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FUNDAMENTAL SOLUTIONS TO THE COSSERAT TYPE MODEL OF HONEYCOMB GRID IN A PLANE-STRESS STATE

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In the paper fundamental solutions to the continuum Woźniak's-type model of elastic grid disk of honeycomb structure are found; the singularities of displacements, strains, stresses as well as couple stresses are displayed and examined. In a particular case when constitutive equations are decoupled, the solutions obtained reduce to the well-known components of the Green tensor for an isotropic and centrosymmetric Cosserat medium in a plane stress state.

1. Introduction

The subject of this paper are fundamental solutions to the theory of elastic plane grids of hexagonal (honeycomb) structure. A Woźniak's approach is applied: the structure's behaviour (in a plane state of stress) is described by means of equations of a two-dimensional Cosserat medium with a fibrous structure, cf. [1–4]. A characteristic feature of a honeycomb plate is its noncentrosymmetry; constitutive equations are coupled by a so-called **B** tensor. Thus, the governing equations, i.e. the equilibrium equations expressed in terms of displacements and rotations have nontrivial form, more complex than that well-known (see [5]) from the theory of a plane stress state of an isotropic and centrosymmetric Cosserat media. The equations considered in this paper have not been the subject of consideration in the hitherto existing literature; in particular the relevant fundamental solutions (describing a response of an infinite plate under point loads) have not been examined either.

2. Governing equations

Consider an infinite hexagonal grid (Fig. 1) constructed of elastic bars connected by rigid nodes. Internode distance is denoted by 1. A family of so-called main nodes is chosen as in Fig. 1; the choice is made under a fixed observation, i.e. at the fixed global coordinate system x^σ. The functions $u^\alpha(x^\sigma)$, $\varphi(x^\sigma)$ are assumed to stand for displacements and rotations of nodes, respectively. A set of equilibrium equations in terms of displacements takes the form [2]

$$(2.1)\qquad [(2\mu+\lambda)\,\partial_1^2+(\mu+\alpha)\,\partial_2^2]\,u^1+(\lambda+\mu-\alpha)\,\partial_1\,\partial_2\,u^2+[B\,(\partial_1^2-\partial_2^2)+2\alpha\,\partial_2]\,\varphi+ +p^1=0,$$

(2.1) cont.]
$$(\lambda+\mu-\alpha)\,\partial_1\,\partial_2\,u^1+[(2\mu+\lambda)\,\partial_2^2+(\mu+\alpha)\,\partial_1^2]\,u^2+[-2B\partial_1\,\partial_2-2\alpha\,\partial_1]\,\varphi+ +p^2=0,$$

$$[B\,(\partial_1^2-\partial_2^2)-2\alpha\,\partial_2]\,u^1+[-2B\,\partial_1\,\partial_2+2\alpha\,\partial_1]\,u^2+[C\,(\partial_1^2+\partial_2^2)-4\alpha]\,\varphi+ +Y^3=0,$$

where the densities of external forces and couples subjected to main nodes are denoted by p^α and Y^3, respectively. It is assumed here that the other nodes are free of loads.

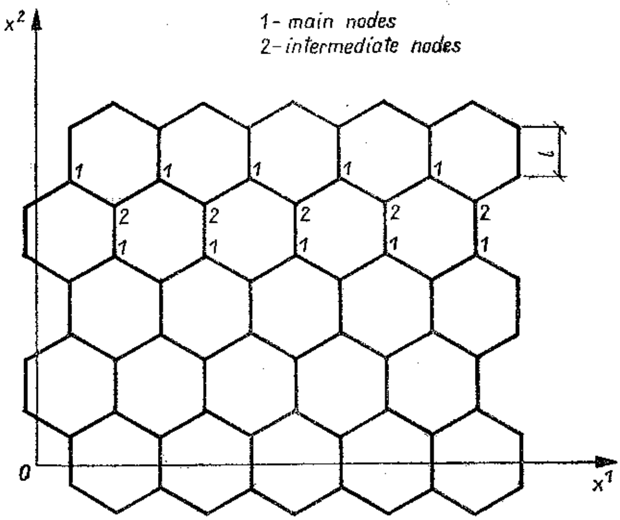

FIG. 1.

The moduli λ, μ, α, B and C characterize elastic properties of the structure. The constants λ, μ and α depend on the slenderness ratio of lattice rods only, whereas the moduli B and C are proportional to l and l^2, respectively. Therefore, if the slenderness of rods is constant and l tends to zero, both moduli B and C vanish. The effective moduli are assumed to satisfy the inequalities

(2.2)
$$\mu>0, \quad \alpha>0, \quad \mu+\lambda>0, \quad C>0, \quad B^2<C\mu$$

resulting from positive definiteness of a strain energy of a lattice, see [2].

By substituting $B=0$ into the system (2.1), one obtains known equations describing a plane stress problem of an isotropic and centrosymmetric Cosserat medium.

In order to simplify the further procedure, it is useful decouple the system (2.1). After tiresome rearrangements we arrive at equations of increased order:

(2.3)
$$\mathscr{L}u^\beta+\mathscr{S}_{\beta i}\,p^i=0, \quad \mathscr{L}\varphi+\mathscr{S}_{3i}\,p^i=0, \quad \beta=1,2, \quad i=1,2,3,$$

where $p^3\equiv Y^3$. The operators $\mathscr{S}_{IJ}$ read

(2.4)
$$\mathscr{S}_{11}\,(\partial_1,\partial_2)=\{-4\alpha\,[(2\mu+\lambda)\,\partial_2^2+\mu\partial_1^2]+C\,(2\mu+\lambda)\,\nabla^2\,(g\partial_1^2+\partial_2^2)- -4B^2\,\partial_1^2\,\partial_2^2\},$$

(2.4) [cont.]

$$\mathcal{S}_{22}(\partial_1, \partial_2)=\{-4\alpha[(2\mu+\lambda)\partial_1^2+\mu\partial_2^2]+C(2\mu+\lambda)\nabla^2(g\partial_2^2+\partial_1^2)-B^2(\partial_2^2-\partial_1^2)^2\},$$

$$\mathcal{S}_{33}(\partial_1, \partial_2)=(2\mu+\lambda)(\mu+\alpha)\nabla^4,$$

$$\mathcal{S}_{12}(\partial_1, \partial_2)=\{2\alpha BA+4\alpha(\lambda+\mu)\partial_1\partial_2-C(\lambda+\mu-\alpha)\nabla^2\partial_1\partial_2+2B^2\partial_1\partial_2(\partial_2^2-\partial_1^2)\},$$

$$\mathcal{S}_{21}(\partial_1, \partial_2)=\{-2\alpha BA+4\alpha(\lambda+\mu)\partial_1\partial_2-C(\lambda+\mu-\alpha)\nabla^2\partial_1\partial_2+2B^2\partial_1\partial_2(\partial_2^2-\partial_1^2)\},$$

$$\mathcal{S}_{13}(\partial_1, \partial_2)=-2\alpha(2\mu+\lambda)\partial_2\nabla^2+B[-(\mu+\alpha)\partial_1 A+(2\mu+\lambda)\partial_2 {}^*A],$$

$$\mathcal{S}_{31}(\partial_1, \partial_2)=2\alpha(2\mu+\lambda)\partial_2\nabla^2+B[-(\mu+\alpha)\partial_1 A+(2\mu+\lambda)\partial_2 {}^*A],$$

$$\mathcal{S}_{23}(\partial_1, \partial_2)=2\alpha(2\mu+\lambda)\partial_1\nabla^2-B[(2\mu+\lambda)\partial_1 {}^*A+(\mu+\alpha)\partial_2 A],$$

$$\mathcal{S}_{32}(\partial_1, \partial_2)=-2\alpha(2\mu+\lambda)\partial_1\nabla^2-B[(2\mu+\lambda)\partial_1 {}^*A+(\mu+\alpha)\partial_2 A],$$

where

$$A=\partial_1(\partial_1^2-3\partial_2^2),\quad {}^*A=\partial_2(\partial_2^2-3\partial_1^2),\quad \nabla^2=\partial_1^2+\partial_2^2,\quad g=\frac{\mu+\alpha}{2\mu+\lambda}.$$

The high order operators $\mathcal{S}_{ij}$ impose strong regularity restrictions on the functions p^α, Y^3 which stand for densities of external loads subjected to the nodes of the grid.

