ROZPRAWY INŻYNIERSKIE · ENGINEERING TRANSACTIONS · 31, 2, 267—277, 1983
Polska Akademia Nauk · Instytut Podstawowych Problemów Techniki

DEFORMATION OF A NONHOMOGENEOUS VISCO-ELASTIC HOLLOW SPHERE

A. M. EL-KARAMANY (TRIPOLI)

Symmetric deformation of a non-homogeneous visco-elastic hollow sphere subjected to internal and external pressures is investigated. The problem is solved by the elastic-visco-elastic analogy. An exact solution is obtained. A numerical example is given. The non-homogeneity is shown to decrease he maximum stresses.

1. Introduction

The influence of nonhomogeneity has great importance in the study of visco-elastic materials such as structural elements made from composite materials and fibre glasses. The torsion problem of a circular bar of nonhomogeneous visco-elastic material is given in [1]. The method of solution of the homogeneous visco-elastic problems is discussed in [2].

In this work we extend the method adopted in [2] to solve the problem of symmetric deformation of a nonhomogeneous viscoelastic hollow sphere subjected to internal and external pressures. Assuming the creep function to be a function of time and coordinates, the problem is solved by using the elastic-visco-elastic analogy. Graphs have been drawn to demonstrate the variations of stress and strain components.

2. Formulation of the nonhomogeneous visco-elastic problem

Let us consider the small deformation of a nonhomogeneous visco-elastic body. It is assumed that the loading is quasistatic and the relaxation effects of the volume properties of the material are ignored. The stress-strain relations of relaxation type can be written, according to Boltzman [3], as follows:

(2.1)
$$S_{ij} = \int_0^t R(t-\tau, x, y, z)\, de_{ij}(\tau),$$
$$\sigma(t) = K\theta(t).$$

The creep-type stress-strain relations are

$$e_{ij} = \int_0^t \Pi\,(t-\tau, x, y, \dot{z})\, dS_{ij}\ (\tau),$$

(2.2)

$$\theta\,(t) = \frac{\sigma\,(t)}{K},$$

where

$$\sigma = \frac{\sigma_{kk}}{3}, \quad \theta = \varepsilon_{kk}, \quad S_{ij} = \sigma_{ij} - \sigma\delta_{ij}, \quad e_{ij} = \varepsilon_{ij} - \frac{\theta}{3}\,\delta_{ij},$$

$\sigma_{ij}, \varepsilon_{ij}$ denote the stress and strain tensors, respectively, $\mathrm{R}\,(t, x, y, \dot{z})$, $\Pi\,(t, x, y, \dot{z})$ are the relaxation and creep functions, respectively, K—the bulk modulus, $(x, y, \dot{z})$ are the coordinates of an arbitrary point M situated inside the body and t is the time.

Consider first the following boundary-value problem:

(2.3) $$\sigma_{ij}\, n_j|_{S_\sigma} = q_i\,(x_s, y_s, \dot{z}_s, t), \qquad u_i|_{S_u} = \varphi_i\,(x_s, y_s, \dot{z}_s, t)$$

and

(2.4) $$\frac{\partial \sigma_{ij}}{\partial x_j} + \varrho F_i = 0.$$

If the problem is to be solved in terms of displacements, the Cauchy relations must be used

(2.5) $$\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}),$$

and if it is to be solved in terms of stresses, the compatibility equations should be applied

(2.6) $$\epsilon_{ijk}\,\epsilon_{lmn}\,\frac{\partial^2\,\varepsilon_{km}}{\partial x_j\,\partial x_n} = 0,$$

where ϵ_{ijk} is the permutation symbol.

