoting. The iterations are continued until $\omega_J(0)$, $\psi'_J(0)$ and $\psi_J(1)$ satisfy a relative convergence criterion of 0.1% accuracy. After getting the converged solution at step J, the procesure is repaeted for the subsequent steps.

3. Result and discussion

3.1. *Convergence study*

A convergence study conducted for the shell parameter $K=3$ and clamped edge is presented in Table 1. It can be noticed from Table 1 that the Gaussian collocation method has the fastest rate of convergence. Even for lower order approximations this method gives quite accurate results. The equidistant collocation is the slowest to converge. The methods based on maxima and zeros of a Chebyshev polynomial are competitive with the Gaussian collocation method but their results for lower order approximation are not as accurate as the Gaussian method. All collocation methods yield accurate and consistent results for higher order approximations.

Table 1. Convergence study, $K=3$, CE, $\nu=0.3$ $qa^4/Eh^4=12$.

	$\omega^*(0)/h$				$\sigma_r^{b*}(1)\,a^2/Eh^2$			
N	Equidistant	Chebyshev maxima	Chebyshev method	Gaussian method	Equidistant	Chebyshev maxima	Chebyshev method	Gaussian method
1	2.029	2.029	2.029	2.029	8.917	8.917	8.917	8.917
2	—	2.705	2.342	2.496	—	6.607	6.824	6.318
3	—	2.593	2.246	2.427	—	9.188	9.097	9.168
4	2.532	2.422	2.379	2.400	8.616	8.927	8.741	8.871
5	2.374	2.398	2.411	2.405	9.025	8.924	8.964	8.933
6	2.381	2.401	2.403	2.403	8.987	8.928	8.931	8.930
7	2.404	2.403	2.403	2.403	8.922	8.930	8.930	8.930

With these comparative results in view, the convergence study for a larger shell parameter $K=21$ has also been conducted using Chebyshev and Gaussian collocation methods and the results are given in Table 2. A similar conclusion can also be drawn from this table. The buckling pressure q of $0.71 p_{CL}$ based on the convergence failure in 100 iterations for the characteristic (average) deflection has been obtained for the shell parameter $K=21$, using the Gaussian collocation method. It agrees well with the values of $0.71 p_{CL}$ obtained by BUDIANSKY [8].

Table 2. Convergence study for $K=21$, $\nu=.3$ CE, $q=0.5 p_{CL}$

	$W^*(o)/h$		W^*_{avg}/h	
N	Chebyshev method	Gaussian method	Chebyshev method	Gaussian method
5	0.3845	0.3779	1.140	1.171
6	0.2691	0.2906	1.205	1.200
7	0.3061	0.2977	1.193	1.199
8	0.2895	0.2900	1.192	1.192
9	0,2912	0.2908	1.192	1.193
10	0.2905	0.2905	1.192	1.192

3.2. *Static results*

In order to have a check on the accuracy of the Gaussian collocation method, typical results obtained by this method using a lower order approximation with only 4 collocation points have been presented and compared with the results available for $K=0, 1, 2, 3$ for both clamped and simply supported edges. There is a close agreement among the present plate $K=0$ results (Figs. 1 to 3) and those of WAY [9], FEDERHOFER [9] and ALWAR and NATH [10] for load parameter up to 20, but there is some deriviation from the results of Alwar and Nath [10] for higher loads.

The results for shallow spherical caps for $K=1, 2, 3$ are presented in Figs. 4 and 5 for CE and SE, respectively. The results compare very well with those of NATH

Table 3. Static results for shell.

Edge condition	Shell parameter	$2a^4/Eh^4$	Present Gaussian	W^*/h Kornishin	Nath *et al.* [6]	Kanematsu et al. [11]
	1	5	0.862	0.86	0.856	0.938
CE	2	5	0.840	0.84	0.837	0.925
	3	5	0.402	0.40	0.372	0.352
	1	10	1.91	1.91	1.92	1.89
SE	2	1	0.160	0.160	0.150	0.142
	3	3	0.241	0.240	0.224	0.206

