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A NOTE ON THE PERTURBATION APPROACH TO NONLINEAR STATIC ANALYSIS OF ELASTIC PLANE TRUSSES

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A perturbation procedure for the incremental finite element analysis of large deformation problems is presented. The proposed approach takes into account the nonlinearity of kinematics at the incremental step. Explicit forms of the finite element "higher-order" matrices for truss element and numerical examples are given. Comparison of the results obtained by the perturbation approach, tangential stiffness method and closed-form solution are carried out. The appropriate computational cost analyses are presented implying conditions for the economical use of the method.

1. Introduction

The applications of the perturbation approach to computational nonlinear mechanics have been discussed in a number of papers over the last decade [1–10]. In spite of a formal elegance and theoretical attractivity of the approach, no such computational gains have been achieved that could compensate for the increased numerical effor. However, due to the limited experiences with the method no final conclusion on its applicability can be drawn yet and some further studies seem desirable. Such an effort has been undertaken and its preliminary results are reported in this note. In particular we give explicit forms of the finite element „higher order" matrices for a truss element. A simple two-bar truss plays a central role in the numerical evaluation of the approach discussed. The formulation can be understood as an alternative to the routinely used incremental Newton–Raphson iteration schemes.

2. Problem formulation

The fundamental matrix equation describing the continuing equilibrium of any finite element under large displacements can be written as [10, 11]

$$(2.1)\qquad {}^{(1)}k_{ij}\,\Delta u_j + {}^{(2)}k_{ijk}\,\Delta u_j\,\Delta u_k + {}^{(3)}k_{ijkl}\,\Delta u_j\,\Delta u_k\,\Delta u_l = \Delta U_i, \qquad i, j, k, l = 1, 2, \ldots, n,$$

where ${}^{(1)}k_{ij} = k_{ij}^{(c)} + k_{ij}^{(\sigma)} + k_{ij}^{(id)}$ is the "first-order" stiffness matrix, $k_{ij}^{(c)}$ the constitutive stiffness matrix, $k_{ij}^{(\sigma)}$ the initial stress stiffness matrix, $k_{ij}^{(id)}$ the initial displacement matrix, ${}^{(2)}k_{ijk}$ and ${}^{(3)}k_{ijkl}$ are the "second"- and "third-order" stiffness matrices respectively, Δu_i is the incremental generalized displacement vector and n stands for the number of the degrees of freedom for the element. In the traditional approach, Eq. (2.1) is linearized, i.e. only the first term on the left-hand side is retained. In the present paper a perturbation procedure for solving the nonlinear equalibrium equation (2.1) is used as follows. Let us assume all the variables in Eq. (2.1) to be functions of an arbitrary single parameter t, which has its origin at the known point on the equilibrium path, i.e.

$$\Delta u_i(t)|_{t=0}=0, \quad \Delta U_i(t)|_{t=0}=0. \tag{2.2}$$

The physical meaning of the parameter t is not yet delineated. It is to be chosen in a form most suitable for the numerical computation.

The incremental displacements and nodal forces are each approximated by a Taylor series expansion in t about the known configuration at which $t=0$

$$\Delta u_i(t)=\Delta u_i^{(0)}(0)+\Delta u_i^{(1)}(0)\,t+\frac{1}{2}\Delta u_i^{(2)}(0)\,t^2+\ldots=\sum_{m=1}\frac{\Delta u_i^{(m)}}{m!}t^m \tag{2.3}$$

and

$$\Delta U_i(t)=\Delta U_i^{(0)}(0)+\Delta U_i^{(1)}(0)\,t+\frac{1}{2}\Delta U^{(2)}(0)\,t^2+\ldots=\sum_{m=1}\frac{\Delta U_i^{(m)}}{m!}t^m. \tag{2.4}$$

In deriving Eqs. (2.3) and (2.4) it was assumed that the functions $\Delta u_i(t)$, $\Delta U_i(t)$ are single-valued analytical functions of t for a finite (even if small) range $0\leqslant t\leqslant t^*$. Under the same assumption we write

$$\begin{aligned} {}^{(1)}k_{ij}(t)&=k_{ij}^{(0)}+k_{ij}^{(1)}t+\frac{1}{2}k_{ij}^{(2)}t^2+\ldots=\sum_{m=0}\frac{k_{ij}^{(m)}}{m!}t^m,\\ {}^{(2)}k_{ijk}(t)&=\sum_{m=0}\frac{k_{ijk}^{(m)}}{m!}t^m,\\ {}^{(3)}k_{ijkl}(t)&=\sum_{m=0}\frac{k_{ijkl}^{(m)}}{m!}t^m. \end{aligned} \tag{2.5}$$