We shall use the Laplace-Carson transform with real parameter p. The image of $f(t)$ is $\overline{f(p)}$ defined by [4],

(2.7) $$\overline{f(p)} = p \int_0^\infty e^{-pt} f(t)\, dt.$$

Taking the Laplace-Carson transform of (2.1), (2.3) and (2.5) we obtain the following boundary-value problem in terms of images

$$\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} + \varrho \bar{F}_i = 0,$$

$$\bar{\varepsilon}_{ij} = \frac{1}{2}\,(\bar{u}_{i,j} + \bar{u}_{j,i}),$$

(2.8)

$$\bar{S}_{ij} = \bar{R}\bar{e}_{ij}, \qquad \bar{\sigma} = K\bar{\theta},$$

$$\bar{\sigma}_{ij}\, n_j|_{S_\sigma} = \bar{q}_i, \qquad \bar{u}_i|_{S_u} = \bar{\varphi}_i.$$

If the problem is to be solved in terms of stresses, we have

$$\begin{aligned}
&\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} + \varrho \bar{F}_i = 0, \\
&\epsilon_{ijk}\, \epsilon_{lmn} \frac{\partial^2 \bar{\varepsilon}_{kk}}{\partial x_j\, \partial x_n} = 0, \\
&\bar{e}_{ij} = \bar{\Pi} \bar{S}_{ij}, \qquad \bar{\theta} = \frac{\bar{\sigma}}{K}, \\
&\bar{\sigma}_{ij}\, n_j|_{S_\sigma} = \bar{q}_i, \qquad \bar{u}_i|_{S_u} = \bar{\varphi}_i,
\end{aligned} \tag{2.9}$$

where

$$\bar{\Pi} \bar{R} = \bar{R} \bar{\Pi} = 1. \tag{2.10}$$

3. The method of solution

We shall consider the materials for which the creep function Π can be written as

$$\Pi\,(t, x, y, z) = \Pi_0\,(t)\, g\,(x, y, z), \tag{3.1}$$

where

$$g\,(x, y, z) \neq 0$$

everywhere inside or at the boundary of the body.

According to Eq. (2.10) the relaxation function can be expressed as

$$R\,(t, x, y, z) = \frac{R_0\,(t)}{g\,(x, y, z)},$$

where

$$\bar{R}_0\, \bar{\Pi}_0 = \bar{\Pi}_0\, \bar{R}_0 = 1.$$

Let the functions $R_0\,(t)$ and $\Pi_0\,(t)$ be given by

$$\begin{aligned}
R_0\,(t) &= 2G_0 \left[1 - \int_0^t T(t)\, dt\right], \\
\Pi_0\,(t) &= \frac{1}{2G_0} \left[1 + \int_0^t L(t)\, dt\right],
\end{aligned} \tag{3.2}$$

where

$$T(t) = Ae^{-\beta t}\, t^{\alpha - 1}, \qquad L(t) = \frac{e^{-\beta t}}{t} \sum_{n=1}^{\infty} \frac{[A\Gamma\,(\alpha)]^n}{\Gamma\,(\alpha n)}\, t^{\alpha n},$$

A, β, α are empirical constants, $\Gamma\,(\alpha)$ is the gamma function, G_0 is the shear modulus which is constant for the homogeneous body and $R_0\,(0) = 2G_0$, $\Pi_0\,(0) = \dfrac{1}{2G_0}$.

Using Eqs. (3.2) in (2.1) and (2.2) we get

$$S_{ij}=\frac{1}{g(x,y,\dot{z})}\int_0^t R_0(t-\tau)\,de_{ij}(\tau), \tag{3.3}$$

$$e_{ij}=g(x,y,\dot{z})\int_0^t \Pi_0(t-\tau)\,dS_{ij}(\tau). \tag{3.4}$$

From equations (2.7), (3.3) and (3.4) we obtain

$$\bar{S}_{ij}=\frac{1}{g}\bar{R}_0\,\bar{e}_{ij}, \tag{3.5}$$

$$\bar{e}_{ij}=g\bar{\Pi}_0\,\bar{S}_{ij}. \tag{3.6}$$

In addition to he problem described either by Eqs. (2.8) and (3.5) or by (2.9) and (3.6), let us consider the problem of the nonhomogeneous theory of elasticity with the following nonhomogeneity law:

$$\begin{aligned} G&=\frac{G_0}{g(x,y,z)},\\ K&=\text{const}, \end{aligned} \tag{3.7}$$

where G is the shear modulus.