The canonical operator $\mathcal{L}$ defined as determinant of differential operators involved in the system (2.1) can be rearranged to the form

$$\mathcal{L}=(2\mu+\lambda)(\mu+\alpha)C\nabla^6-B^2[(\mu+\alpha)A^2+(2\mu+\lambda){}^*A^2]-4\alpha\mu(2\mu+\lambda)\nabla^4. \tag{2.5}$$

Because of the essential importance of the $\mathcal{L}$ operator, it is worth displaying below its alternative forms. Bearing in mind that

$$\nabla^6=A^2+{}^*A^2,$$

the definition (2.5) can be written in the three following equivalent forms:

(2.6)

$$\mathcal{L}=(\mu+\alpha)[C(2\mu+\lambda)-B^2]A^2+(2\mu+\lambda)[C(\mu+\alpha)-B^2]{}^*A^2-4\alpha\mu(2\mu+\lambda)\nabla^4,$$

$$\mathcal{L}=(\mu+\alpha)[C(2\mu+\lambda)-B^2][\nabla^6-\omega\,{}^*A^2-\varepsilon^2\nabla^4],$$

$$\mathcal{L}=(2\mu+\lambda)[C(\mu+\alpha)-B^2][\nabla^6+\omega' A^2-(\varepsilon')^2\nabla^4],$$

where

$$\omega=\frac{B^2(\lambda+\mu-\alpha)}{(\mu+\alpha)[C(2\mu+\lambda)-B^2]},\qquad \omega'=\frac{B^2(\lambda+\mu-\alpha)}{(2\mu+\lambda)[C(\mu+\alpha)-B^2]},$$

$$\varepsilon^2=\frac{4\alpha\mu(2\mu+\lambda)}{(\mu+\alpha)[C(2\mu+\lambda)-B^2]},\qquad (\varepsilon')^2=\frac{4\alpha\mu}{C(\mu+\alpha)-B^2}.$$

By virtue of the inequalities (2.2) the quantities ω, ω', ε and ε' exist(¹). Moreover the estimations

$$\omega<1, \qquad \omega'>1 \tag{2.7}$$

hold true.

The geometry of the structure of the grid is invariant under rotations at angles $2/3\,\pi n$, $n=\pm 1, \pm 2, \ldots$. The operator $\mathscr{L}$ possesses the same kind of invariance property. To prove this it is sufficient to show that the operators Λ and $^*\Lambda$ do not change their forms under rotations of the coordinate system x^α at angles $2/3\,\pi n$. This is easy to disclose when the complex coordinates $z=x^1+ix^2$, $z=x^1-ix^2$ are introduced, namely

$$\Lambda=4\,(\partial_z^3+\partial_{\bar z}^3), \qquad {}^*\Lambda=-4i\,(\partial_z^3-\partial_{\bar z}^3),$$

where

$$\partial_z=\frac{1}{2}\,(\partial_1-i\partial_2), \qquad \partial_{\bar z}=\frac{1}{2}\,(\partial_1+i\partial_2).$$

In the rotated coordinate system $x^{\alpha'}$ we have

$$z'=ze^{\frac{2}{3}\pi n i}, \qquad \bar z'=\bar z e^{-\frac{2}{3}\pi n i},$$

$$\partial_z^3=(\partial z'/\partial z)^3\,\partial_z^{3\,\prime}=e^{2\pi n i}\,\partial_z^{3\,\prime}=\partial_z^{3\,\prime}$$

$$\partial_{\bar z}^3=(\partial \bar z'/\partial \bar z)^3\,\partial_{\bar z}^{3\,\prime}=e^{-2\pi n i}\,\partial_{\bar z}^{3\,\prime}=\partial_{\bar z}^{3\,\prime}$$

and finally $\Lambda='\Lambda$, $^*\Lambda={}^*\Lambda'$. Thus the desired invariance property of $\mathscr{L}$ has been proven.

A new look at the $\mathscr{L}$ operator yields from its definition expressed in terms of complex differentials

$$\begin{aligned}
&\mathscr{L}=16\,[B^2\,(\lambda+\mu-\alpha)\,(\partial_z^6+2\rho\partial_z^3\,\partial_{\bar z}^3+\partial_{\bar z}^6)-4\alpha\mu\,(2\mu+\lambda)\,\partial_z^2\,\partial_{\bar z}^2],\\
&\rho=\{2C\,(2\mu+\lambda)\,(\mu+\alpha)-B^2\,(3\mu+\lambda+\alpha)\}/[B^2\,(\lambda+\mu-\alpha)], \qquad \alpha\neq\mu+\lambda
\end{aligned} \tag{2.8}$$

and its real counterpart

$$\mathscr{L}=\frac{1}{2}\,B^2\,(\lambda+\mu-\alpha)\,[(\rho+1)\,\partial_1^6+3\,(\rho-5)\,\partial_1^4\,\partial_2^2+3\,(\rho+5)\,\partial_1^2\,\partial_2^4+(\rho-1)\,\partial_2^6]-4\alpha\mu\,(2\mu+\lambda)\,\nabla^4.$$

The latter formula can also be obtained from Eq. (2.5) by a more complicated procedure.

Notice that the complex definition (2.8) is invariant under the interchange of complex variables: $\bar z\to z$, $z\to\bar z$. Thus the complex description seems to be more suitable to the considered problem than the approach involving real variables.

(¹) Notice that ε is a generalization of l^{-1} constant, which is widely used in the literature on micropolar isotropic media.

3. The fundamental solutions

Consider an infinite plane, hexagonal grid subjected to concentrated tangent forces $p^\alpha = F^\alpha \delta(x^1)\delta(x^2)$ and normal moment $Y^3 = M\delta(x^1)\delta(x^2)$ in point (0, 0). The displacements caused by the force F^β and moment M are denoted by $(u_\alpha^\beta, \varphi^\beta)$ and (u_α^3, φ^3), respectively. In order to obtain the fundamental solutions (u_α^K, φ^K), $K=1, 2, 3$, the standard method based on the integral Fourier transform will be applied.

By performing the transformation of Eqs. (2.3), then solving the obtained algebraic equations and carrying out the inverse transformation, the sought set of functions

$$\begin{Bmatrix} u_\alpha^K \\ \varphi^K \end{Bmatrix}(x^1, x^2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\begin{Bmatrix} \tilde{u}_\alpha^K(\alpha_1, \alpha_2) \\ \tilde{\varphi}^K(\alpha_1, \alpha_2) \end{Bmatrix} \bar{e}^{i(\alpha_1 x^1 + \alpha_2 x^2)}\, d\alpha_1\, d\alpha_2, \qquad K=1, 2, 3, \tag{3.1}$$

are finally found. The Fourier transforms of displacements due to the tangent forces F^α and normal moment M

$$\tilde{u}_\alpha^K = \frac{\mathscr{S}_{\alpha K}(\alpha_1, \alpha_2)}{W(\alpha_1, \alpha_2)}\tilde{p}^K, \qquad \tilde{\varphi}^K = \frac{\mathscr{S}_{3K}(\alpha_1, \alpha_2)}{W(\alpha_1, \alpha_2)}\tilde{p}^K, \qquad \alpha, K=1, 2, 3,$$

where $\mathscr{S}_{\alpha\beta}(\alpha_1, \alpha_2) = \mathscr{S}_{\alpha\beta}(\partial_1 \to -i\alpha_1, \partial_2 \to -i\alpha_2)$ and $W(\alpha_1, \alpha_2) = -\mathscr{L}(\partial_1 \to -i\alpha_1, \partial_2 \to -i\alpha_2)$, play a role of integrands in Eq. (3.1). The following form of the function W will be used (cf. $(2.6)_2$):

$$W(\alpha_1, \alpha_2) = (\mu+\alpha)\,[C(2\mu+\lambda) - B^2]\,\{(\alpha_1^2+\alpha_2^2)^3 - \omega(\alpha_2^3 - 3\alpha_1^2\alpha_2)^2 + \varepsilon^2(\alpha_1^2+\alpha_2^2)^2\}.$$

