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IMPROVED ERROR ESTIMATES OF SOLUTIONS IN THE LINEAR THEORY OF THIN ELASTIC SHELLS

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DANIELSON [2] has shown that the difference between the classical linear shell theory solution and the solution of the three-dimensional elasticity theory is in the L_2 norm a quantity of relative order ε. In this report that difference is proved to be of relative order $\delta \leq \varepsilon$, and in some cases of practical interest even $\delta \ll \varepsilon$.

1. INTRODUCTION

A rigorous mathematical proof that the classical linear theory of thin elastic shells provides a valid approximation to the threedimensional elasticity theory has been given by KOITER [1] and DANIELSON [2]. From the supposedly-known solution of the twodimensional equations of the shell theory, these authors constructed a three-dimensional statically admissible stress field and a kinematically admissible displacement field. The latter was in [1] distributed linearly across the shell thickness, in accordance with a so-called modified Kirchhoff–Love hypothesis, and was in [2] a third-order polynomial with respect to the thickness coordinate. Now the question at issue was to find how close the kinematically admissible displacement field approximated the actual (unknown) solution of the threedimensional elasticity problem. This was answered using the complementary variational principles. Both the energetic and the L_2 norms of the error displacement field were shown to be of the relative order $\sqrt{\varepsilon}$ according to [1], and ε in accordance with [2]. Here $\varepsilon = h/R + h^2/L^2$ is a small parameter related to the shell thickness h, to a characteristic radius of curvature of the middle surface R and to a characteristic wave length of the deformation pattern of that surface L. A major advantage of these results lies in the fact that they make possible a precise quantitative comparison of the shell theory (approximate) solution with the elasticity theory (exact) solution without knowing the latter. In fact, given h and R, ε is fully defined after obtaining L from the shell theory solution.

At a glance the estimates [1, 2] seem to be the best possible for the classical shell theory. They state that the shell theory fails to yield an adequate description of the shell behaviour when h/R or/and h/L are close to unity, i.e. for thick shells or/and shells whose deformation variation is rapid. This is claimed to hold true irrespective of the bending to membrane strain ratio $h\rho/\gamma$, i.e. for the membrane, bending and isometric deformation. Consequently, all the most significant modes of the shell behaviour appear to be covered.

In this presentation we shall prove that the error estimates [1, 2] can still be improved. Our argument is based on the simple observation that, in general, the characteristic wave length of the membrane strains L_N and the characteristic wave length of the bending strains L_M need not be equal. Then a generalized, small parameter δ is introduced related to ε by $\delta \leqslant \varepsilon$. Moreover, in some cases of practical interest a strong inequality $\delta \ll \varepsilon = 1$ is proved to be true. In such a case the shell theory solution fails in terms of ε as an approximation to the solution of the three-dimensional elasticity theory problem. At the same time, in terms of δ this approximation is satisfactory.

2. Equations of classical shell theory

The notation used in [2] is generally retained here. We shall consider the same classical variant of the linear theory of thin elastic, isotropic shells as in [1, 2]. Nevertheless, other variants can be treated similarly.

The shell theory in question is specified by the virtual work principle

$$\int_S (n^{\alpha\beta}\gamma_{\alpha\beta}+m^{\alpha\beta}\rho_{\alpha\beta})\,dS=\int_S (p^\alpha u_\alpha^* +p^3 u_3^*)\,dS+\int_{\partial S}[n^{\alpha\beta}u_\alpha^*+Q^\beta u_3^*+m^{\alpha\beta}\times \times(u^*_{3,\alpha}+b^\lambda_\alpha u^*_\lambda)]n_\beta\,ds, \tag{2.1}$$

written in the usual normal coordinate system (x^α, z) ascribed to the midsurface S with the bounding curve ∂S and the unit vector n_β of the outward normal to ∂S; $b^\lambda_\varkappa$ is the tensor of curvature of S and the commas denote partial differentiation with respect to x^α. The reduced tangential and normal loadings per unit area of S are denoted by p^α and p^3; the symmetric tangential stress resultants $n_{\alpha\beta}$, the transverse shear stress resultants Q_β and the symmetric stress couples $m_{\alpha\beta}$ are assumed to be prescribed at ∂S.