Substitution of Eqs. (2.3)-(2.5) into Eq. (2.1) furnishes

$$\begin{aligned} &\sum_{s=1}\sum_{r=1}^{s}\frac{{}^{(1)}k_{ij}^{(sr)}}{(s-r)!}\,\frac{\Delta u_j^{(s-r)}}{r!}\,t^s+\sum_{s=2}\sum_{v=2}^{s}\sum_{r=1}^{v-1}\frac{{}^{(2)}k_{ijk}^{(s-v)}}{(s-v)!}\,\frac{\Delta u_j^{(v-r)}}{(v-r)!}\,\frac{\Delta u_k^{r}}{r!}\,t^s+\\ &+\sum_{s=3}\sum_{w=3}^{s}\sum_{v=2}^{w-1}\sum_{r=1}^{v-1}\frac{{}^{(3)}k_{ijkl}^{(s-w)}}{(s-w)!}\,\frac{\Delta u_j^{(w-v)}}{(w-v)!}\,\frac{\Delta u_k^{(v-r)}}{(v-r)!}\,\frac{\Delta u_l^{(r)}}{r!}\,t^s=\sum_{s=1}\frac{\Delta U_i^{(s)}}{s!}\,t^s. \end{aligned} \tag{2.6}$$

Arranging the terms according to each power of t and noting that Eqs. (2.8) must hold for any $t\in[0, t^*]$, we can write

$${}^{(1)}k_{ij}^{(0)}\frac{\Delta u_j^{(m)}}{m!}=\frac{\Delta U_i^{(m)}}{m!}-\Delta\hat{U}_i^{(m)}, \tag{2.7}$$

where

$$\Delta \hat{U}_i^{(1)}=0,$$

(2.8)
$$\Delta \hat{U}_i^{(m)}=\sum_{s=1}^{m-1} \frac{{}^{(1)}k_{ij}^{(m-s)}}{(m-s)!} \frac{\Delta u_j^{(s)}}{s!}+\sum_{v=2}^{m}\sum_{s=1}^{v-1} \frac{{}^{(2)}k_{ijk}^{(m-v)}}{(m-v)!} \frac{\Delta u_j^{(v-s)}}{(v-s)!} \frac{\Delta u_k^{(s)}}{s!}+$$
$$+\sum_{w=3}^{m}\sum_{v=2}^{w-1}\sum_{s=1}^{v-1} \frac{{}^{(3)}k_{ijkl}^{(m-w)}}{(m-w)!} \frac{\Delta u_j^{(w-v)}}{(w-v)!} \frac{\Delta u_k^{(v-s)}}{(v-s)!} \frac{\Delta u_l^{(s)}}{s!} \quad \text{for} \quad m\geqslant 2.$$

For $m=1, 2, 3$ the first three equations for each element take the form

(2.9)
$$ {}^{(1)}k_{ij}^{(0)}\, \Delta u_j=\Delta U_i^{(1)},$$
$$ {}^{(1)}k_{ij}^{(0)}\, \Delta u_j^{(2)}+2\,{}^{(1)}k_{ij}^{(1)}\, \Delta u_j^{(1)}+2\,{}^{(2)}k_{ijk}^{(0)}\, \Delta u_j^{(1)}\, \Delta u_k^{(1)}=\Delta U_i^{(2)},$$
$$ {}^{(1)}k_{ij}^{(0)}\, \Delta u_j^{(3)}+3\,{}^{(1)}k_{ij}^{(1)}\, \Delta u_j^{(2)}+3\,{}^{(1)}k_{ij}^{(2)}\, \Delta u_j^{(1)}+3\,{}^{(2)}k_{ijk}^{(0)}\, \Delta u_j^{(2)}\, \Delta u_k^{(1)}+$$
$$+6\,{}^{(2)}k_{ijk}^{(1)}\, \Delta u_j^{(1)}\, \Delta u_k^{(1)}+3\,{}^{(2)}k_{ijk}^{(0)}\, \Delta u_j^{(1)}\, \Delta u_k^{(2)}+$$
$$+6\,{}^{(3)}k_{ijkl}^{(0)}\, \Delta u_j^{(1)}\, \Delta u_k^{(1)}\, \Delta u_e^{(1)}=\Delta U_i^{(3)}.$$

For any explicit dependence of the stiffness matrices ${}^{(1)}k_{ij}$, ${}^{(2)}k_{ijk}$, ${}^{(3)}k_{ijkl}$ upon their arguments the terms of the expansions (2.5) can be found as functions of the corresponding lower-order terms of the argument expansions. If we write, for instance,

(2.10)
$$ {}^{(1)}k_{ij}={}^{(1)}k_{ij}(\mathbf{u}, \boldsymbol{\sigma}),$$
$$ {}^{(2)}k_{ijk}={}^{(2)}k_{ijk}(\mathbf{u}, \boldsymbol{\sigma}),$$
$$ {}^{(3)}k_{ijkl}={}^{(3)}k_{ijkl}(\mathbf{u}, \boldsymbol{\sigma}),$$

where $\mathbf{u}$ is the displacement vector and $\boldsymbol{\sigma}$ is a stress vector, then with the expansions