In this case we have the following boundary-value problem [5]:

$$\begin{aligned} &\frac{\partial\sigma_{ij}}{\partial x_j}+\varrho F_i=0,\\ &\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\\ &S_{ij}=\frac{2G_0}{g}e_{ij},\qquad \sigma=\theta K,\\ &\sigma_{ij}n_j|_{S_\sigma}=q_i,\qquad u_i=\varphi_i, \end{aligned} \tag{3.8}$$

or

$$\begin{aligned} &\frac{\partial\sigma_{ij}}{\partial x_j}+\varrho F_i=0,\\ &\epsilon_{ijk}\epsilon_{lmn}\frac{\partial^2\varepsilon_{km}}{\partial x_j\,\partial x_n}=0,\qquad \theta=\frac{\sigma}{K},\\ &\sigma_{ij}n_j|_{S_\sigma}=q_i,\qquad u_i|_{S_u}=\varphi_i. \end{aligned} \tag{3.9}$$

From Eqs. (2.8), (2.9), (3.8) and (3.9) it follows that the present nonhomogeneous visco-elastic problem in terms of the images is identical to the corresponding nonhomogeneous elastic problem with condition (3.7) and the following substitutions:

(3.10)
$$2G_0 \to \bar{R}_0, \frac{1}{2G_0} \to \bar{\Pi}_0, \quad S_{ij} \to \bar{S}_{ij}, \quad \sigma \to \bar{\sigma},$$
$$e_{ij} \to \bar{e}_{ij}, \quad \varepsilon_{ij} \to \bar{\varepsilon}_{ij}, \quad \sigma_{ij} \to \bar{\sigma}_{ij}, \quad \theta \to \bar{\theta},$$
$$F_i \to \bar{F}_i, \quad q_i \to \bar{q}_i, \quad \varphi_i \to \bar{\varphi}_i, \quad u_i \to \bar{u}_i.$$

Therefore the solution of the nonhomogeneous visco-elastic problem can be determined if the solution of the corresponding nonhomogeneous elastic problem is known.

4. Deformation of nonhomogeneous visco-elastic hollow sphere

Let us study the symmetric deformation of a nonhomogeneous visco-elastic hollow sphere of radii a and b $(a < b)$ subjected to internal and external pressures $P_a(t)$ and $P_b(t)$.

4.1. *Solution of the elastic problem*

First we shall solve the corresponding nonhomogeneous elastic problem assuming that K is a constant and G is a differentiable function of r, the radial coordinate of a spherical coordinate system (r, θ, φ).

For $\sigma_r, \sigma_\varphi = \sigma_\theta, \varepsilon_r, \varepsilon_\varphi = \varepsilon_\theta$ we have the following boundary-value problem

(4.1)
$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\varphi) = 0,$$

(4.2)
$$\frac{d\varepsilon_r}{dr} + \frac{1}{r}(\varepsilon_\varphi - \varepsilon_r) = 0,$$

(4.3)
$$\varepsilon_\varphi = \frac{1}{9K}(2\sigma_\varphi + \sigma_r) + \frac{1}{6G}(\sigma_\varphi - \sigma_r),$$
$$\varepsilon_r = \frac{1}{9K}(2\sigma_\varphi + \sigma_r) + \frac{1}{3G}(\sigma_r - \sigma_\varphi),$$