Let us define the nondimensional variables $\bar{\alpha}, \bar{\beta}$ by means of the formulae $\bar{\alpha} = \alpha_1/\varepsilon$, $\bar{\beta} = \alpha_2/\varepsilon$ and then introduce the polar coordinates $\bar{\alpha} = \gamma\cos\psi$, $\bar{\beta} = \gamma\sin\psi$. Now the function $W(\gamma, \psi)$ can be displayed in the form

$$W(\gamma, \psi) = (\mu+\alpha)\,[C(2\mu+\lambda) - B^2]\,\varepsilon^6\gamma^4\,[1+\gamma^2\varkappa(\omega, \psi)],$$

$$\varkappa(\omega, \psi) = 1 - \omega\sin^2 3\psi.$$

By virtue of the inequality (2.7) the function $\varkappa(\omega, \psi)$ is positive for any ω and $\psi \in (0, 2\pi)$. After introducing the polar coordinates new expressions of $\mathscr{S}_{\alpha\beta}$ $(\alpha_1 \to \varepsilon\gamma\cos\psi, \alpha_2 \to \varepsilon\gamma\sin\psi)$ determine the transforms $\tilde{u}_\alpha^K, \tilde{\varphi}^K$ dependent on γ and ψ. Equations (3.1) take the form

$$\begin{Bmatrix} u_\alpha^K \\ \varphi^K \end{Bmatrix}(r, \vartheta) = \frac{\varepsilon^2}{2\pi}\int_0^{2\pi}\int_0^{\infty}\begin{Bmatrix} \tilde{u}_\alpha^K \\ \tilde{\varphi}^K \end{Bmatrix}(\gamma, \psi)\, \bar{e}^{i\gamma\bar{r}\cos(\psi-\vartheta)}\,\gamma\, d\gamma\, d\psi, \tag{3.2}$$

where $x^1 = r\cos\vartheta$, $x^2 = r\sin\vartheta$, $\bar{r} = r\varepsilon$.

If one inserts the transforms u_α^K and $\tilde{\varphi}^K$ into Eq. (3.2), the fundamental solutions associated with the force $p^1=\delta(x^1)\,\delta(x^2)$

$$u_1^1(r,\vartheta)=k\,[hC_0(r,\vartheta)-C_2(r,\vartheta)]+\frac{C}{2gA}\,[-(1+g)\,J_0(r,\vartheta)+(1-g)\,J_2(r,\vartheta)]-$$
$$-\frac{B^2}{2(\mu+\alpha)\,A}\,[J_4(r,\vartheta)-J_0(r,\vartheta)],$$

$$u_2^1(r,\vartheta)=-kS_2(r,\vartheta)+\frac{C(1-g)}{2gA}\,K_2(r,\vartheta)+\frac{B^2}{2(\mu+\alpha)\,A}\,K_4(r,\vartheta)+$$
$$+\frac{B\varepsilon}{8\pi^2\,\mu\,(2\mu+\lambda)}\,J_3(r,\vartheta),$$

$$\varphi^1(r,\vartheta)=\frac{-\varepsilon}{8\pi^2\,\mu}\,K_1(r,\vartheta)+\frac{\delta\varepsilon^2}{32\pi^2\,\mu}\,[(g-1)\,J_4(r,\vartheta)+(g+1)\,J_2(r,\vartheta)],$$

the force $p^2=\delta(x^1)\,\delta(x^2)$

$$u_1^2(r,\vartheta)=-kS_2(r,\vartheta)+\frac{C(1-g)}{2gA}\,K_2(r,\vartheta)+\frac{B^2}{2(\mu+\alpha)\,A}\,K_4(r,\vartheta)+$$
$$+\frac{-B\varepsilon}{8\pi^2\,\mu\,(2\mu+\lambda)}\,J_3(r,\vartheta),$$

$$u_2^2(r,\vartheta)=k\,[hC_0(r,\vartheta)+C_2(r,\vartheta)]-\frac{C}{2gA}\,[(1+g)\,J_0(r,\vartheta)+(1-g)\,J_2(r,\vartheta)]+$$
$$+\frac{B^2}{2(\mu+\alpha)\,A}\,[J_0(r,\vartheta)+J_4(r,\vartheta)],$$

$$\varphi^2(r,\vartheta)=\frac{\varepsilon}{8\pi^2\,\mu}\,J_1(r,\vartheta)+\frac{\delta\varepsilon^2}{32\pi^2\,\mu}\,[(g-1)\,K_4(r,\vartheta)-(1+g)\,K_2(r,\vartheta)]$$

and the moment $Y^3=\delta(x^1)\,\delta(x^2)$

$$u_1^3(r,\vartheta)=\frac{\varepsilon}{8\pi^2\,\mu}\,K_1(r,\vartheta)+\frac{\delta\varepsilon^2}{32\pi^2\,\mu}\,[(g-1)\,J_4(r,\vartheta)+(g+1)\,J_2(r,\vartheta)],$$

$$u_2^3(r,\vartheta)=\frac{-\varepsilon}{8\pi^2\,\mu}\,J_1(r,\vartheta)+\frac{\delta\varepsilon^2}{32\pi^2\,\mu}\,[(g-1)\,K_4(r,\vartheta)-(g+1)\,K_2(r,\vartheta)].$$

$$\varphi^3(r,\vartheta)=\frac{-(2\mu+\lambda)}{A}\,J_0(r,\vartheta)$$

are finally found, where

$$k=\frac{\mu+\lambda}{16\pi^2\,\mu\,(2\mu+\lambda)},\qquad g=\frac{\mu+\alpha}{2\mu+\lambda},\qquad h=\frac{3\mu+\lambda}{\mu+\lambda},\qquad \delta=B/\alpha,$$
$$A=4\pi^2\,[C(2\mu+\lambda)-B^2]$$

and

$$\begin{Bmatrix} J_{2n} \\ K_{2n} \end{Bmatrix}(r,\vartheta) = -\int_0^\infty \int_0^{2\pi} \begin{Bmatrix} \cos 2n\psi \\ \sin 2n\psi \end{Bmatrix} \frac{e^{i\gamma \bar{r}\cos(\psi-\vartheta)}}{1+\varkappa(\omega,\psi)\gamma^2}\, \gamma\, d\gamma\, d\psi,$$

$$(3.3)\qquad \begin{Bmatrix} J_{2n+1} \\ K_{2n+1} \end{Bmatrix}(r,\vartheta) = -\int_0^\infty \int_0^{2\pi} \begin{Bmatrix} \cos(2n+1)\psi \\ \sin(2n+1)\psi \end{Bmatrix} \frac{e^{i\gamma \bar{r}\cos(\psi-\vartheta)}}{1+\varkappa(\omega,\psi)\gamma^2}\, d\gamma\, d\psi,$$

$$\begin{Bmatrix} C_{2n} \\ S_{2n} \end{Bmatrix}(r,\vartheta) = \int_0^\infty \int_0^{2\pi} \begin{Bmatrix} \cos 2n\psi \\ \sin 2n\psi \end{Bmatrix} \frac{e^{i\gamma \bar{r}\cos(\psi-\vartheta)}}{\gamma\,[1+\varkappa(\omega,\psi)\gamma^2]}\, d\gamma\, d\psi.$$