(2.11)
$$\mathbf{u}=\sum_{m=0} \frac{\mathbf{u}^{(m)}}{m!} t^m,$$
$$\boldsymbol{\sigma}=\sum_{m=0} \frac{\boldsymbol{\sigma}^{(m)}}{m!} t^m.$$

the following relations hold:

(2.12)
$$\mathbf{k}^{(0)}=\mathbf{k}|_{t=0}=\mathbf{k}(\mathbf{u}^{(0)}, \boldsymbol{\sigma}^{(0)}),$$
$$\mathbf{k}^{(1)}=\left.\frac{d\mathbf{k}}{dt}\right|_{t=0},$$
$$\mathbf{k}^{(2)}=\left.\frac{d^2\mathbf{k}}{dt^2}\right|_{t=0}.$$

As these relations hold for the "first- second- and third-order" stiffness matrices, the appropriate left-hand side subscripts are omitted.

Having decribed the element we pass to the relations for the whole assembly of elements following the standard finite element derivation. To this aim we replace in Eqs. (2.9) $\Delta\mathbf{u}$ for $\Delta\mathbf{r}$, $\Delta\mathbf{U}$ for $\Delta\mathbf{R}$ and define an incremental load parameter $\Delta\lambda$ by the relation

$$\Delta Ra = \Delta\lambda \bar{R}_a, \tag{2.13}$$

where $\bar{R}_a$ is a given reference load.

Noting the relations

$$\begin{aligned} \Delta\lambda(t) &= \sum_{m=1} \frac{\Delta\lambda^{(m)}}{m!} t^m, \\ \Delta r_i(t) &= \sum_{m=1} \frac{\Delta r_i^{(m)}}{m!} t^m. \end{aligned} \tag{2.14}$$

we choose the load increment as the path parameter so that $\Delta\lambda(t)=t$, $\Delta\lambda^{(1)}=1$, $\Delta\lambda^{(2)}=0$. Using the relations (2.9) the following sequence of equations is easily obtained:

$$\begin{aligned} {}^{(1)}K_{ab}{}^{(0)}\, \Delta r_b{}^{(1)} &= \bar{R}_a, \\ {}^{(1)}K_{ab}{}^{(0)}\, \Delta r_b{}^{(2)} &= -2\,{}^{(1)}K_{ab}{}^{(1)}\, \Delta r_b{}^{(1)} - 2\,{}^{(2)}K_{abc}{}^{(0)}\, \Delta r_b{}^{(1)}\, \Delta r_c{}^{(1)}, \\ {}^{(1)}K_{ab}{}^{(0)}\, \Delta r_b{}^{(3)} &= -3\,{}^{(1)}K_{ab}{}^{(1)}\, \Delta r_b{}^{(2)} - 3\,{}^{(1)}K_{ab}{}^{(2)}\, \Delta r_b{}^{(1)} - 3\,{}^{(2)}K_{abc}{}^{(0)} \Delta r_b{}^{(2)}\, \Delta r_c{}^{(1)} - \\ &\quad - 3\,{}^{(2)}K_{abc}{}^{(0)}\, \Delta r_b{}^{(1)}\, \Delta r_c{}^{(2)} - 6\,{}^{(2)}K_{abc}{}^{(1)}\, \Delta r_b{}^{(1)}\, \Delta r_c{}^{(1)} - \\ &\quad - 6\,{}^{(3)}K_{abcd}{}^{(0)}\, \Delta r_b{}^{(1)}\, \Delta r_c{}^{(1)}\, \Delta r_d{}^{(1)}, \end{aligned} \tag{2.15}$$

where the global matrices K are assembled using the corresponding elemental matrices. Thus, instead of the one matrix nonlinear equilibrium equation we arrived at a sequence of three matrix linear equations to be solved in succession using subsequent results to built the right-hand sides of the equations to follow. Another point to be noted is that once the inverse matrix $[{}^{(1)}K_{ab}{}^{(0)}]^{-1}$ is calculated in the process of solution it can be used for obtaining the higher derivatives of Δr_i in succession, which may save a lot of effort.

3. Truss element

In the present chapter the explicit form of the matrices for a truss element are developed. We assume the truss element shown in Fig. 1 to be a bar of the uniform cross-section and uniform properties along its length with two nodes at the ends. The displacement field is described by the relation

$$\mathbf{u}(\mathbf{r}) = [N_i\, N_j]\, \mathbf{u}, \tag{3.1}$$

where $N_i = 1 - r/l$ and $N_j = r/l$ are the shape functions of the element. We also assume the updated Lagrangian formalism and the linear elastic material response.