(4.4)
$$\sigma_r|_{r=a} = -Pa, \quad \sigma_r|_{r=b} = -Pb.$$

Assuming

(4.5)
$$S = \varepsilon_\varphi - \varepsilon_r = \frac{1}{2G(r)}(\sigma_\varphi - \sigma_r),$$
$$q = \sigma_\varphi - \sigma_r = 2G(r)(\varepsilon_\varphi - \varepsilon_r),$$

we obtain

(4.6)
$$q = 2G(r)\, S.$$

Using Eqs. (4.3) in (4.2) we obtain

(4.7) $$\frac{1}{9K}\left(3\frac{d\sigma_\varphi}{dr}-\frac{dq}{dr}\right)+\frac{1}{3}\frac{dS}{dr}+\frac{S}{r}=0.$$

Taking $\sigma_r=\sigma_\varphi-q$, the equilibrium equation (4.1) can be written as

(4.8) $$\frac{d\sigma_\varphi}{dr}-\frac{dq}{dr}-\frac{2q}{r}=0.$$

Substituting (4.5) into Eqs. (4.7) and (4.8), we get the equation

$$\frac{dq}{dr}+\frac{3}{r}q+\frac{3K}{2}\frac{dS}{dr}+\frac{q}{2}\frac{KS}{r}=0,$$

which can be expressed in the form

$$\frac{d}{dr}\left(qr^3+\frac{3}{2}KSr^3\right)=0.$$

The solution of the above differential equation is $q+\frac{3}{2}KS=\frac{C}{r^3}$ (C is a constant) which, by means of Eq. (4.6), yields

(4.9) $$S=\frac{C}{2\left(G+\frac{3}{4}K\right)r^3},\qquad q=\frac{C}{\left(1+\frac{3}{4}\frac{K}{G}\right)r^3}.$$

Integrating Eq. (4.1) and using Eqs. (4.5), (4.9) and (4.4), we obtain

(4.10) $$\sigma_r=-P_a+(P_a-P_b)\frac{I(r)}{I(b)},\qquad \sigma_\varphi=-P_b+(P_a-P_b)\frac{[I(r)+F(r)]}{I(b)},$$

where

(4.11) $$F(r)=\frac{1}{2r^3(1+3K/4G)},\qquad I(r)=\int_a^r\frac{dx}{x^4[1+3K/4G(x)]}.$$

From Hooke's law (4.3) the strain components are determined,

(4.12) $$\varepsilon_\varphi=\frac{1}{9K}M(r)+\frac{N(r)}{12G(r)},\qquad \varepsilon_r=\frac{M(r)}{9K}-\frac{N(r)}{6G},$$

where

$$M(r)=-3P_a+\frac{P_a-P_b}{I(b)}[3I(r)+2F(r)],$$

and

$$N(r)=2(P_a-P_b)\frac{F(r)}{I(b)}.$$

Assuming

(4.13) $$G = G_0/g(r), \qquad K = K_0 G_0,$$

equation (4.11) yields

(4.14) $$I(r) = \int_a^r \frac{dx}{x^4 [1+3K_0 g(x)/4]}, \qquad F(r) = \frac{1}{2r^3 \left(1+\frac{3}{4} K_0 g(r)\right)},$$

where $g(r) \neq 0$ is independent of G_0, and K_0 is a dimensionless constant.

It follows from the above equations that $I(r)$, $F(r)$ $M(r)$ and $N(r)$ are independent of G_0.

In view of Eqs. (4.13) and (4.14), the strain components given by (4.12) can be expressed as

(4.15) $$\begin{aligned} \varepsilon_\varphi &= -\frac{2}{3K_0} P_a \left(\frac{1}{2G_0}\right) + \frac{1}{2G_0} \frac{P_a - P_b}{3I(b)} \left\{ \frac{2}{K_0} I(r) + F(r) \left[g(r) + \frac{4}{3K_0} \right] \right\}, \\ \varepsilon_r &= -\frac{2}{3K_0} \left(\frac{1}{2G_0}\right) + \frac{1}{2G_0} \frac{P_a - P_b}{3I(b)} \left\{ \frac{2}{K_0} I(r) + F(r) \left[\frac{4}{3K_0} - 2g(r) \right] \right\}. \end{aligned}$$

The stress components given by Eq. (4.10) are independent of G_0.