The symmetric tensors of the membrane strains $\gamma_{\alpha\beta}$ and the bending strains $\rho_{\alpha\beta}$ are related to the tangential and normal displacements u^*_α, u^*_3 of S as follows:

$$\gamma_{\alpha\beta}=\frac{1}{2}(u^*_{\alpha|\beta}+u^*_{\beta|\alpha})-b_{\alpha\beta}u^*_3,\qquad \rho_{\alpha\beta}=u^*_{3|\alpha\beta}-b^\lambda_\alpha b_{\lambda\beta}u^*_3+b^\lambda_\alpha u^*_{\lambda|\beta}+ +b^\lambda_\beta u^*_{\lambda|\alpha}+b^\lambda_{\alpha|\beta}u^*_\lambda, \tag{2.2}$$

where a vertical bar denotes covariant differentiation with respect to the metric of undeformed S.

The constitutive equations have the form

$$\gamma_{\alpha\beta}=\frac{1}{Eh}[(1+\nu)n_{\alpha\beta}-\nu a_{\alpha\beta}n^\lambda_\lambda],\qquad \rho_{\alpha\beta}=\frac{12}{Eh^3}[(1+\nu)m_{\alpha\beta}-\nu a_{\alpha\beta}m^\lambda_\lambda], \tag{2.3}$$

where E is Young's modulus, ν — Poisson's ratio, h — the shell thickness and $a_{\alpha\beta}$ is the first metric tensor of S.

The equations of equilibrum and the static boundary conditions can be deduced from Eqs. (2.1) and (2.2) by standard technique. It is pertinent to note that these equations are fully exact in the sense that they ensure the overall equilibrium of a shell element of finite thickness h. The constitutive equations (2.3) are evidently approximate, but for our purposes there is no need to know the error involved.

Let the surface coordinates x^α have the dimensions of length such that

$$|a_{\alpha\beta}|\leqslant 1, \quad |a^{\alpha\beta}|\leqslant 1. \tag{2.4}$$

Then both the covariant and contravariant components of a surface tensor have the same dimensions as the physical components. A generalized radius of curvature R is now defined as the largest number such that

$$|b_{\alpha\beta}|\leqslant\frac{1}{R}, \quad |b_{\alpha\beta|\eta}|\leqslant\frac{1}{R\sqrt{hR}}, \quad |h_{,\alpha}|\leqslant\sqrt{\frac{h}{R}}. \tag{2.5}$$

In practice, R is equal to the minimum principal radius of curvature of the mid-surface. The inequalities $(2.5)_{2,3}$ express the assumption that the wave length characterizing the curvature and thickness variations is not less than $\sqrt{hR}$ in order of magnitude.

The stress resultants, couples and the strains can be represented by their maximum absolute values n, m, γ, ρ, i.e.

$$|n_{\alpha\beta}|\leqslant n, \quad |m_{\alpha\beta}|\leqslant m, \quad |\gamma_{\alpha\beta}|\leqslant\gamma, \quad |\rho_{\alpha\beta}|\leqslant\rho. \tag{2.6}$$

Owing to the constitutive equations (2.3), these values are interrelated in the form

$$n=O(Eh\gamma), \quad m=O(Eh^3\rho), \tag{2.7}$$

where the symbol $A=O(B)$ means that there exists a dimensionless positive constant k such that $|A|\leqslant k|B|$.

The characteristic wave lengths of the membrane and bending strains L_N and L_M are the largest numbers satisfying the following inequalities:

$$|\gamma_{\alpha\beta|\eta}|\leqslant\frac{\gamma}{L_N}, \quad |\gamma_{\alpha\beta|\eta\lambda}|\leqslant\frac{\gamma}{L_N^2}, \quad |\rho_{\alpha\beta|\eta}|\leqslant\frac{\rho}{L_M}, \quad |\rho_{\alpha\beta|\eta\lambda}|\leqslant\frac{\rho}{L_M^2}. \tag{2.8}$$