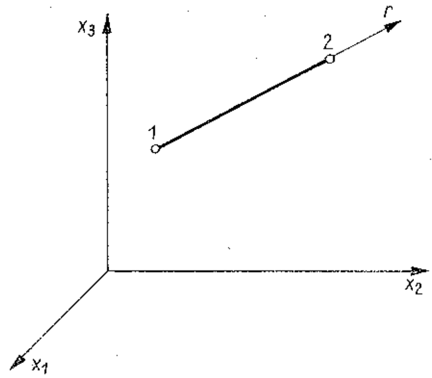

FIG. 1.

Let us specify the forms of matrices ${}^{(1)}k_{ij}$, ${}^{(2)}k_{ijk}$, ${}^{(3)}k_{ijkl}$ and their expansion terms as follows. The nonlinear incremental strain-displacement relation reads

$$\Delta\varepsilon_a = \Delta\varepsilon_a^L + \Delta\varepsilon_a^N = B_{ai}^L\, \Delta U_i + B_{aij}^N\, \Delta u_i\, \Delta u_j, \tag{3.2}$$

where $\Delta\varepsilon_a^L$, B_{ai}^L refer to the linear part of the strain increment, and $\Delta\varepsilon_a^N$, B_{aij}^N to its nonlinear part. The matrix ${}^{(1)}k_{ij}{}^{(c)}$ is obtained as

$$ {}^{(1)}k_{ij}^{(c)} = \int_v C_{ab}\, B_{ai}^L\, B_{bj}^L\, dV, \qquad i,j=1,2 \tag{3.3}$$

which, by uniform cross-section assumption, leads to

$$ {}^{(1)}k_{ij}^{(c)} = \frac{EA}{l} \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}. \tag{3.4}$$

Adding the known form of the initial stress matrix $\mathbf{k}^{(\sigma)}$ (see Appendix) we obtain the full form of the "first-order" matrix ${}^{(1)}k_{ij}$.

Similary we define the "second-order" matrix as

$$ {}^{(2)}k_{ijk} = \int_v C_{ab}\, (B_{ai}^L\, B_{bjk}^N + B_{ajk}^N\, B_{bk}^L)\, dV, \qquad i,j,k=1,2, \tag{3.5}$$

and the "third-order" matrix as

$$ {}^{(3)}k_{ijkl} = \int_v C_{ab}\, B_{aij}^N\, B_{bkl}^N\, dV, \qquad i,j,k,l=1,2. \tag{3.6}$$

The explicit forms of the matrices ${}^{(1)}k_{ij}$, ${}^{(2)}k_{ijk}$, ${}^{(3)}k_{ijkl}$ are given in Appendix.

Let us now focus our interest on the calculation of the particular terms in the expansions (2.6)–(2.8). According to the linear elastic material assumption and the updated Lagrangian description we obtain

$$\begin{gathered} E^{(0)} = E = \text{const}, \qquad E^{(1)} = 0, \\ \mathbf{k} = \mathbf{k}\left(l(t),\, N(t)\right). \end{gathered} \tag{3.7}$$

In general, the actual length of the element and the actual axial force can be written as

$$l=l_0+\Delta l, \quad N=N_0+\Delta N, \tag{3.8}$$

so that

$$l^{(1)}=l_0+\Delta l^{(1)}\, t, \quad N^{(1)}=N_0+\Delta N^{(1)}\, t \tag{3.9}$$

and

$$\begin{aligned} l^{(2)}&=l_0+\Delta l^{(1)}\, t+\frac{1}{2}\,\Delta l^{(2)}\, t^2,\\ N^{(2)}&=N_0+\Delta N^{(1)}\, t+\frac{1}{2}\,\Delta N^{(2)}\, t^2. \end{aligned} \tag{3.10}$$

Substituting Eqs. (3.7)–(3.10) into Eq. (2.12) leads to

$$\begin{aligned} \mathbf{k}^{(0)}&=\mathbf{k}|_{t=0}=\mathbf{k}\,(l^{(0)}, N^{(0)}),\\ \mathbf{k}^{(1)}&=\left.\frac{d\mathbf{k}}{dt}\right|_{t=0}=\frac{\partial \mathbf{k}^{(0)}}{\partial l^{(0)}}\,\Delta l^{(1)}+\frac{\partial \mathbf{k}^{(0)}}{\partial N^{(0)}}\,\Delta N^{(1)},\\ \mathbf{k}^{(2)}&=\left.\frac{d^2\mathbf{k}}{dt^2}\right|_{t=0}=\frac{\partial^2 \mathbf{k}^{(0)}}{\partial l^{(0)}\,\partial l^{(0)}}\,\Delta l^{(1)}\,\Delta l^{(1)}+\frac{\partial^2 \mathbf{k}^{(0)}}{\partial N^{(0)}\,\partial N^{(0)}}\,\Delta N^{(1)}\,\Delta N^{(1)}+\\ &\quad+2\,\frac{\partial^2 \mathbf{k}^{(0)}}{\partial l^{(0)}\,\partial N^{(0)}}\,\Delta N^{(1)}\,\Delta l^{(1)}+\frac{\partial \mathbf{k}^{(0)}}{\partial l^{(0)}}\,\Delta l^{(2)}+\frac{\partial \mathbf{k}^{(0)}}{\partial N^{(0)}}\,\Delta N^{(2)}. \end{aligned} \tag{3.11}$$