The integrals C_{2n} and S_{2n} ougth to be understood in a generalized sense. Simple rearrangements lead to the formulae

$$\begin{Bmatrix} C_{2n} \\ S_{2n} \end{Bmatrix} = \begin{Bmatrix} C^1_{2n} \\ S^1_{2n} \end{Bmatrix} + \left(1-\frac{\omega}{2}\right)\begin{Bmatrix} J_{2n} \\ K_{2n} \end{Bmatrix} + \frac{\omega}{4}\begin{Bmatrix} J_{6-2n} \\ -K_{6-2n} \end{Bmatrix} + \frac{\omega}{4}\begin{Bmatrix} J_{6+2n} \\ K_{6+2n} \end{Bmatrix},$$

where the integrals

$$\begin{Bmatrix} C^1_{2n} \\ S^1_{2n} \end{Bmatrix} = \mathrm{Re}\int_0^\infty \int_0^{2\pi} \frac{e^{i\gamma \bar{r}\cos(\psi-\vartheta)}}{\gamma} \begin{Bmatrix} \cos 2n\psi \\ \sin 2n\psi \end{Bmatrix} d\gamma\, d\psi$$

are divergent (in a classical meaning) parts of C_{2n} and S_{2n}. By interchanging an order of integration one obtains

$$\begin{Bmatrix} C^1_{2n} \\ S^1_{2n} \end{Bmatrix} = \int_0^{2\pi} f(\gamma, r, \psi, \vartheta) \begin{Bmatrix} \cos 2n\psi \\ \sin 2n\psi \end{Bmatrix} d\psi,$$

where

$$f(\gamma, r, \psi, \vartheta) = \mathrm{Re}\int_0^\infty \frac{e^{i\gamma(-\bar{r}\cos(\psi-\vartheta))}}{\gamma}\, d\gamma = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{i\gamma(-\bar{r}\cos(\psi-\vartheta))}}{|\gamma|}\, d\gamma =$$
$$= -\ln|\bar{r}\cos(\psi-\vartheta)| + \mathrm{const}.$$

The last equality results as a particular case of the known formula of the theory of Fourier transform, cf. [6]. Tables, Eq. (27), p. 449. Hence

$$\begin{Bmatrix} C^1_{2n} \\ S^1_{2n} \end{Bmatrix} = \begin{Bmatrix} -2\pi\delta_{2n,0}\ln\bar{r} \\ 0 \end{Bmatrix} - 2\pi\, b_{-n} \begin{Bmatrix} \cos 2n\vartheta \\ \sin 2n\vartheta \end{Bmatrix},$$

where δ_{ij} denotes Kronecker delta and b_n — coefficients in the expansion (cf. [7], p. 383)

$$\ln|\cos(\psi-\vartheta)| = \sum_{m=-\infty}^{+\infty} b_m e^{2im(\psi-\vartheta)}, \qquad b_m = \begin{cases} -\ln 2, & m=0 \\ 0.5\,(-1)^{m+1}/|m|, & m\neq 0. \end{cases}$$

The above considerations allow us to express the fundamental solutions u_α^K, φ^K in terms of the functions J_m and K_m.

$$
\begin{aligned}
&u_1^1(r,\vartheta)=\zeta\ln r+\pi_0\cos 2\vartheta+\sum_{m=0}^{4}{}^{(2m)}\xi_1^1\,J_{2m}(r,\vartheta)+\text{const},\\
&u_2^1(r,\vartheta)=\pi_0\sin 2\vartheta+{}^{(3)}\xi_2^1\,J_3(r,\vartheta)+\sum_{m=1}^{4}{}^{(2m)}\eta_2^1\,K_{2m}(r,\vartheta)+\text{const},\\
&\varphi^1(r,\vartheta)={}^{(1)}\eta_3^1\,K_1(r,\vartheta)+\sum_{m=1}^{2}{}^{(2m)}\xi_3^1\,J_{2m}(r,\vartheta),\\
&u_1^2(r,\vartheta)=\pi_0\sin 2\vartheta+{}^{(3)}\xi_1^2\,J_3(r,\vartheta)+\sum_{m=1}^{4}{}^{(2m)}\eta_1^2\,K_{2m}(r,\vartheta)+\text{const},\\
&u_2^2(r,\vartheta)=\zeta\ln r-\pi_0\cos 2\vartheta+\sum_{m=0}^{4}{}^{(2m)}\xi_2^2\,J_{2m}(r,\vartheta)+\text{const},\\
&\varphi^2(r,\vartheta)={}^{(1)}\xi_3^2\,J_1(r,\vartheta)+\sum_{m=1}^{2}{}^{(2m)}\eta_3^2\,K_{2m}(r,\vartheta),\\
&u_1^3(r,\vartheta)={}^{(1)}\eta_1^3\,K_1(r,\vartheta)+\sum_{m=1}^{2}{}^{(2m)}\xi_1^3\,J_{2m}(r,\vartheta),\\
&u_2^3(r,\vartheta)={}^{(1)}\xi_2^3\,J_1(r,\vartheta)+\sum_{m=1}^{2}{}^{(2m)}\eta_2^3\,K_{2m}(r,\vartheta),\\
&\varphi^3(r,\vartheta)={}^{(0)}\xi_3^3\,J_0(r,\vartheta),
\end{aligned}
\tag{3.4}
$$

where the coefficients involved in the relations (3.4) are defined as follows:

$$
\zeta=\frac{-(3\mu+\lambda)}{4\pi\mu(2\mu+\lambda)},\qquad \pi_0=\frac{\mu+\lambda}{8\pi\mu(2\mu+\lambda)},\qquad {}^{(0)}\xi_3^3=\frac{-(2\mu+\lambda)}{4\pi^2[C(2\mu+\lambda)-B^2]},
$$

$$
{}^{(0)}\xi_1^1={}^{(0)}\xi_2^2=\{2\alpha C(2\mu+\lambda)^2-B^2[(\mu+\lambda)(2\mu+\lambda)+(\mu+\alpha)(3\mu+\lambda)]\}/N,
$$

$$
{}^{(2)}\xi_1^1=-{}^{(2)}\xi_1^2={}^{(2)}\eta_1^2=\{B^2(\mu+\lambda)(3\mu+\lambda+\alpha)-2C\alpha(2\mu+\lambda)^2\}/N,
$$

$$
{}^{(4)}\xi_1^1=-{}^{(4)}\xi_2^2=-{}^{(4)}\eta_1^2=-\frac{1}{2}B^2[3\mu-\alpha)(\mu+\lambda)+\lambda^2]/N,
$$

$$
{}^{(6)}\xi_1^1={}^{(6)}\xi_2^2=\frac{(3\mu+\lambda)\omega}{16\pi^2\mu(2\mu+\lambda)},\qquad N=16\pi^2\mu(2\mu+\lambda)(\mu+\alpha)[C(2\mu+\lambda)-B^2],
$$

$$
{}^{(8)}\xi_1^1=-{}^{(8)}\xi_2^2=-{}^{(8)}\eta_1^2=\frac{-(\mu+\lambda)\omega}{32\pi^2\mu(2\mu+\lambda)},
$$

$$
{}^{(3)}\xi_1^2=\frac{-B\varepsilon}{8\pi^2\mu(2\mu+\lambda)},\qquad {}^{(2)}\xi_1^3=-{}^{(2)}\eta_2^3=\frac{\delta\varepsilon^2(1+g)}{32\pi^2\mu},
$$

$$
{}^{(4)}\xi_1^3={}^{(4)}\eta_2^3=\frac{\delta\varepsilon^2(g-1)}{32\pi^2\mu},\qquad {}^{(1)}\xi_2^3=-{}^{(1)}\eta_1^3=\frac{-\varepsilon}{8\pi^2\mu}.
$$

The other coefficients can be calculated by means of the formulae

$$^{(m)}\eta^i_i=0, \quad ^{(m)}\xi^i_j=(-1)^{m}\,{}^{(m)}\xi^j_i, \quad ^{(m)}\eta^i_j=(-1)^{m}\,{}^{(m)}\eta^j_i.$$

In order to make the paper clear, the procedure of expanding J_m and K_m functions into power series with respect to r and trigonometrical series with respect to the angular variable ϑ is presented in the Appendix.

It can be shown that in the case of $B=0$ the functions $u^K_\alpha\,(B=0)$, $\varphi^K\,(B=0)$ take the form of the well-known fundamental solutions of the theory of a plane stress state of isotropic and centrosymmetric micropolar medium (see e.g. [5]. Section 3.11). Moreover the functions $u^\beta_\alpha\,(B=0)$ tend to fundamental solutions of the couple-stress theory, provided $\alpha\to\infty$. Such limiting cases are of theoretical interest only because the B modulus cannot tend to zero, while C remains constant. As it was pointed out in the 2-nd section, the modulus C vanishes faster than B if the internode distance l diminishes. Nevertheless, this paper is not devoted to hexagonal grids only, but its aim is rather to examine a mathematical model which can be useful for another media with microstructure, like perforated disks of triangular layout, for instance.