In view of Eqs. (2.3), (2.6) and (2.7), it becomes evident that the variations of the stress resultants and couples can be evaluated by the same wave lengths. Thus we have

$$|n_{\alpha\beta|\eta}|\leqslant\frac{n}{L_N}, \quad |n_{\alpha\beta|\eta\lambda}|\leqslant\frac{n}{L_N^2}, \quad |m_{\alpha\beta|\eta}|\leqslant\frac{m}{L_M}, \quad m_{\alpha\beta|\eta\lambda}|\leqslant\frac{m}{L_M^2}. \tag{2.9}$$

To shorten notation we also introduce two auxiliary nondimensional parameters of the form

$$\varepsilon_N=\frac{h}{R}+\frac{h^2}{L_N^2}, \quad \varepsilon_M=\frac{h}{R}+\frac{h^2}{L_M^2}. \tag{2.10}$$

In accordance with Eqs. (2.5)–(2.9) the two-dimensional shell theory problem and its solution have been approximately characterized by a set of numbers containing h, R, L_N, L_M, γ (or n), ρ (or m). Given h and R, we find from Eqs. (2.6)–(2.8) the remaining quantities after the solution of the shell theory equations. In contrast to the previous approach due to KOITER [1] and DANIELSON [2], our description allows for two different wave lengths of the membrane and bending strains L_N and L_M. It is readily verified that the characteristic wave length L of [1, 2] is the minimum of (L_N, L_M).

3. Statically and kinematically admissible stress fields

All expressions appearing in this section were derived in [1] or [2]. However, their error terms are estimated here more precisely. This is achieved by the use of the two independent wave lengths L_N and L_M instead of one quantity L.

The three-dimensional statically admissible stress field $\tilde{\sigma}_{ij}(x^\delta, z)$ constructed by Koiter [1] is rewritten in a slightly modified form:

$$\tilde{\sigma}_{\alpha\beta}(x^\delta, z)=\frac{n_{\alpha\beta}}{h}-\frac{12zm_{\alpha\beta}}{h^3},$$

$$\tilde{\sigma}_{\alpha 3}(x^\delta, z)=-\frac{z}{h}n^\beta_{\alpha|\beta}+\frac{3}{2h}\left(\frac{4z^2}{h^2}-1\right)m^\beta_{\alpha|\beta}+\frac{3z}{h}\left(\frac{8z^2}{3h^2}-1\right)b^\delta_\alpha m^\beta_{\delta|\beta}-\frac{4z^3}{h^3}(b^\delta_\alpha m^\beta_\delta)|_\beta,$$

$$\tilde{\sigma}_{33}(x^\delta, z)=\tilde{\sigma}_{33}(x^\delta, 0)+\frac{z^2}{2h}n^{\alpha\beta}|_{\alpha\beta}-\frac{z}{h}b_{\alpha\beta}n^{\alpha\beta}+\frac{6z^2}{h^3}b_{\alpha\beta}m^{\alpha\beta}-\frac{4z^3}{h^3}b^\lambda_\alpha b_{\lambda\beta}m^{\alpha\beta}-\frac{3z}{2h}\left(\frac{4z^2}{3h^2}-1\right)m^{\alpha\beta}|_{\alpha\beta}. \tag{3.1}$$

The expressions (3.1) are valid for no body forces. It is also assumed that the reduction of the loads to the middle surface introduces no surface couples. The error involved in the relations (3.1) is of relative order $O(h/R)$, [1]. The same error results from our replacement of Koiter's pseudo stresses by the actual stresses $\tilde{\sigma}_{ij}$. For future use it is sufficient to take the relations (3.1) in a simplified version [2]

$$\tilde{\sigma}_{\alpha\beta}(x^\delta, z)=\frac{n_{\alpha\beta}}{h}-\frac{12zm_{\alpha\beta}}{h^3},\qquad \tilde{\sigma}_{33}(x^\delta, z)=O\left(\frac{n\varepsilon_N}{h}, \frac{m\varepsilon_M}{h^2}\right),$$

$$\tilde{\sigma}_{\alpha 3}(x^\delta, z)=-\frac{z}{h}n^\beta_{\alpha|\beta}+\frac{3}{2h}\left(\frac{4z^2}{h^2}-1\right)m^\beta_{\alpha|\beta}+O\left(\frac{hn}{RL_N}, \frac{m}{RL_M}\right). \tag{3.2}$$

It must be stressed that the statically admissible stress field in a three-dimensional body is entirely defined by the shell theory solution.