Finally, using explicit forms of the matrices shown in Appendix, the relations (3.11) take the following form:

for terms of the "first-order" matrix:

$$\begin{aligned} {}^{(1)}k_{ij}{}^{(0)}&=\frac{N^{(0)}}{l^{(0)}}\,{}^{(1)}W^{n}_{ij}+\frac{EA}{l^{(0)}}\,{}^{(1)}W^{e}_{ij},\\ {}^{(1)}k_{ij}{}^{(1)}&=-\frac{\Delta l^{(1)}}{l^{(0)}}\,{}^{(1)}k^{0}_{ij}+\frac{\Delta N^{(1)}}{l^{(0)}}\,{}^{(1)}W^{n}_{ij},\\ {}^{(1)}k_{ij}{}^{(2)}&=2\,\frac{\Delta l^{(1)}}{l^{(0)}}\,{}^{(1)}k_{ij}{}^{(1)}-\frac{\Delta l^{(2)}}{l^{(0)}}\,{}^{(1)}k_{ij}{}^{(0)}+\frac{\Delta N^{(2)}}{l^{(0)}}\,{}^{(1)}W^{n}_{ij}; \end{aligned} \tag{3.12}$$

for terms of the "second-order" matrix

$$\begin{aligned} {}^{(2)}k_{ijk}{}^{(0)}&=\frac{EA}{(l^{(0)})^2}\,{}^{(2)}W_{ijk},\\ {}^{(2)}k_{ijk}{}^{(1)}&=-2\,\frac{\Delta l^{(1)}}{l^{(0)}}\,{}^{(2)}k_{ijk}{}^{(0)}; \end{aligned} \tag{3.13}$$

for terms of the "third-order" matrix

$${}^{(3)}k_{ijkl}{}^{(0)}=\frac{EA}{4\,(l^{(0)})^3}\,{}^{(3)}W_{ijkl}, \tag{3.14}$$

where ${}^{(1)}\mathbf{W}$, ${}^{(2)}\mathbf{W}$, ${}^{(3)}\mathbf{W}$ are given in Appendix. Values of $\Delta l^{(1)}$, $\Delta l^{(2)}$ are calculated using Eqs. $(3.9)_1$ and $(3.10)_1$ after arriving at $l^{(0)}$, $l^{(1)}$, $l^{(2)}$ from current values of nodal coordinates at each level of the perturbation expansion. The values $\Delta N^{(1)}$, $\Delta N^{(2)}$ are obtained by transforming the terms of the expansions of the nodal force increments to the axial force. The matrix of the transformation is updated at each level of the perturbation expansion.

4. Results

On the basis of the derivation discussed above an algorithm has been worked out and numerical calculations carried out. As an example, the simple problem of a two-bar truss (Fig 2) is chosen due to its simplicity and the existence of the

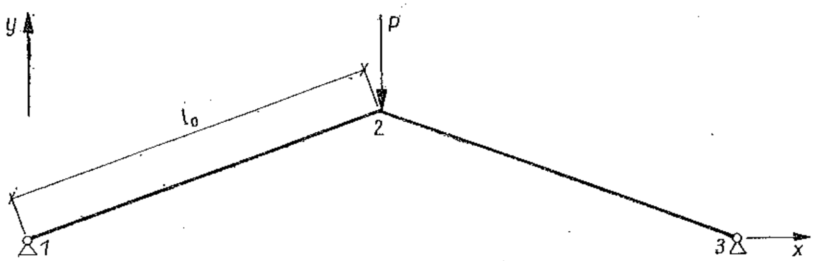

Fig. 2.

closed-form solution. Input data are: initial lenght $l_0=100$, coordinates $\mathbf{x}^T=|0\ 93.97\ 187.97|$, $y^T=|0 34.20|$, Young's modulus $E=50=$const, reference load $\bar{P}=1$, cross--sectional area $A=1=$const.

First, a series of numerical calculations for a different number of incremental steps is carried out. For the sake of comparison the perturbation approach (PA) and the tangential stiffness method (TS) are used. The exact closed-form solution yields the maximum force $P_{\max}=0.8535$. For this value of P from PA we obtain in two incremental steps vertical displacement $u=10.71$ represented by point B in Fig. 3.