4.2. Solution of the nonhomogeneous visco-elastic problem

Using the analogy (3.10) we obtain the solution of the nonhomogeneous visco-elastic problem in terms of images as follows.

(4.16) $$\begin{aligned} \bar{\sigma}_r &= -\bar{P}_a + \frac{\bar{P}_a - \bar{P}_b}{I(b)} I(r), \qquad \bar{\sigma}_\varphi = -\bar{P}_a + \frac{\bar{P}_a - \bar{P}_b}{I(b)} [I(r) + F(r)], \\ \bar{\varepsilon}_\varphi &= -\frac{2}{3K_0} \bar{\Pi}_0 \bar{P}_a + \bar{\Pi}_0 (\bar{P}_a - \bar{P}_b) \Psi_1(r), \\ \bar{\varepsilon}_r &= -\frac{2}{K_0} \bar{\Pi}_0 \bar{P}_a + \bar{\Pi}_0 (\bar{P}_a - \bar{P}_b) \Psi_2(r), \end{aligned}$$

where

$$\begin{aligned} \Psi_1 &= \frac{1}{3I(b)} \left\{ \frac{2}{K_0} I(r) + F(r) \left[\frac{4}{3K_0} + g(r) \right] \right\}, \\ \Psi_2 &= \frac{1}{3I(b)} \left\{ \frac{2}{K_0} I(r) + F(r) \left[\frac{4}{3K_0} - 2g(r) \right] \right\} \end{aligned}$$

are independent of G_0

Applying the inverse Laplace-Carson transform, the exact solution is obtained from Eq. (4.16) as

$$\sigma_r = -P_a(t) + \frac{P_a(t) - P_b(t)}{I(b)} I(r),$$

$$\sigma_\varphi = -P_a(t) + \frac{P_a(t) - P_b(t)}{I(b)} [I(r) + F(r)],$$

(4.17)
$$\varepsilon_\varphi = -\frac{2}{3K_0} \int_0^t \Pi_0(t-\tau)\, dP_a(\tau) + \Psi_1(r) \left[\int_0^t \Pi_0(t-\tau)\, dP_a(\tau) - \int_0^t \Pi_0(t-\tau)\, dP_b(\tau) \right],$$

$$\varepsilon_r = -\frac{2}{3K_0} \int_0^t \Pi_0(t-\tau)\, dP_a(\tau) + \Psi_2 \left[\int_0^t \Pi_0(t-\tau)\, dP_a(\tau) - \int_0^t \Pi_0(t-\tau)\, dP_b(\tau) \right].$$

5. Numerical example

As an example of nonhomogeneity let us take

(5.1) $$g(r) = (b/r)^n, \quad (n \neq -1).$$

Using relation (3.1) and $\Pi(t, r) = \Pi_0(t)\, g(r)$, $G = \dfrac{G_0}{g(r)}$ in Eq. (4.14) we obtain

(5.2) $$I(r) = I_n(r) = \int_a^r \frac{dx}{x^4 \left(1 + \dfrac{nC}{x^n}\right)}, \qquad F(r) = F_n(r) = \frac{1}{2r^3 \left(1 + \dfrac{nC}{r^n}\right)},$$

where

$$C_n = \frac{3}{4} K_0\, b^n.$$

Setting $n = 1, \frac{3}{2}, 2$, we obtain from Eq. (5.2)

$$I_1(r) = \frac{1}{C_1^2}\left(\frac{1}{r} - \frac{1}{a}\right) - \frac{1}{2C_1}\left(\frac{1}{r^2} - \frac{1}{a^2}\right) + \frac{1}{C_1^3} \ln\left(\frac{r(a + C_1)}{a(r + C_1)}\right),$$