Danielson [2] constructed from the shell theory solution a kinematically admissible displacement field $\hat{u}_i(x^\delta, z)$ that we record in a more explicit form:

$$\hat{u}_\alpha(x^\delta, z)=u^*_\alpha-z(u^*_{3,\alpha}+b^\beta_\alpha u^*_\beta-\frac{3(1+\nu)z}{Eh}m^\beta_{\alpha|\beta}-\frac{z^2}{2Eh}[2(1+\nu)n^\beta_{\alpha|\beta}-\nu n^\beta_{\beta,\alpha}]+\frac{2z^3}{Eh^3}[2(1+\nu)m^\beta_{\alpha|\beta}-\nu m^\beta_{\beta,\alpha}],$$

$$\hat{u}_3(x^\delta, z)=u^*_3-\frac{\nu}{1-\nu}\left[z\gamma^\alpha_\alpha-\frac{1}{2}z^2\rho^\alpha_\alpha\right]. \tag{3.3}$$

when the terms containing $n_{\alpha\beta}$ and $m_{\alpha\beta}$ are absent, then Eqs. (3.3) define KOITER'S [1] so-called modified Kirchhoff–Love displacement field. If also γ_α and ρ_α are rejected, we get the original Kirchhoff–Love hypothesis.

A suitable form of the strain-displacement relations in the three-dimensional body reads [2]

$$(3.4)\qquad \begin{aligned} \hat{\gamma}_{\alpha\beta}(x^\delta, z) &= \frac{1}{2}(\hat{u}_{\alpha|\beta}+\hat{u}_{\beta|\alpha}) - b_{\alpha\beta}\,\hat{u}_3 + O\left(\frac{\hat{u}}{R^2}\right)z + O\left(\frac{\hat{u}}{R^3}\right)z^2, \\ \hat{\gamma}_{\alpha 3}(x^\delta, z) &= \frac{1}{2}(\hat{\gamma}_{\alpha,3}+\hat{u}_{3,\alpha}) + b^\beta_\alpha\,\hat{u}_\beta + O\left(\frac{\hat{u}}{R^2}\right)z, \qquad \hat{\gamma}_{33}(x^\delta, z) = \hat{u}_{3,3}, \end{aligned}$$

where $\hat{u}$ denotes the maximum value of $|\hat{u}_i(x^\delta, z)|$. Substituting from Eqs. (3.4) into Eqs. (3.3) and bearing in mind Eq. (2.2), a kinematically admissible deformation field $\hat{\gamma}_{ij}(x^\delta, z)$ is obtained:

$$\hat{\gamma}_{\alpha\beta}(x^\delta, z) = \gamma_{\alpha\beta} - z\rho_{\alpha\beta} + O(\gamma\varepsilon_N, h\rho\varepsilon_M), \qquad \hat{\gamma}_{33}(x^\delta, z) = -\frac{\nu}{1+\nu}[\gamma^\beta_\beta - z\rho^\beta_\beta],$$

$$(3.5)\qquad \begin{aligned} \hat{\gamma}_{\alpha 3}(x^\delta, z) &= \frac{\nu}{2(1-\nu)}\left[-z\gamma^\beta_{\beta,\alpha} + \frac{1}{2}z^2\rho^\beta_{\beta,\alpha}\right] - \frac{3(1+\nu)}{2Eh}m^\beta_{\alpha|\beta} - \frac{z}{2Eh}\times \\ &\times[2(1+\nu)n^\beta_{\alpha|\beta} - \nu n^\beta_{\beta,\alpha}] + \frac{3z^2}{Eh^3}[2(1+\nu)m^\beta_{\alpha|\beta} - \nu m^\beta_{\beta,\alpha}] + O\left(\frac{nh}{ERL_N}, \frac{m}{ERL_M}\right). \end{aligned}$$