4. Singularities of fundamental solutions and relevant to them: strain, stress and couple stress components

From both physical and mathematical points of view it is worth considering singularities of displacements, strains and stresses due to concentrated forces and couples. The main aim of the analysis is to disclose the influence of the noncentrosymmetry of the lattice on the singularity intensity factors.

Let us focus our attention on the behaviour of the components u^K_α, φ^K of the Green tensor in the vicinity of point $r=0$; the first terms of the expansions (3.4) read

$$\begin{bmatrix} u^1_1 & u^2_1 & u^3_1 \\ u^1_2 & u^2_2 & u^3_2 \\ \varphi^1 & \varphi^2 & \varphi^3 \end{bmatrix} \sim \begin{bmatrix} 'A^1_1 \ln r & 'A^2_1 r^2 \ln r \sin 2\vartheta & 'A^3_1 r \ln r \sin\vartheta \\ 'A^1_2 r^2 \ln r \sin 2\vartheta & 'A^2_2 \ln r & 'A^3_2 r \ln r \cos\vartheta \\ 'A^1_3 r \ln r \sin\vartheta & 'A^2_3 r \ln r \cos\vartheta & 'A^3_3 \ln r \end{bmatrix},$$

where

$$'A^1_1='A^2_2=\zeta+\frac{2\pi}{(1-\omega)^{1/2}}\left\{^{(0)}\xi^1_1-\left[\left(\frac{2}{\omega}-1\right)-\frac{2}{\omega}(1-\omega)^{1/2}\right]{}^{(6)}\xi^1_1\right\},$$

$$'A^1_2='A^2_1=\frac{\pi\varepsilon^2\,\omega}{8\,(1-\omega)^{3/2}}\left[\frac{2-\omega}{\omega}\,{}^{(2)}\eta^2_1+{}^{(4)}\eta^2_1-{}^{(8)}\eta^2_1\right],$$

$$'A^1_3=-'A^3_1=+'A^3_2=-'A^2_3=\frac{-\pi\varepsilon}{(1-\omega)^{1/2}}\,{}^{(1)}\eta^3_1, \qquad 'A^3_3=\frac{2\pi}{(1-\omega)^{1/2}}\,{}^{(0)}\xi^3_3.$$

If B tends to zero the quantities $'A^\alpha_\beta$ converge to the following values:

$$\lim_{B\to 0}\,'A^1_1=\lim_{B\to 0}\,'A^2_2=\frac{1}{4\pi}\left(\frac{1}{2\mu+\lambda}+\frac{1}{\mu+\alpha}\right),$$

$$\lim_{B\to 0} {'A}_2^1 = \lim_{B\to 0} {'A}_1^2 = -\frac{\alpha^2}{8\pi C(\mu+\alpha)^2}, \qquad \lim_{B\to 0} {'A}_3^3 = -1/2\pi C,$$

$$\lim_{B\to 0} {'A}_3^1 = -\lim_{B\to 0} {'A}_1^3 = -\lim_{B\to 0} {'A}_3^2 = \lim_{B\to 0} {'A}_2^3 = -\frac{\alpha}{2\pi C(\mu+\alpha)}$$

identical with the results yielded from the theory of a micropolar isotropic medium (cf. [5]).

Consider the singularities of strains and stresses due to the concentrated force $p^1 = \delta(x^1)\,\delta(x^2)$. Appropriate calculations give

$$\gamma_{11} \approx {'A}_1^1 \frac{\cos\vartheta}{r} + \frac{\sin\vartheta\, k(\vartheta)}{r}, \qquad \gamma_{12} \approx \frac{-\sin\vartheta\, l(\vartheta)}{r},$$

$$\gamma_{21} \approx {'A}_1^1 \frac{\sin\vartheta}{r} - \frac{\cos\vartheta\, k(\vartheta)}{r}, \qquad \gamma_{22} \approx \frac{\cos\vartheta\, l(\vartheta)}{r},$$

$$\varkappa_1 \approx \frac{\sin\vartheta\, s(\vartheta)}{r}, \qquad \varkappa_2 \approx \frac{-\cos\vartheta\, s(\vartheta)}{r} + {'A}_3^1 \ln r,$$

where

$$k(\vartheta) = 2\sum_{m=1}^{\infty} b_{1,m}^1\, m \sin(2m\vartheta), \qquad b_{1,m}^1 = \delta_{1,m}\,\pi_0 + \sum_{p=0}^{4} {}^{(2p)}\xi_1^1 \left(\check{J}_{0,2m}^{2p} + \check{J}_{0,-2m}^{2p}\right),$$

$$l(\vartheta) = 2\sum_{m=1}^{\infty} b_{2,m}^1\, m \cos 2m\vartheta, \qquad b_{2,m}^1 = \pi_0\,\delta_{1,m} + \sum_{p=1}^{4} {}^{(2p)}\eta_2^1\, \check{K}_{0,2m}^{2p} - \check{K}_{0,-2m}^{2p}),$$

$$s(\vartheta) = 2\sum_{m=1}^{\infty} b_{3,m}^1\, m \sin 2m\vartheta, \qquad b_{3,m}^1 = \sum_{p=1}^{2} {}^{(2p)}\xi_3^1 \left(\check{J}_{0,2m}^{2p} + \check{J}_{0,-2m}^{2p}\right).$$

The introduced functions k, l, s, are dimensional: $[k] = m/N$, $[l] = m/N$, $[s] = 1/N$. Finally, by using constitutive equations [1, 2] we arrive at

$$\sigma_{11} \sim \{(2\mu+\lambda)\, {'A}_1^1 \cos\vartheta + (2\mu+\lambda)\sin\vartheta\, k(\vartheta) + \lambda\cos\vartheta\, l(\vartheta) + B\sin\vartheta\, s(\vartheta)\}\frac{1}{r},$$

$$\sigma_{12} \sim \{-(\mu+\alpha)\sin\vartheta\, l(\vartheta) + (\mu-\alpha)\, {'A}_1^1 \sin\vartheta - (\mu-\alpha)\cos\vartheta\, k(\vartheta) + B\cos\vartheta\, s(\vartheta)\}\frac{1}{r} - B\, {'A}_3^1 \ln r,$$

$$\sigma_{21} \approx \{-(\mu+\alpha)\, k(\vartheta)\cos\vartheta + (\mu+\alpha)\, {'A}_1^1 \sin\vartheta - (\mu-\alpha)\sin\vartheta\, l(\vartheta) + B\cos\vartheta\, s(\vartheta)\}\frac{1}{r} - B\, {'A}_3^1 \ln r,$$

$$\sigma_{22} \approx \{(2\mu+\lambda)\, l(\vartheta)\cos\vartheta + \lambda\, {'A}_1^1 \cos\vartheta + \lambda\, k(\vartheta)\sin\vartheta - Bs(\vartheta)\sin\vartheta\}\frac{1}{r},$$

$$m_1 \approx \{C\, s(\vartheta)\sin\vartheta + B\,[{'A}_1^1 \cos\vartheta + k(\vartheta)\sin\vartheta - \cos\vartheta\, l(\vartheta)]\}\frac{1}{r},$$

$$m_2 \approx \{-C\, s(\vartheta)\cos\vartheta + B\,[-{'A}_1^1 \sin\vartheta + l(\vartheta)\sin\vartheta + k(\vartheta)\cos\vartheta]\}\frac{1}{r} + C\, {'A}_3^1 \ln r.$$

The coefficients standing at the right-hand sides of the above formulae are called singularity intensity factors. The coefficients underlined are dimensional, whereas the other are nondimensional. The underlined factors $B\,'A_3^1$ relevant to tangent stresses depend explicitly on ε^{-1} and B/μ constants which have dimensions of length. The factor $B\,'A_3^1$ itself has the dimension of m^{-1}. The underlined factors associated with couple stresses have the dimensions of m.