$$F_1(r)=\frac{1}{2r^2(r+C_1)},\qquad C_1=\frac{3}{4}K_0\,b,$$

$$I_{3/2}(r)=\frac{2}{3C_{3/2}}\left(\frac{1}{a^{3/2}}-\frac{1}{r^{3/2}}\right)-\frac{2}{3C_{3/2}^2}\ln\left(\frac{1+C_{3/2}/a^{3/2}}{1+C_{3/2}/r^{3/2}}\right),$$

$$F_{3/2}(r)=\frac{1}{2r^{3/2}(r^{3/2}+C_{3/2})},\qquad C_{3/2}=\frac{3}{4}K_0\,b^{3/2},$$

$$I_2=-\frac{1}{C_2}\left(\frac{1}{r}-\frac{1}{a}\right)-\frac{1}{C_2^{3/2}}\arctan\left(\frac{r-a}{\sqrt{C_2}\left(1+\frac{ar}{C_2}\right)}\right),$$

$$F_2(r)=\frac{1}{2r(r^2+C_2)},\qquad C_2=\frac{3}{4}K_0\,b^2.$$

It may be observed that the homogeneous problems may be considered obtained as special cases when $n=0$.

Substituting $P_a(t)=P_a^0\,h(t)$, $P_b(t)=P_b^0(t)\,h(t)$ into Eq. (4.17) we find

$$\sigma_r=-P_a^0\,h(t)+(P_a^0-P_b^0)\frac{I(r)}{I(b)}h(t),$$

$$\sigma_\varphi=-P_a^0\,h(t)+(P_a^0-P_b^0)\frac{[I(r)+F(r)]}{I(b)}h(t),$$

$$\varepsilon_\varphi=-\frac{2}{3K_0}\Pi_0(t)\,P_a^0\,h(t)+\Psi_1\cdot(P_a^0-P_b^0)\,\Pi_0(t)\,h(t),$$

$$\varepsilon_r=-\frac{2}{3K_0}\Pi_0(t)\,P_a^0\,h(t)+\Psi_2\cdot(P_a^0-P_b^0)\,\Pi_0(t)\,h(t),$$

where P_a^0, P_b^0 are constantes and

$$h(t)=\begin{cases}0, & t\leq 0,\\ 1, & t>0\end{cases}$$

is the Heaviside unit step function [6].

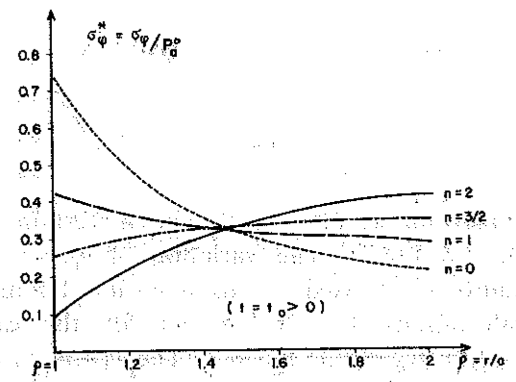

FIG. 1.

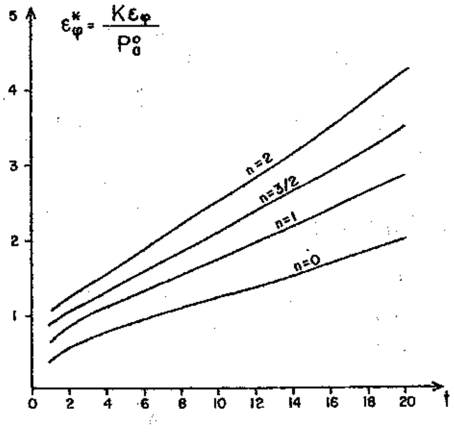

FIG. 2.