The well-known Hooke's law of elasticity theory reads

$$(3.6)\qquad \hat{\sigma}_{ij}(x^\delta, z) = \frac{E}{1+\nu}\left[\hat{\gamma}_{ij} + \frac{\nu}{1-2\nu}g_{ij}\,\hat{\gamma}^k_k\right],$$

where $g_{ij}(x^\delta, z)$ is the metric tensor in the three-dimensional shell space. Within an error of relative order $O(h/R)$, g_{ij} can be replaced by the midsurface metric tensor, i.e. $g_{\alpha\beta} = a_{\alpha\beta} + O(h/R)$, $g_{33} = 1$. Then, substituting from Eq. (3.5) into Eq. (3.6) we get a kinematically admissible stress field $\hat{\sigma}_{ij}(x^\delta, z)$. It is no use recording that lengthy expression. Comparing it to the statically admissible stress field (3.2), we find that

$$(3.7)\qquad \hat{\sigma}_{ij}(x^\delta, z) - \tilde{\sigma}_{ij}(x^\delta, z) = O\left(\frac{n\varepsilon_N}{h}, \frac{m\varepsilon_M}{h^2}\right).$$

Setting $\varepsilon = \varepsilon_N = \varepsilon_M$ this result reduces to that of [2].

4. Error estimates of shell theory solutions

To accomplish our work we need some obvious inequalities for energetic and L_2 norms in function space. Following [1, 2] it is assumed that the actual distributions of the stress $\sigma_{ij}(x^\delta, z)$ and displacement $u_i(x^\delta, z)$ in the three-dimensional body coincide at the bounding surface with the statically admissible stress field $\tilde{\sigma}_{ij}$ and the kinematically admissible displacement field $\hat{u}_i$

$$(4.1)\qquad \sigma^{ij}n_j = \tilde{\sigma}^{ij}n_j, \qquad u_i = \hat{u}_i,$$

where n_j are the components of the unit vector of the outward normal to the bounding surface. In other words, the boundary conditions (4.1) are "regular" [1]. Assuming the relations (4.1) the following inequalities hold true [2]:

$$C_2[\hat{\sigma}_{ij}-\sigma_{ij}]\leqslant C_2[\hat{\sigma}_{ij}-\tilde{\sigma}_{ij}], \quad P_2[\hat{u}_i-u_i]\leqslant C_2[\hat{\sigma}_{ij}-\tilde{\sigma}_{ij}], \tag{4.2}$$

where for any symmetric tensor $S_{ij}(x^\delta, z)$ the complementary energy functional $C_2[S_{ij}]$ reads

$$C_2[S_{ij}]=\frac{1}{2E}\int_V [(1+\nu)S^i_j S^j_i-\nu S^i_i S^j_j]\,dV, \tag{4.3}$$

V being the volume of the body. The elastic energy functional $P_2[v_i]$ can be obtained from Eq. (4.3) using Eqs. (3.6) and (3.4) with $v_i=\hat{u}_i$, $S_{ij}=\hat{\sigma}_{ij}$. Since $C_2[S_{ij}]$ is a positive definite homogenous quadratic functional, it can be chosen as an energetic norm for the stresses. Similarly, $P_2[v_i]$ becomes an energetic norm for the displacement provided rigid body motions are deleted. It follows that the estimates (4.2) for the energetic norms can be transformed into inequalities for the L_2 norms of the error displacement and stress fields

$$a\|\hat{\sigma}_{ij}-\sigma_{ij}\|^2\leqslant C_2[\hat{\sigma}_{ij}-\tilde{\sigma}_{ij}], \quad b\|\hat{u}_i-u_i\|^2\leqslant C_2[\hat{\sigma}_{ij}-\tilde{\sigma}_{ij}], \tag{4.4}$$

where, for any S_{ij} and v_i, we have

$$\|S_{ij}\|^2=\int_V S^i_j S^j_i\,dV, \quad \|v_i\|^2=\int_V v^i v_i\,dV \tag{4.5}$$