Thus it can be seen that the noncentrosymmetry of the structure does not increase an order of stress singularities, but it changes their character. In particular, logarithmic terms appear. However, a coupling of constitutive equations (resulting from the noncentrosymmetry of the lattice) increases an order of couple stress singularities since new terms of the order $O(r^{-1})$ occur. Nevertheless, these terms are multiplied by the coefficients of the dimension of length; thus they vanish, provided the parameter l tends to zero.

Similar formulae and conclusions result from an analysis of the grid subjected to the concentrated force in the x^2 direction. Thus this case will not be considered here and now an influence of the concentrated couple $Y^3=\delta(x^1)\,\delta(x^2)$ on the strain and stress distributions will be examined. It is reasonable to analyse strains and stresses referred to the polar coordinate system (r, ϑ). Appropriate computations lead to the following formulae which express deformation patterns in the vicinity of point $r=0$:

$$\lim_{r\to 0}\gamma_{rr} \text{ is finite}, \qquad \gamma_{\vartheta r}\approx('A_1^3+'A_3^3)\ln r+\frac{w(\vartheta)}{r}, \qquad \gamma_{r\vartheta}\approx-('A_1^3+'A_3^3)\ln r,$$

(4.1)
$$\gamma_{\vartheta\vartheta}\approx\frac{1}{r}\,t(\vartheta), \qquad \varkappa_r\approx 'A_3^3\,\frac{1}{r}, \qquad \varkappa_\vartheta\approx p(\vartheta)\,\frac{1}{r},$$

where

$$w(\vartheta)=-\sum_{j=-\infty}^{+\infty}[(6j-3)\,u_{r,j}+u_{\vartheta,j}]\sin(6_j-3)\,\vartheta,$$

$$t(\vartheta)=\sum_{j=-\infty}^{+\infty}[(6j-3)\,u_{\vartheta,j}+u_{r,j}]\cos(6j-3)\,\vartheta,$$

$$u_{r,j}={}^{(2)}\xi_1^3\,\check{J}^2_{0,6j-2}+{}^{(4)}\xi_1^3\,\check{J}^4_{0,6j-4},$$

$$u_{\vartheta,j}={}^{(2)}\xi_1^3\,\check{J}^2_{0,6j-2}-{}^{(4)}\xi_1^3\,\check{J}^4_{0,6j-4},$$

$$p(\vartheta)=6\sum_{m=1}^{\infty}b^3_{3,m}\,m\sin 6m\,\vartheta, \qquad b^3_{3,m}={}^{(0)}\xi_3^3\,(\check{J}^0_{6,m}+\check{J}^0_{0,-6m}).$$

By substituting the expressions (4.1) into constitutive equations, singular terms of stresses

$$\sigma_{rr}\approx\frac{1}{r}\,[\lambda\,t(\vartheta)+B\,'A_3^3\cos 3\vartheta-B\,p(\vartheta)\sin 3\vartheta],$$

(4.2)

$$\sigma_{r\vartheta}\approx\frac{1}{r}\,[(\mu-\alpha)\,w(\vartheta)-B\,'A_3^3\sin 3\vartheta-B\,p(\vartheta)\cos 3\vartheta]-\underline{2\alpha\,('A_1^3+'A_3^3)\ln r},$$

(4.2) [cont.]
$$\sigma_{\vartheta r} \approx \frac{1}{r}\,[(\mu+\alpha)\,w(\vartheta) - B\,'A_3^3 \sin 3\vartheta - B\,p(\vartheta)\cos 3\vartheta] + \underline{2\alpha\,('A_1^3 + 'A_3^3)\ln r},$$

$$\sigma_{\vartheta\vartheta} \approx \frac{1}{r}\,[(2\mu+\lambda)\,t(\vartheta) - B\,'A_3^3 \cos 3\vartheta + B\,p(\vartheta)\sin 3\vartheta],$$

and couple stresses

$$m_r \approx \frac{1}{r}\,[\underline{C\,'A_3^3} - B\,t(\vartheta)\cos 3\vartheta - B\,w(\vartheta)\sin 3\vartheta],$$

$$m_\vartheta \approx \frac{1}{r}\,[C\,p(\vartheta) + B\,t(\vartheta)\sin 3\vartheta - B\,w(\vartheta)\cos 3\vartheta]$$

are found. The singularity intensity factors of couple stresses are nondimensional, the other coefficients which determine the singularities of stresses $\sigma_{\alpha\beta}$ are of dimensions m^{-1} and m^{-2}. The latter factors $\pm 2\alpha\,('A_1^3 + 'A_3^3)$ depend explicity on the parameter ε

$$2\alpha\,('A_1^3 + 'A_3^3) = \frac{-\varepsilon^2}{4\pi\,(1-\omega)^{1/2}}, \qquad [\varepsilon] = m^{-1},$$

the others depend on ε^{-1} and (B/μ) constants, which have dimensions of length.

If one inserts $B=0$ into the RHS of Eqs. (4.2) and (4.3), all the not underlined terms disappear. Thus the existence of $B \neq 0$ introduces new singularities of order $O(r^{-1})$ to stress components and to circumferential stress couple.

In the case of $B=0$ the stresses are axially symmetrical, whereas the terms resulting from noncentrosymmetry of the structure produce stress patterns which are of triple symmetry, what immediately results from the definitions of functions $w(\vartheta)$, $t(\vartheta)$ and $p(\vartheta)$.

Concluding remarks

The results of this paper can be treated as a generalization of the fundamental solutions to the theory of isotropic, centrosymmetric micropolar medium- in the case of a specific medium (in a plane-stress state) with a noncentrosymmetrical honeycomb microstructure. One aim of the work is to examine the influence of the lack of centrosymmetry of the medium on singularities of displacements, strains and stresses relevant to tangent concentrated forces and a normal concentrated couple.

The triple symmetry, which is revealed by coupling of constitutive equations, does not increase the order of displacement singularities; however, it essentially increases the orders of stress and strain singularities. In particular, a tangent force F^1 produces a singularity of the couple stress m_1 of order $O(r^{-1})$, while in the isotropic medium this couple stress is finite in a vicinity of the point in which the force is applied. Similarly a concentrated couple creates a singular state of stress $\sigma_{\alpha\beta}$ and singular circumferential couple stress m_ϑ of order $O(r^{-1})$, whereas in the isotropic case the same point load induces weaker logarithmic singularities to shear stresses $\sigma_{r\vartheta}$, $\sigma_{\vartheta r}$ and a singularity of order $O(r^{-1})$ to the couple stress m_r.

The effects of singularities which result from the triple symmetry of the medium are to some extent weakened by the fact that their intensity factors are dimensional ones; they depend on effective moduli of dimensions of length and they vanish when the internode distance l tends to zero.

The results obtained are characterized by a triple symmetry (which comes from a similar symmetry of the "microstructure"), hence they are far more complicated than the axially symmetric results following from the theory of an isotropic axi-symmetric medium. Thus one can reduce to closed forms neither the fundamental solutions nor their singular parts.

A Cosserat approach applied in the paper is the simplest well-established mathematical model among other models of a better accuracy than the zero-order asymptotic theory, cf. [3]. It is thought appropriate to formulate a high order approach which would not introduce singularities to Green tensor components because the fundamental solutions for the initial discrete model take finite values only.

Appendix

1. *The integrals* $S^p_{m,n}$

$$S^p_{m,n}=\frac{1}{2\pi i}\int_\gamma \frac{z^p\,dz}{(z-z_1)^m(z-z_2)^n},$$

where $z_1=-e^{-v}$, $z_2=-e^{v}$, $v>0$, $p, m, n\in N$, γ denote a contour which traverses around a point z_1 in the clockwise direction and does not surround a point z_2, cf. Fig. 2.