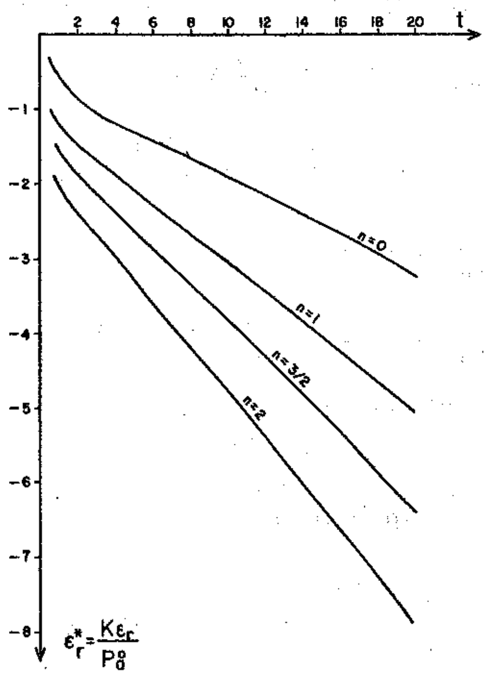

FIG. 3.

The stress distribution $\sigma^*_\varphi = \sigma_\varphi / P^0_a$, at a certain fixed instant $t = t_0$ ($t_0 > 0$) is shown in Fig. 1. The variation of $\varepsilon^*_i = \varepsilon_i K/P^0_a$ ($i = r, \varphi$) with time t (in minutes) for a fixed radius $\varrho = r/a = 1.4$ are shown in Figs. 2 and 3. The calculations are carried out for the case of free external surface ($P^0_b = 0$) and $K_0 = 8/3$, $b/a = 2$. The creep function $\Pi_0(t)$ is taken in the form given in Eq. (3.2) when $\alpha = 0.4$, $A = 0.1572$ and $\beta = 0.05$.

Conclusion

It is seen that nonhomogeneity decreases the maximum values of the stress components as compared o those in homogeneous bodies, and renders the distribution of the stresses more uniform.

References

1. A. M. El-Karamany, *On the problem of the non-homogeneous viscoelastic bodies*, Mekhanika Polimerov, 5, 826–831, Riga, USSR 1978.
2. A. A. Ilioushin, *The approximation method of calculating the constructures by linear thermal visco-elastic theory*, Mekhanika Polimerov, 2, 210–221, Riga, USSR, 1968.
3. M. E. Gurtin and E. Sternberg, *On the linear theory of visco-elasticity*, Arch. Ration. Mech. Anal., **11**, 291, 1962.
4. V. A. Ditkin and A. P. Prudnikov, *Internal transforms and operational calculus*, Pergamon Press, Oxford–New York 1965.
5. V. A. Lomakin, *The theory of elasticity of the non-homogeneous bodies*, Moscow, 1976.
6. R. M. Christensen, *Theory of viscoelasticity, an introduction*, Academic Press, New York–London, 1971.

Резюме

ДЕФОРМАЦИИ ПОЛОГО ШАРА ИЗГОТОВЛЕННОГО ИЗ НЕОДНОРОДНОГО ВЯЗКОУПРУГОГО МАТЕРИАЛА

Исследованы симетричные деформации неоднородного вязкоупругого полого шара, подвергнутого внутренним и внешним давлениям. Проблема решена, послуживаясь упругой-вязкоупругой аналогией. Получено точное решение и приведен числовой пример. Показано, что неоднородность приводит к снижению максимальных напряжений.

Streszczenie

ODKSZTAŁCENIE WYDRĄŻONEJ KULI WYKONANEJ Z NIEJEDNORODNEGO MATERIAŁU LEPKOSPRĘŻYSTEGO

Zbadano symetryczne odkształcenie niejednorodnej lepkosprężystej kuli wydrążonej poddanej ciśnieniu wewnętrznemu i zewnętrznemu. Zagadnienie rozwiązano posługując się analogią sprężysto-lepkosprężystą. Otrzymano rozwiązanie ścisłe i podano przykład liczbowy. Wykazano, że niejednorodność prowadzi do obniżenia naprężeń maksymalnych.

DEPARTMENT OF MATHEMATICS
AL-FATEH UNIVERSITY, TRIPOLI, LIBYA.

Received December 1, 1982.