To arrive at the similar value of the displacement the TS method requires 8 incremental steps (Fig. 3. curve C_1, $u=10.78$). Point A ($u=7.2$) in Fig. 3 represents the linear solution while the curve C_3 corresponds to the closed-form nonlinear solution. The curve C_2 illustrates the improvement of the results obtained by using 3 steps of PA ($u=11.5$) and 12 steps of TS ($u=11.6$). The smaller number of steps to achieve a comparable accuracy in PA is clearly seen also in this case. Figure 4 presents a comparison of the results obtained using the same number of the incremental steps in TS as well as in PA. The TS method represented by the curve C_1 yielded for $P=0.8535$ the displacement $u=10.6$ comparing to $u=13.7$ resulting from PA (curve C_2). This gives a difference of about 24%. Curve C_3 in Fig. 4 represents the closed-form solution.

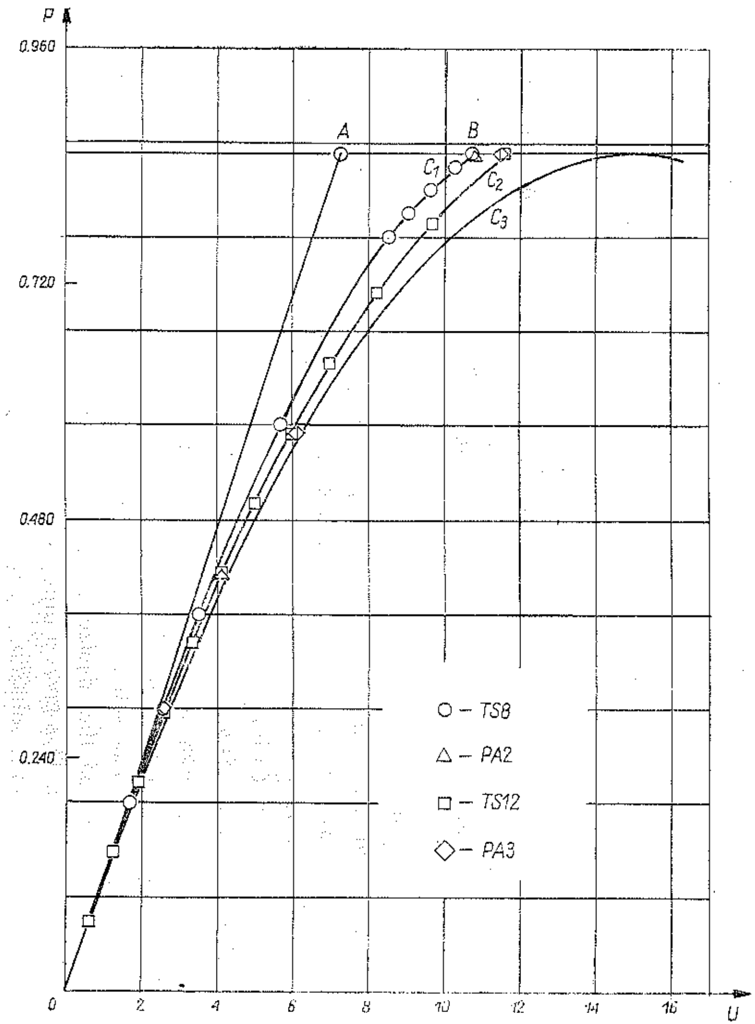
P
0.960
0.720
0.480
0.240
0
2
4
6
8
10
12
14
16
U
A
B
C1
C2
C3
○ – TS8
△ – PA2
□ – TS12
◇ – PA3

FIG. 3.

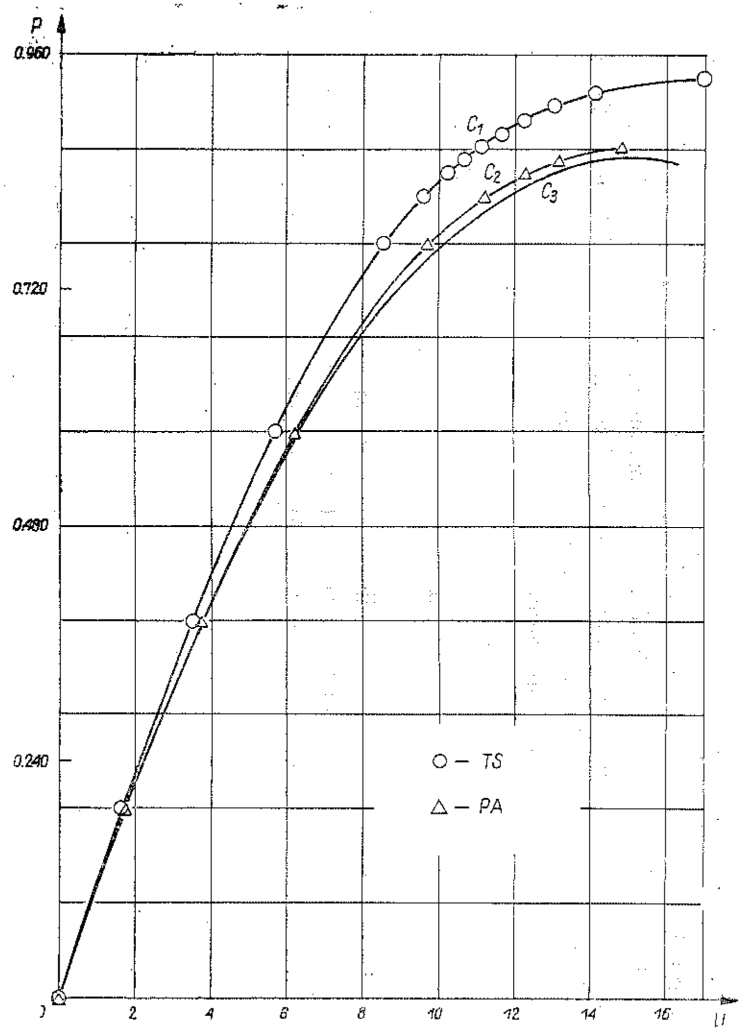
P
0.960
0.720
0.480
0.240
2
4
6
8
10
12
14
16
U
C1
C2
C3
○ – TS
△ – PA