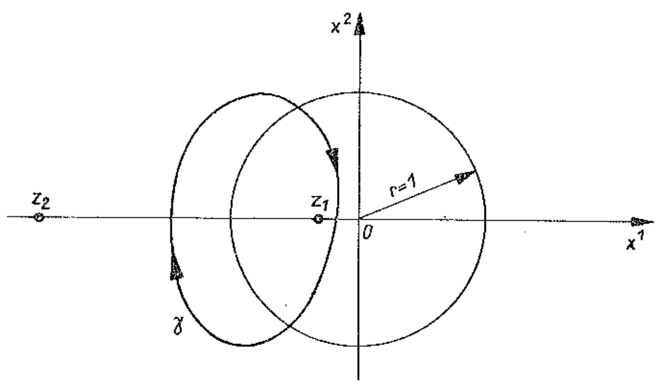

FIG. 2.

With the aid of the residue theory (for details see [9]) we find

$$S^p_{m,n}=\frac{1}{(m-1)!}\sum_{l=0}^{m-1}\binom{m-1}{l}(p,l)(-n,m-1-l)(-e^{-v})^{p-l}(2Shv)^{-n-m+1+l},$$

where

$$(a,b)=\begin{cases}\prod_{s=0}^{b-1}(a-s) & \text{for } b\geq 1\\ 1 & \text{for } b=0\end{cases}, \quad a\in R, \quad b\in N\cup\{0\}.$$

2. *The integrals* K_n^{2m}

$$K_n^{2m}=\int_0^{2\pi}\frac{e^{2mi\psi}\,d\psi}{(1-\omega\sin^2 3\psi)^n}, \quad n\in N, \quad m\in N\cup\{0\}, \quad \omega<1.$$

It is easily seen that

a) $\operatorname{Im} K_n^{2m}=0$, b) $K_n^{2m}=K_n^{-2m}$, c) $K_n^{2m}=0$ for $m/3\notin N\cup\{0\}$

and therefore it is assumed further that $m=3p$, $p\in N\cup\{0\}$. The integral

$$\text{(A.1)} \qquad K_n^{6p}=\left(\frac{2}{\omega}\right)^n\int_0^{2\pi}\frac{e^{6pi\psi}\,d\psi}{(a+\cos 6\psi)^n}=\left(\frac{2}{\omega}\right)^n\int_{-\pi}^{\pi}\frac{e^{pi\psi}\,d\psi}{(a+\cos\psi)^n}$$

can be found by the residue method. Introduce a complex variable $z=e^{i\psi}$ which lies on an arc C, provided $\psi\in(-\pi,\pi)$. The denominator of the integrand can be rearranged as follows:

$$(a+\cos\psi)^n=\frac{1}{(2z)^n}(z^2+2az+1)^n=\frac{1}{(2z)^n}(z-z_1)^n(z-z_2)^n,$$

where

$$z_1=-e^{-v}, \quad z_2=-e^{v}, \quad \operatorname{Ch} v=a, \quad \operatorname{Sh} v=\frac{2}{\omega}(1-\omega)^{1/2},$$

$$e^{-v}=\frac{2-\omega}{\omega}-\frac{2}{\omega}(1-\omega)^{1/2}, \quad e^{v}=\frac{2-\omega}{\omega}+\frac{2}{\omega}(1-\omega)^{1/2}.$$

Let ω be positive, hence $a>1$ (if $\omega<0$, then one can set $\operatorname{Ch} v=-a$, $\operatorname{Sh} v=(a^2-1)^{1/2}$; z_1, z_2 are defined as before). Thus $z_\alpha\in R_-$, $|z_1|<1$ and $|z_2|>1$. The integral (A.1) reads

$$K_n^{6p}=2\pi\left(\frac{4}{\omega}\right)^n\hat{S}_{n,n}^{p+n-1}, \quad \hat{S}_{n,n}^{p+n-1}=\frac{1}{2\pi i}\int_C\frac{z^{p+n-1}}{(z-z_1)^n(z-z_2)^n}\,dz.$$

By virtue of the single-valuedness of the integrand, the integration along the arc C can be substituted by the integration along the γ contour, particularly for γ a unit circle $|z|=1$ can be chosen. Thus $\hat{S}_{n,n}^r=S_{n,n}^r$ and hence

$$K_n^{6p}=2\pi\left(\frac{4}{\omega}\right)^n S_{n,n}^{p+n-1}.$$

3. *The integrals* L_n^{2m}

$$L_n^{2m}=\int_0^{2\pi}\frac{e^{2mi\psi}\ln(1-\omega\sin^2 3\psi)}{(1-\omega\sin^2 3\psi)^n}\,d\psi.$$

By using the expansion

$$\text{(A.2)} \qquad \ln(1-\omega\sin^2 3\psi) = -\sum_{n=1}^{\infty} \frac{\omega^n \sin^{2n} 3\psi}{n}$$

the definition of K_n^{2m}, and assuming that the series (A.2) is integrable term-by-term, we arrive at

$$L_n^{2m} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} Q_m^n K_n^{4m} \frac{\omega^n}{n} (-1)^{m+1}, \qquad Q_m^n = 2^{-2n} \binom{2n}{m+n}.$$

The integral considered can also be reduced to a closed form (cf. [9]) which will not be reported here because of its complexity.

4. *Expansions of the functions* $J_p(r, \vartheta)$ and $K_p(r, \vartheta)$

In order to perform γ-integration in the RHS of the relations (3.3), the following formulae will be applied:

$$-\mathrm{Re} \int_0^{\infty} \frac{e^{ipx}\, x dx}{x^2+q^2} = E_s(pq), \qquad -\mathrm{Re} \int_0^{\infty} \frac{i e^{ipx}\, dx}{x^2+q^2} = q^{-1} E_a(pq), \qquad q>0,$$

where

$$E_s(x) = 0.5\, [\mathrm{Ei}(x)\, e^{-x} + \mathrm{Ei}(-x)\, e^{x}],$$

$$E_a\, x = 0.5\, [\mathrm{Ei}(x)\, e^{-x} - \mathrm{Ei}(-x)\, e^{x}].$$

Ei $(\cdot)$ denoted the exponential integral function, cf. [8]. As a result of γ-integration in the relations (3.3), we have

$$\begin{Bmatrix} J_{2n} \\ K_{2n} \end{Bmatrix} (r, \vartheta) = \int_0^{2\pi} \begin{Bmatrix} \cos 2n\psi \\ \sin 2n\psi \end{Bmatrix} \frac{F_s(r, \vartheta; \psi)}{\varkappa(\omega, \psi)}\, d\psi, \qquad n=0, 1, 2, \ldots,$$

$$\begin{Bmatrix} J_{2n+1} \\ K_{2n+1} \end{Bmatrix} (r, \vartheta) = \begin{Bmatrix} \cos(2n+1)\psi \\ \sin(2n+1)\psi \end{Bmatrix} \frac{F_a(r, \vartheta; \psi)}{[\varkappa(\omega, \psi)]^{1/2}}\, d\psi,$$

where

$$F_t(r, \vartheta; \psi) = E_t\left(-\frac{r\cos(\psi-\vartheta)}{[\varkappa(\omega, \psi)]^{1/2}}\right), \qquad t=a, s.$$

By integrating the RHS of Eqs. (3) term-by-term the following expansions

$$J_{2p} + i K_{2p} = \sum_{k=0, 2, 4} \frac{\bar{r}^k}{k!} [(\gamma_E + \ln \bar{r} + f_k\, k!)\, A_k^{2p}(\vartheta) + B_k^{2p}(\vartheta) - 0.5\, C_k^{2p}(\vartheta)],$$

$$J_{2p-1} + i K_{2p-1} = \sum_{k=1, 3, 5} \frac{\bar{r}^k}{k!} [(\gamma_E + \ln \bar{r} - f_k\, k!)\, A_k^{2p-1}(\vartheta) + B_k^{2p-1}(\vartheta) - 0.5\, C_k^{2p-1}(\vartheta)].$$

are found, were γ_E means Euler's constant and

$$f_0=0,\, f_1=1, \quad \text{and} \quad f_{n+1}=\frac{1}{(n+1)\,!\,(n+1)}+\sum_{k=1}^{n}\frac{1}{k\,k!}\,\frac{(-1)^{n-k+1}}{(n-k+1)\,!}.$$