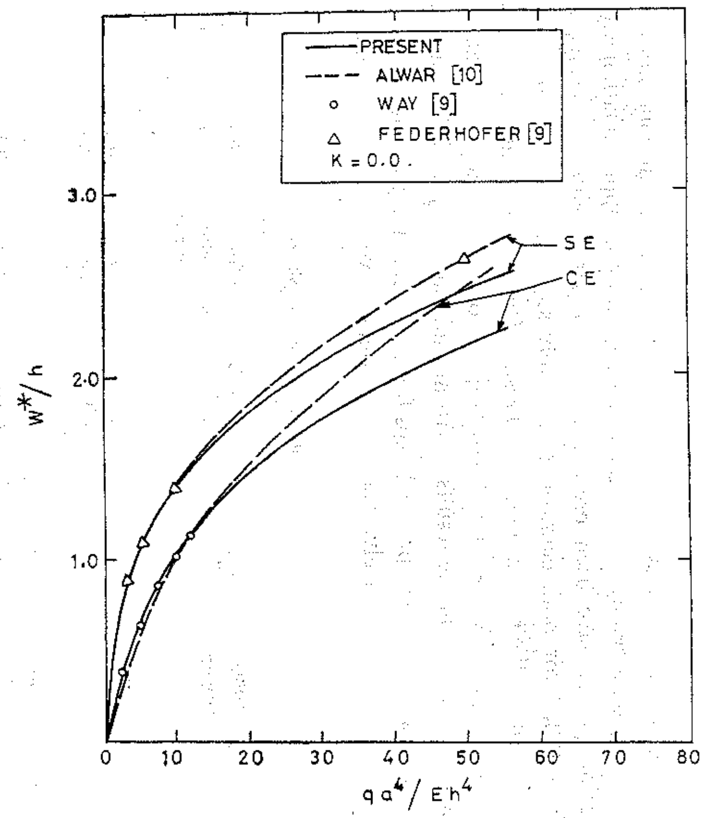

FIG. 1. Central deflection *Vs* load.

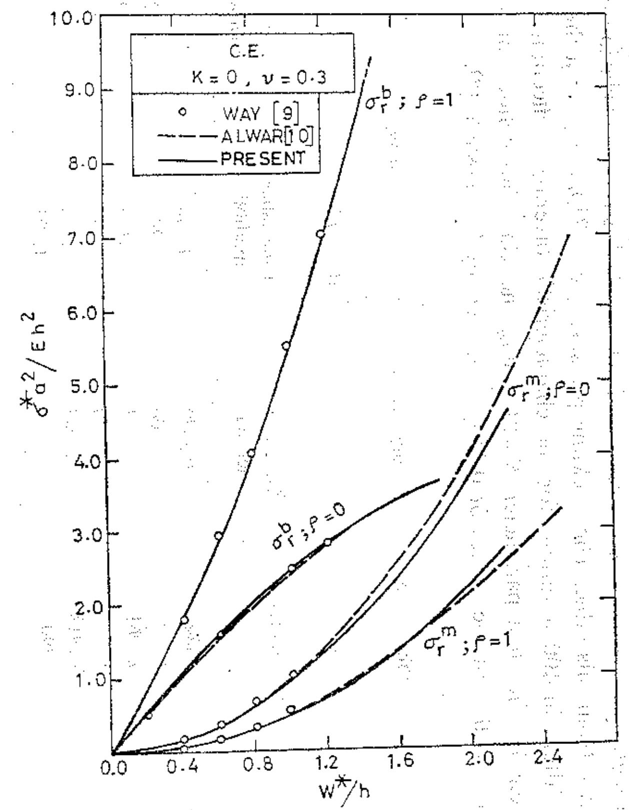

FIG. 2. Membrane and bending stresses *Vs* central deflection.