and

$$a=\inf C_2[\ ], \quad b=\inf P_2[\ ]. \tag{4.6}$$

Substituting from Eq. (3.7) into Eq. (4.3), we find that

$$C_2[\hat{\sigma}_{ij}-\tilde{\sigma}_{ij}]=O(\varepsilon_N^2 n\gamma A,\ \varepsilon_M^2 mp A), \tag{4.7}$$

where A is the area of the midsurface. Now, Eqs. (4.7), (4.2) and (4.4) imply that both the energetic and the L_2 norms of the error displacement and stress fields are of relative order δ where

$$\begin{aligned} \delta&=\frac{h}{R}+\frac{h^2}{L_N^2}+\frac{hp}{\gamma}\frac{h^2}{L_M^2} \quad &\text{for}\quad \gamma>hp,\\ \delta&=\frac{h}{R}+\frac{h^2}{L_M^2}+\frac{\gamma}{hp}\frac{h^2}{L_N^2} \quad &\text{for}\quad \gamma<hp,\\ \delta&=\frac{h}{R}+\frac{h^2}{L_N^2}+\frac{h^2}{L_M^2} \quad &\text{for}\quad \gamma=hp. \end{aligned} \tag{4.8}$$

In order to elucidate this final result we recall the error estimate obtained for the same variant of the shell theory by Danielson [2]. It reads

$$\varepsilon=\frac{h}{R}+\frac{h^2}{L^2}, \quad L=\min(L_N, L_M). \tag{4.9}$$

A general conclusion to be drawn from comparison of the relations (4.8) and (4.9) states that in any case $\delta \leqslant \varepsilon$, i.e. our estimates are never worse than Danielson's. Actually, these estimates are equivalent only when $\gamma = hp$, i.e. in the case of bending deformation or/and if $L_N = L_M$. Moreover, there can be the shell theory solutions of practical interest such that $\delta \ll \varepsilon = 1$. In terms of ε these solutions fail to yield an adequate description of the shell behaviour whereas in terms of δ their accuracy is still sufficient. As an example to this effect we can choose the case ($h/R \ll 1$, $hp/\gamma \ll 1$, $h^2/L_N^2 \ll 1$, $h^2/L_M^2 = 1$) of a thin shell in the state of nearly membrane deformation, with a slow variation of the membrane strains and a rapid variation of the bending strains.

5. Concluding remarks

The error estimates obtained in this paper of the solutions in the linear classical theory of shells are the best possible for the invariant form of equations that have been dealt with. In contrast to the form of [1, 2], our estimates are precise for any deformation of the shell. It is readily verified that the Sanders–Koiter shell theory solutions bear the same relative error δ as the solutions of the classical theory. Also, when instead of the kinematically admissible displacement field due to Danielson [2] we take a modified Kirchhoff–Love displacement field due to Koiter [1], the error will again be given by the relations (4.8) with δ replaced by δ^2.

References

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Streszczenie

ULEPSZONE SZACOWANIE BŁĘDU ROZWIĄZANIA W LINIOWEJ TEORII WIOTKICH POWŁOK SPRĘŻYSTYCH

Danielson [2] wykazał, że różnicę pomiędzy rozwiązaniem zagadnienia klasycznej liniowej teorii powłok a rozwiązaniem przestrzennego zagadnienia teorii sprężystości charakteryzuje w normie L_2 błąd względny rzędu ε. W niniejszej pracy wykazujemy, że błąd ten jest rzędu $\delta \leqslant \varepsilon$, a w niektórych praktycznie ważnych przypadkach zachodzi oszacowanie $\delta \ll \varepsilon$.

Резюме

УЛУЧШЕННАЯ ОЦЕНКА ОШИБКИ РЕШЕНИЯ В ЛИНЕЙНОЙ ТЕОРИИ ГИБКИХ УПРУГИХ ОБОЛОЧЕК

Данельсон [2] показал, что разницу между решением задачи классической линейной теории оболочек и точным решением трехмерной задачи теории упругости характеризует в среднеквадратичной норме L_2 относительная ошибка порядка ε. В данной работе показываем, что эта ошибка порядка $\delta \leqslant \varepsilon$, а в некоторых, практически важных случаях имеет место оценка $\delta \leqslant \varepsilon$.

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