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MAGNETOELASTIC STABILITY OF A CURRENT-CARRYING ROD IN AN EXTERNAL MAGNETIC FIELD

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A theoretical investigation of the effects of a longitudinal magnetic field upon the stability of a current-carrying rod is made. From general nonlinear equations of rod static equilibrium the linearised system of stability equations are obtained. It is shown that for almost all common boundary conditions the governing system equations are self-adjoint. By means of the stability static approach the critical values of Ampère force are obtained, beyond which the rod becomes unstable.

NOTATIONS

L	length of rod,
r	radius of rod,
$\mathbf{B}_0$	vector of magnetic field induction
J_0	total current in rod,
j_0	density of current,
x, y, z	spatial coordinates,
$\mathbf{F}$	vector force of internal stresses,
$\mathbf{M}$	vector of moment of force of internal stresses,
$\mathbf{R}$	vector of Ampère force (per unit of length),
$\mathbf{t}$	unit vector of the tangent to central-line of rod,
$\hat{\mathbf{i}}_x$	unit vector along x-direction,
l	arc length of deformed rod central-line,
u	displacement along z-direction,
v	displacement along y-direction,
E	elasticity modulus,
C_1, C_2, C_3	constants,
d_1, d_2, d_3, d_4	constants.

1. INTRODUCTION

In the present paper the stability problem for a current-carrying rod in an external longitudinal magnetic field is considered. It is shown that the interaction of a rod current with an external magnetic field can bring about the spatial buckling of a rod, analogous to the buckling of a rod under twisting couple [1–2]. In the case when there is not an external magnetic field and a rod carries high currents, buckling due to its own magnetic field also takes place [3–4]. A survey of investigations concerning current-carrying rods stability is given in [4].

The effects of the longitudinal external magnetic field on elastic waves propagation in a bar with surface current are studied in [5]. A study of the stability of current-carrying plates is presented in the works [6–7] where the finite plate solution is given. This study also included results concerning critical values of plate current beyond which the plate becomes unstable. The stability of current-carrying cylindrical shells is investigated in [8–9]. When there is an external magnetic field perpendicular to the middle surface of a current-carrying plate in [10–12], the experimental and theoretical results concerning critical values of Ampère force are given.

2. Statement of the problem and governing equations

We consider a thick, elastic, conducting rod of a circular cross section of radius r and length L. The rod is a part of the electrical circuit of the current J_0. The current in the rod is uniformly distributed over a cross section. There is an external magnetic field B_0, the direction of which coincides with the direction of the current of the unstrained rod. We suppose that the rod's current J_0 is sufficiently small compared with the current critical value beyond which the rod becomes unstable due to its own magnetic field only [4]. This supposition makes it possible to neglect both the rod's own magnetic action and thermal effects. We also assume that the rod is free from mechanical loads.

In the primary unstrained state the external magnetic field does not interact with rod current. When the rod is bent, the interaction of a current with an external magnetic field takes place, and the Ampère force $\mathbf{R}$ (per unit of length of the rod central-line), acts on the rod:

$$\mathbf{R} = (\mathbf{J}_0 \times \mathbf{B}_0). \tag{2.1}$$

Let us consider the stability of the rod under action of the force $\mathbf{R}$ by means of the static method of stability.

We take a Cartesian system of fixed axes (x, y, z) of which axis of x coincides with the central line of the unstrained rod.

We shall investigate the problem using the following nonlinear Kirchhoff's equations which describe the static equilibrium of the deformed rod [1]:

$$\frac{d\mathbf{F}}{dl} = -\mathbf{R}, \qquad \frac{d\mathbf{M}}{dl} = [\mathbf{F} \times \mathbf{t}]. \tag{2.2}$$

The vector $\mathbf{F}$ is a force of rod internal stresses, the vector $\mathbf{M}$ is a moment of force of internal stresses, l is the arc length of the deformed rod central-line measured from the point of origin $x = y = z = 0$, $\mathbf{t}$ is a unit vector directed along the tangent of the deformed rod central-line $\mathbf{t} = (dx/dl, dy/dl, dz/dl)$.

In the deformed state of the rod we have

$$\mathbf{J}_0 = J_0\,\mathbf{t}. \tag{2.3}$$

The vector of magnetic field induction which is parallel to the axis x can be written as

$$\mathbf{B}_0 = B_0\,\hat{\mathbf{i}}_x, \tag{2.4}$$

where $\hat{\mathbf{i}}_x$ is the unit vector directed along the axis x.