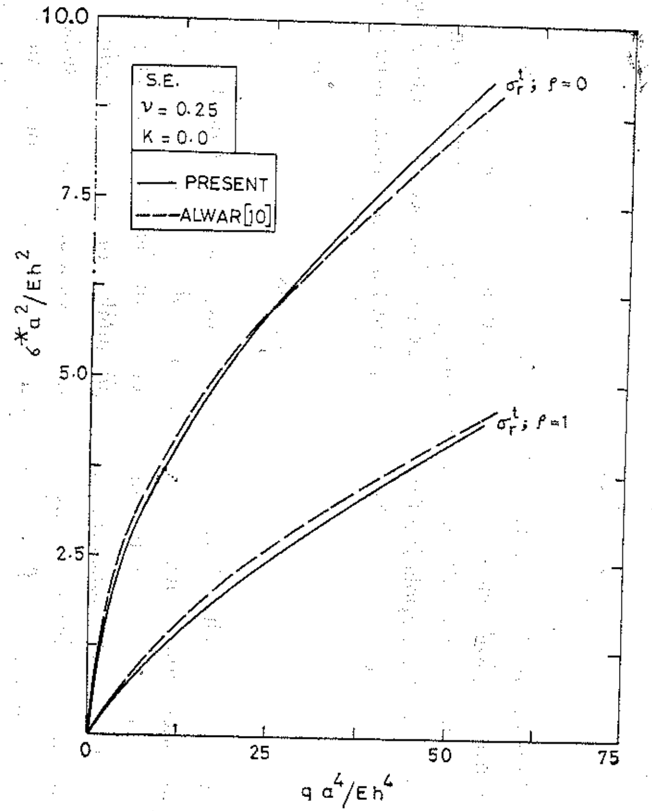

FIG. 3. Variation of total stress *Vs* load.

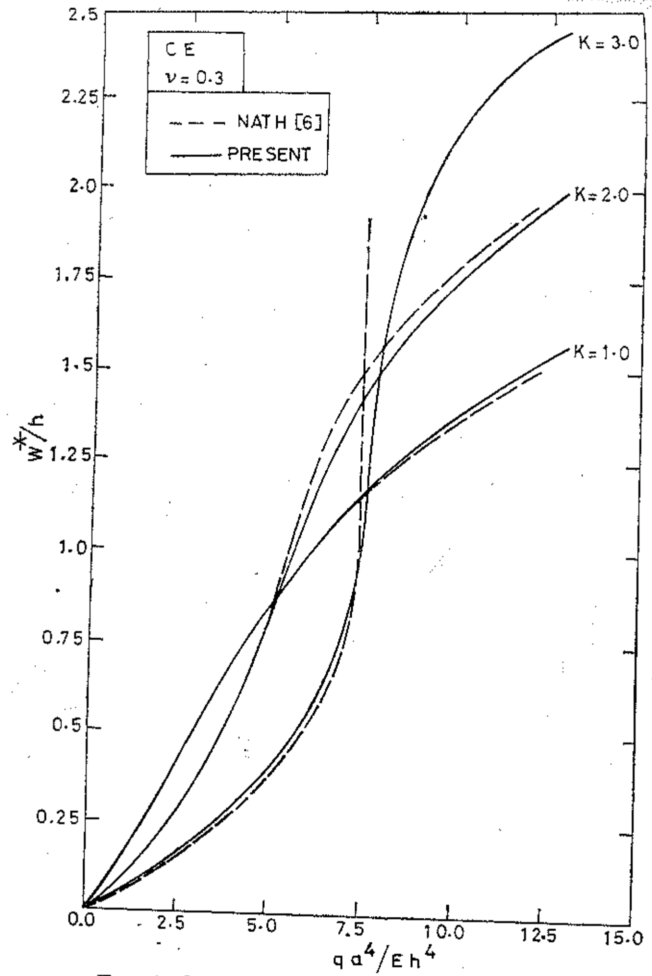

FIG. 4. Central deflection *Vs* external pressure.