FIG. 4.

5. Operation counts

An operation is defined as either a multiplication or a division. Let us denote by n and s the numbers of structural and elemental degrees of freedom, respectively, and by p the number of elements.

In order to count the number of operations of TS and PA algorithms it is convenient to divide it into three parts. The first part concerns the solution of the matrix equilibrium equations. The number of operations for one incremental step for TS in solving Eq. (2.15) is given by

$$T_k(n)=n^3+n^2 \tag{5.1}$$

and for PA in solving Eqs. $(2.14)_2$ and (2.15) by

$$P_k(n)=n^3+(n^2)+(3n^2+n)+(8n^2+2n)+(3n+3)=n^3+12n^2+6n+3. \tag{5.2}$$

In the second part computations needed for the evaluation of the nodal forces are considered. The number of operations for TS in Eq. $(2.9)_1$ is given by

$$T_N(s,p)=s^2p \tag{5.3}$$

and for PA in (2.9) by

$$P_N(s,p)=(s^3+12s^2+6s+3)p. \tag{5.4}$$

The third part concerns the generation of element stiffness matrices and their transformation to the global coordinates. In this case the number of operations for TS in transforming ${}^{(1)}k^{(0)}$ is given by

$$T_E(s,p)=(2s^2)p \tag{5.5}$$

and for PA in transforming ${}^{(1)}k^{(0)}$, ${}^{(1)}k^{(1)}$, ${}^{(1)}k^{(2)}$, ${}^{(2)}k^{(0)}$, ${}^{(2)}k^{(1)}$, ${}^{(3)}k^{(0)}$ is given by

$$P_E(s,p)=[(2s^2)+(3s^2)+(5s^2)+(3s^3)+(s^3)+(4s^4)]p=(4s^4+4s^3+10s^2)p. \tag{5.6}$$

Using Eqs. (5.1)-(5.6) the total cost is

$$\begin{aligned} T(n,s,p)&=T_k+T_N+T_E,\\ P(n,s,p)&=P_k+P_N+P_E. \end{aligned} \tag{5.7}$$

For $n=1$, $s=1$, $p=1$ we obtain

$$\begin{aligned} T&=2+1+2=5,\\ P&=22+21+18=61,\\ P/T&=12, \end{aligned} \tag{5.8}$$

where the components of Eq. (5.8) correspond to the cost of the three algorithm parts, respectively.

In the case of the example presented in Sect. 4, i.e. for $n=2$, $s=4$, $p=2$, we have

$$
\begin{aligned}
T&=12+32+64=108,\\
P&=71+438+2880=3389,\\
P/T&=31.
\end{aligned}
\tag{5.9}
$$

A similar ratio was obtained by comparing the execution times for both TS and PA algorithms.

For a more realistic data of, say, $n=102$, $s=4$, $p=102$ we obtain

$$
\begin{aligned}
T&=1071612+1632+3264=1076508,\\
P&=1186671+22338+146880=1355889,\\
P/T&=1.26.
\end{aligned}
\tag{5.10}
$$

The cost ratio of PA to TS methods tends to decrease with the growing complexity of the problem.

Let us now find values of the cost parameters for which a comparable accuracy in PA and TS can be achieved. We assume a truss with 2 nodes elements ($s=4$) and $p=n$ which, using Eq. (5.7), leads to

$$
\begin{aligned}
T(n)&=n^3+n^2+48n,\\
P(n)&=n^3+12n^2+1665n+3.
\end{aligned}
\tag{5.11}
$$

As shown in Sect. 4 TS needs four times more the incremental steps as compared to PA. Thus we conclude that

$$
P(n)\leqslant 4T(n)
\tag{5.12}
$$

for $n\geqslant 23$ which means that the PA algorithm becomes competitive for trusses having more than 23 d.o.f.