The functions A_i^j, B_i^j and C_i^j are expressed by means of the integrals

$$A_j^s=\int_0^{2\pi}\frac{\cos^j(\psi-\vartheta)\exp(is\psi)\,d\psi}{[\varkappa(\omega,\psi)]^{[j/2]+1}},$$

$$B_j^s=\int_0^{2\pi}\frac{\cos^j(\psi-\vartheta)\ln|\cos(\psi-\vartheta)|\exp(is\psi)\,d\psi}{[\varkappa(\omega,\psi)]^{[j/2]+1}},$$

$$C_j^s=\int_0^{2\pi}\frac{\cos^j(\psi-\vartheta)\ln|\varkappa(\omega,\psi)|\exp(is\psi)\,d\psi}{[\varkappa(\omega,\psi)]^{[j/2]+1}},$$

and can be rearranged to the form

$$A_{2n}^{2p}=\sum_{m=-n}^{n}Q_m^n\,K_{n+1}^{2(m+p)}\,e^{-2im\vartheta},$$

$$A_{2n-1}^{2p-1}=\sum_{m=-n}^{n-1}P_m^n\,K_n^{2(m+p)}\,e^{-(2m+1)i\vartheta},\qquad P_m^n=2^{-2n+1}\binom{2n-1}{m+n},$$

$$B_{2n}^{2p}=\sum_{m=-\infty}^{\infty}d_m^n\,K_{n+1}^{2(m+p)}\,e^{-2im\vartheta},\qquad d_m^n=\sum_{p=-n}^{n}Q_p^n\,b_{m-p},$$

$$B_{2n-1}^{2p-1}=\sum_{m=-\infty}^{\infty}e_m^n\,K_n^{2(m+p)}\,e^{-(2m+1)i\vartheta},\qquad e_m^n=\sum_{p=-n}^{n-1}P_p^n\,b_{m-p},$$

$$C_{2n}^{2p}=\sum_{m=-n}^{n}Q_m^n\,L_{n+1}^{2(m+p)}\,e^{-2mi\vartheta},$$

$$C_{2n-1}^{2p-1}=\sum_{m=-n}^{n-1}P_m^n\,L_n^{2(m+p)}\,e^{-(2m+1)i\vartheta},$$

where the coefficients Q_m^n and functions K_i^j, L_i^j are defined and analysed in the preceding sections of the Appendix. Now the functions J_r and K_r can be written as follows:

$$J_{2p}(r,\vartheta)=\sum_{n=0}^{\infty}\frac{\bar{r}^{2n}}{(2n)!}\left[\ln\bar{r}\sum_{m=-\infty}^{+\infty}{}^{*}\,\hat{J}_{2n,\,2m}^{2p}\cos(2m\vartheta)+\sum_{m=-\infty}^{\infty}\check{J}_{2n,\,2m}^{2p}\cos(2m\vartheta)\right],$$

$$J_{2p-1}(r,\vartheta)=\sum_{n=1}^{\infty}\frac{\bar{r}^{2n-1}}{(2n-1)!}\left[\ln\bar{r}\sum_{m=-\infty}^{+\infty}{}^{*}\,\hat{J}_{2n-1,\,2m+1}^{2p-1}\cos(2m+1)\vartheta+\right.$$

$$\left.+\sum_{m=-\infty}^{\infty}\check{J}_{2n-1,\,2m+1}^{2p-1}\cos(2m+1)\vartheta\right],$$

$$K_{2p}(r,\vartheta)=\sum_{n=0}^{\infty}\frac{\bar{r}^{2n}}{(2n)!}\Big[\ln\bar{r}\sum_{m=-\infty}^{+\infty}{}^{*}\hat{K}^{2p}_{2n,2m}\sin 2m\vartheta+\sum_{m=-\infty}^{\infty}\check{K}^{2p}_{2n,2m}\sin 2m\vartheta\Big],$$

$$K_{2p-1}(r,\vartheta)=\sum_{n=1}^{\infty}\frac{\bar{r}^{2n-1}}{(2n-1)!}\Big[\ln\bar{r}\sum_{m=-\infty}^{\infty}{}^{*}\hat{K}^{2p-1}_{2n-1,2m+1}\sin(2m+1)\vartheta+$$

$$+\sum_{m=-\infty}^{\infty}\check{K}^{2p-1}_{2n-1,2m+1}\sin(2m+1)\vartheta\Big],$$

where

$$\hat{J}^{2p}_{2n,2m}=-\hat{K}^{2p}_{2n,2m}=a^{2p}_{2n,2m},\qquad \hat{J}^{2p-1}_{2n-1,2m+1}=-\hat{K}^{2p-1}_{2n-1,2m+1}=a^{2p-1}_{2n-1,2m+1},$$

$$\check{J}^{2p}_{2n,2m}=-\check{K}^{2p}_{2n,2m}=(\gamma_E+2n!\,f_{2n})\,a^{2p}_{2n,2m}+b^{2p}_{2n,2m}-0.5\,c^{2p}_{2n,2m},$$

$$\check{J}^{2p-1}_{2n-1,2m+1}=-\check{K}^{2p-1}_{2n-1,2m+1}=(\gamma_E-(2n-1)!\,f_{2n-1})\,a^{2p-1}_{2n-1,2m+1}+$$

$$+b^{2p-1}_{2n-1,2m+1}-0.5\,c^{2p-1}_{2n-1,2m+1}.$$

An asterisk at the signs of sums denotes a finite summation. The coefficients $a^{i}_{j,m}$, $b^{i}_{j,m}$ and $c^{i}_{j,m}$ are defined below:

$$a^{2p}_{2n,2m}=\begin{cases}Q^{n}_{m}K^{2(m+p)}_{n+1} & \text{for}\quad m\in[-n,n],\\ 0 & \text{for}\quad |m|>n,\end{cases}$$

$$b^{2p}_{2n,2m}=d^{n}_{m}K^{2(m+p)}_{n+1},\qquad c^{2p}_{2n,2m}=\begin{cases}Q^{n}_{m}L^{2(m+p)}_{n+1} & \text{for}\quad m\in[-n,n],\\ 0 & \text{for}\quad |m|>n,\end{cases}$$

$$a^{2p-1}_{2n-1,2m+1}=\begin{cases}P^{n}_{m}K^{2(m+p)}_{n} & \text{for}\quad m\in[-n,n-1]\\ 0 & \text{for}\quad m\notin[-n,n-1]\end{cases},\qquad b^{2p-1}_{2n-1,2m+1}=e^{n}_{m}K^{2(m+p)}_{n},$$

$$c^{2p-1}_{2n-1,2m+1}=\begin{cases}P^{n}_{m}L^{2(m+p)}_{n} & \text{for}\quad m\in[-n,n-1]\\ 0 & \text{for}\quad m\notin[-n,n-1].\end{cases}$$

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STRESZCZENIE

ROZWIĄZANIA PODSTAWOWE W TEORII TYPU COSSERATÓW TARCZY SIATKOWEJ O STRUKTURZE PLASTRA MIODU

W pracy znaleziono rozwiązania podstawowe kontynualnego modelu Woźniaka opisującego zachowanie się tarczy siatkowej o strukturze plastra miodu. Wydzielono i zbadano osobliwości tych rozwiązań oraz osobliwości odkształceń, napięć i napięć momentowych. W szczególnym przypadku rozdzielonych równań konstytutywnych otrzymane rozwiązania sprowadzają się do znanych składowych tensora Greena w teorii izotropowego i centrosymetrycznego ośrodka Cosseratów w płaskim stanie naprężenia.

Резюме

ФУНДАМЕНТАЛЬНЫЕ РЕШЕНИЯ В ТЕОРИИ ТИПА КОССЕРА СЕТОЧНОГО ДИСКА СО СТРУКТУРОЙ ПЧЕЛИНОГО СОТА

В работе найдены фундаментальные решения континуальной модели Возьняка, описывающей поведение сеточного диска со структурой пчелиного сота. Выделены и исследованы особенности этих решений, а также особенности деформаций, напряжений и моментных напряжений. В частном случае распряженных определяющих соотношений полученные решения сводятся к известным составляющим тензора Грина в теории изотропной и центросиметричной среды Коссера в плоском напряженном состоянии.

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