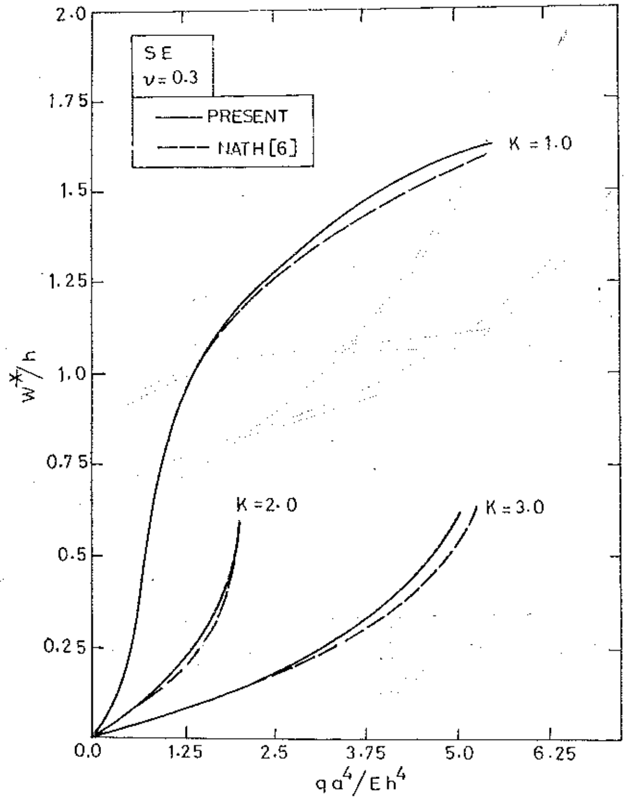

FIG. 5. Central deflection *Vs* external pressure.

and ALWAR [6]. As a further check on the accuracy of the results, the central deflection for typical load are compared with other available results in Table 3. The results agree closely with these of KORNISHIN [11]. The iterative method employed with quadratic extrapolation for the first iteration has been found to be very efficient since mostly only two iterations are sufficient for convergence at each step.

4. CONCLUSION

It can be concluded from the present study that the method of interior global point collocation, with zeros of a Legendre polynomial as collocation points, is the most efficient, accurate and simple method for the nonlinear analysis of circular plates and shallow spherical shells. The number of collocation points depends on the kind of problem investigated. Four collocation points are sufficient to have quite accurate results for the nonlinear response analysis for the shell parameters $K=0$, 1, 2, and 3 while seven collocation points are required for the buckling analysis for $K=21$.

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Streszczenie

WYBÓR FUNKCJI KOLOKACJI DLA OSIOWO-SYMETRYCZNEGO NIELINIOWEGO DWUPUNKTOWEGO PROBLEMU BRZEGOWEGO W STATYCE MAŁOWYNIOSŁYCH POWŁOK KULISTYCH

W pracy rozważa się problem optymalnego doboru punktów kolokacji, który, przy założonej dokładności, prowadzi do minimalnej liczby tych punktów. Przeprowadzono analizę zbieżności dla przypadku osiowo-symetrycznych, małowyniosłych sferycznych powłok poddanych działaniu równomiernego obciążenia przy czterech układach punktów kolokacji: punkty równoległe, kolokacja w maksimach wielomianów Czebyszewa, kolokacja w punktach zerowych wielomianów Czebyszewa i Legendre'a (punkty Gaussa). Stwierdzono, że gaussowska metoda kolokacji prowadzi do najlepszej zbieżności i daje dokładne wyniki nawet przy niewielkiej liczbie punktów kolokacji. Przedstawiono wyniki dotyczące nieliniowej analizy statycznej sprężystych płyt kołowych i powłok kulistych uzyskane metodą Gaussa i stwierdzono ich dobrą zgodność ze znanymi wynikami.

Резюме

ПОДБОР ФУНКЦИИ КОЛЛОКАЦИИ ДЛЯ ОСЕСИММЕТРИЧНОЙ НЕЛИНЕЙНОЙ ДВУХТОЧЕЧНОЙ КРАЕВОЙ ПРОБЛЕМЫ В СТАТИКЕ ПОЛОГИХ СФЕРИЧЕСКИХ ОБОЛОЧЕК

В работе рассматривается проблема оптимального подбора точек коллокации, который при заданной точности, приводит к минимальному количеству этих точек. Проведен анализ сходимости для случая осесимметричных, пологих сферических оболочек, подвергнутых действию равномерной нагрузки, при четырех системах точек коллокации: равноудаленные точки, коллокация максимумов многочленов Чебышева, коллокация в нулевых точках многочленов Чебышева и Лежандра (точки Гаусса) Констратировано, что гауссовский метод коллокации приводит к наилучшей сходимости и дает точные результаты даже при небольшом количестве точек коллокации. Представленны результаты, касающиеся нелинейного статического анализа упругих круговых плит и сферических оболочек, полученные методом Гаусса и констратированно их хорошее совпадение с известными результатами.

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