Using Eqs. (2.3) and (2.4) we obtained the following expressions for components of perturbed Ampère force:

$$R_x = 0, \quad R_y = J_0\,B_0\,\frac{dz}{dl}, \quad R_z = -J_0\,B_0. \tag{2.5}$$

From the relations (2.5) one can conclude that the rod is bent as in the plane (x, y), so in the plane (x, z), i.e. we have the case of spatial buckling of the rod.

By putting the relations (2.5) in the first equation (2.2) and integrating, we obtained

$$F_x = C_1; \quad F_y = -J_0\,B_0\,z + C_2; \quad F_x = J_0\,B_0\,y + C_3. \tag{2.6}$$

Since the rod is free from mechanical load applied at rod ends, $C_1 = 0$. The other constants of integration are to be determined from boundary conditions.

By putting the relations (2.6) in the section equation (2.2) and taking into account that for a circular rod

$$\mathbf{M} = \frac{\pi E r^4}{4}\left[\mathbf{t}\times\frac{d\mathbf{t}}{dl}\right], \tag{2.7}$$

we obtained the following governing nonlinear equations which describe the strain-state the rod (E is an elasticity modulus):

$$\begin{gathered}
\frac{\pi E r^4}{4}\,\frac{d}{dl}\left(\frac{dy}{dl}\,\frac{d^2 z}{dl^2} - \frac{dz}{dl}\,\frac{d^2 y}{dl^2}\right) = \\
= -J_0\,B_0\left(z\frac{dz}{dl} + y\,\frac{dy}{dl}\right) + C_2\,\frac{dz}{dl} - C_3\,\frac{dy}{dl},
\end{gathered}$$

$$\frac{\pi E r^4}{4}\,\frac{d}{dl}\left(\frac{dz}{dl}\,\frac{d^2 x}{dl^2} - \frac{dx}{dl}\,\frac{d^2 z}{dl^2}\right) = J_0\,B_0\,y\,\frac{dx}{dl} + C_3\,\frac{dx}{dl}, \tag{2.8}$$

$$\text{(2.8) [cont.]} \quad \frac{\pi E r^4}{4} \frac{d}{dl}\left(\frac{dx}{dl}\frac{d^2 y}{dl^2} - \frac{dy}{dl}\frac{d^2 x}{dl^2}\right) = J_0 B_0 z \frac{dx}{dl} - C_2 \frac{dx}{dl}.$$

The equations of equilibrium (2.8) are essentially simplified when the rod is slightly deformed ($l = x$, $dy/dl \ll 1$, $dx/dl \approx 1$, $dz/dl \approx 1$).

From Eq. (2.8), neglecting small nonlinear terms, we obtained the following linear system of equations with respect to small displacements of the rod $z = u(\eta)$, $y = v(\eta)$, $\eta = x/L$,

$$\text{(2.9)} \quad \frac{d^4 u}{d\eta^4} + a_0 \frac{dv}{d\eta} = 0, \quad \frac{d^4 v}{d\eta^4} - a_0 \frac{du}{d\eta} = 0, \quad a_0 = \frac{4 J_0 B_0 L^3}{\pi E r^4}.$$

Equations (2.9) are static equations of stability of the current-carrying rod in the presence of an external longitudinal magnetic field.

Equations (2.9) are to be solved with common boundary conditions at rod ends $\eta = 0$, $\eta = 1$.

3. Solution of the problem

Now our task is to solve Eqs. (2.9).

When the boundary conditions at rod ends are symmetrical with respect to displacements u, v it is convenient to define a complex function $w(\eta) = u(\eta) + iv(\eta)$.

Then, instead of Eq. (2.9) we have

$$\text{(3.1)} \quad \frac{d^4 w}{d\eta^4} - i a_0 \frac{dw}{d\eta} = 0.$$

We consider Eq. (3.1) with the following boundary conditions:

$$\text{(3.2)} \quad w(0) = w'(0) = w(1) = w''(1) = 0,$$

$$\text{(3.3)} \quad w(0) = w'(0) = w(1) = w'(1) = 0,$$

$$\text{(3.4)} \quad w(0) = w''(0) = w(1) = w''(1) = 0.$$

In the relations (3.2)–(3.4) the primes denote differentiation with respect to η. Let us now show that the above mentioned boundary value-problems are self-adjoint, i.e. we have to show that the functions w_1, w_2 satisfy the following condition [13]:

$$\text{(3.5)} \quad \langle \hat{D} w_1, \bar{w} \rangle = \langle w_1, \overline{D w_2} \rangle,$$

where $\hat{D} = d^4/d\eta^4 - i a_0\, d/d\eta$, w_1, w_2 are functions which satisfy the boundary conditions, $\langle w_1, w_2\rangle = \int_0^1 w_1, w_2\, d\eta$, $\bar{w}$ is a conjugate complex function w.