It should be noted that no simplifications (such as band-matrices, storing of non-zero entries only etc.) has been accounted for yet. The inclusion of such simplifications is unlikely to change the conclusions significantly, and if not so, the benefits for the PA scheme are likely to be greater than those influencing the TS approach.

6. Conclusions

The paper decribes the preliminary results obtained in the course of a extensive study on solution algorithms as applied to highly nonlinear structural problems. The advantages of using the perturbation scheme over the conventional tangential stiffness method are illustrated with regard to the reduced number of the incremental steps needed to achieve the similar accuracy. The final conclusions are drawn by referring to the simultaneous cost comparisons. Bearing in mind the advantages offered by the perturbation schemes in dealing with singular points along the equilibrium path, the authors feel the need for further detailed evaluation of this method.

Appendix

The explicit form of stiffness matrices are:

for the "first-order" matrix in global coordinates

$$^{(1)}k_{ij}=\frac{N}{l}\,{}^{(1)}W^{n}_{ij}+\frac{EA}{l}\,{}^{(1)}W^{e}_{ij},$$

$$^{(1)}\mathbf{W}^{n}=\begin{vmatrix} \mathbf{I}_2-\mathbf{c}\mathbf{c}^T & -(\mathbf{I}_2-\mathbf{c}\mathbf{c}^T) \\ -(\mathbf{I}_2-\mathbf{c}\mathbf{c}^T) & \mathbf{I}_2-\mathbf{c}\mathbf{c}^T \end{vmatrix},$$

where

$$\mathbf{c}=\begin{vmatrix} c_1 \\ c_2 \end{vmatrix}=\frac{1}{l}\begin{vmatrix} x_2-x_1 \\ y_2-y_1 \end{vmatrix},$$

$$^{(1)}\mathbf{W}^{e}=\begin{vmatrix} \mathbf{c}\mathbf{c}^T & -\mathbf{c}\mathbf{c}^T \\ -\mathbf{c}\mathbf{c}^T & \mathbf{c}\mathbf{c}^T \end{vmatrix};$$

for the "second-order" matrix (in local coordinates)

$$^{(2)}k_{ijk}=\frac{EA}{l^2}\,{}^{(2)}W_{ijk},$$

$$^{(2)}W_{111}=-1,\quad {}^{(2)}W_{112}=1,$$
$$^{(2)}W_{211}=1,\quad {}^{(2)}W_{212}=1,$$
$$^{(2)}W_{121}=1,\quad {}^{(2)}W_{122}=-1,$$
$$^{(2)}W_{221}=-1,\quad {}^{(2)}W_{222}=1;$$

for the "third-order" matrix (in local coordinates)

$$^{(3)}k_{ijkl}=\frac{EA}{4l^3}\,{}^{(3)}W_{ijkl},$$

$$^{(3)}W_{1111}=1,\quad {}^{(3)}W_{1112}=-1,$$
$$^{(3)}W_{2111}=1,\quad {}^{(3)}W_{2112}=1,$$
$$^{(3)}W_{1211}=1,\quad {}^{(3)}W_{1212}=1,$$
$$^{(3)}W_{2211}=-1,\quad {}^{(3)}W_{2212}=-1,$$
$$^{(3)}W_{1121}=-1,\quad {}^{(3)}W_{1122}=-1,$$
$$^{(3)}W_{2121}=1,\quad {}^{(3)}W_{2122}=1,$$
$$^{(3)}W_{1221}=1,\quad {}^{(3)}W_{1222}=1,$$
$$^{(3)}W_{2221}=-1,\quad {}^{(3)}W_{2222}=1.$$

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Streszczenie

O ZASTOSOWANIU METODY PERTURBACJI DO NIELINIOWEJ ANALIZY STATYCZNEJ SPRĘŻYSTYCH KRATOWNIC

W pracy przedstawiono podejście perturbacyjne do przyrostowej skończenie elementowej analizy konstrukcji w zakresie dużych przemieszczeń. Uwzględniono, zwykle pomijaną, nieliniowość geometryczną wewnątrz każdego kroku przyrostowego. Podano jawną postać macierzy sztywności pierwszego i wyższych rzędów. Przeprowadzono porównanie wyników otrzymanych metodą perturbacyjną, metodą zmiennej sztywności i z rozwiązania analitycznego.

Резюме

О ПРИМЕНЕНИИ ПЕРТУРБАЦИОННОГО ПОДХОДА К НЕЛИНЕЙНОМУ СТАТИЧЕСКОМУ АНАЛИЗУ ФЕРМ

В работе представлен пертурбационный подход к анализу в конечных элементах в приростах конструкций в области больших перемещений. В подходе учтена, обычно пренебрегаемая, геометрическая нелинейность внутри каждого шага в приростах. Приведен явный вид матриц жесткости первого и высших порядков. Проведено сравнение результатов полученных пертурбационным подходом методом переменной жесткости с аналитическим решением.

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