By integrating the integral $\langle \hat{D}w_1, \bar{w}\rangle$, we obtained

$$\langle \hat{D}w_1, \bar{w}_2\rangle = [w_1^{\mathrm{III}}\bar{w}_2 - w_1^{\mathrm{II}}\bar{w}_2^{\mathrm{I}} + w_1^{\mathrm{I}}\bar{w}_2^{\mathrm{II}} - w_1\bar{w}_2^{\mathrm{III}} - a_0 i w_1 \bar{w}_2]\big|_{\eta=0}^{\eta=1} + \int_0^1 (w_1\bar{w}_2^{\mathrm{IV}} + i a_0 w_1 \bar{w}_2^{\mathrm{I}})\, d\eta.$$

For each boundary condition the expression in the brackets vanishes and therefore the condition (3.5) in fulfilled.

Since the boundary problems under consideration are self-adjoint, the application of the static method of stability is valid [2].

Note that the boundary value problem corresponding to a cantilevered rod is a non-self-adjoint one.

The general solution of Eq. (3.1) has the form

$$w = d_1 + d_2 \exp(-2ai\eta) + d_3 \exp[(\sqrt{3}+i)a\eta] + d_4 \exp[(-\sqrt{3}+i)a\eta], \quad a = \sqrt[3]{a_0}/2. \tag{3.6}$$

By satisfying the boundary condition for the first boundary problem (3.2), we obtained the following equation determining eigenvalues:

$$\sin a^{(1)} (\cos a^{(1)} - ch\sqrt{3}\, a^{(1)}) = 0.$$

The minimal eigenvalue which corresponds to the critical Ampère force is equal to $a_1^{(1)} = \pi$.

The eigenfunctions corresponding to a_1 have the form

$$u = -2\cos\pi\eta\,[\mathrm{sh}\sqrt{3}\,\pi\eta + \mathrm{sh}\sqrt{3}\,\pi(\eta-1)] + \mathrm{sh}\sqrt{3}\,\pi\cos 2\pi\eta + \sqrt{3}\,(1+\mathrm{ch}\sqrt{3}\,\pi)\sin 2\pi\eta - 3\,\mathrm{sh}\sqrt{3}\,\pi.$$

$$v = -2\sin\pi\eta\,[\mathrm{sh}\sqrt{3}\,\pi\eta + \mathrm{sh}\sqrt{3}\,\pi(\eta-1)] - \mathrm{sh}\sqrt{3}\,\pi\sin 2\pi\eta + \sqrt{3}\,(1+\mathrm{ch}\sqrt{3}\,\pi)(\cos 2\pi\eta - 1).$$

For the second (3.3) and the third (3.4) boundary conditions we have the following equations determining eigenvalues:

$$\cos 2a = \cos a\,\mathrm{ch}\sqrt{3}\,a - \sqrt{3}\sin a\,\mathrm{sh}\sqrt{3}\,a,$$

$$\cos 2a = \cos a\,\mathrm{ch}\sqrt{3}\,a + \sqrt{3}\sin a\,\mathrm{sh}\sqrt{3}\,a.$$

The minimal nontrivial roots of these equations are

$$a_1^{(2)} \approx 3.623, \qquad a_1^{(3)} \approx 2.611.$$

Let us consider the following numerical example concerning a free-supported copper rod ($a_1 \approx 2.611$), $r = 1.6\cdot 10^{-3}$ m, $L = 0.66$ m, $E = 0.87\times 10^{11}$ Pa.

This example was considered in the work [4] where it was shown that such a rod with current the (external magnetic field is absent) is unstable due to its own magnetic field when the value of the critical current J_{0*} is close to 4000 A.

In our case we have the following critical values for current $J_* \ll J_{0*}$ and external magnetic field induction intensity B_{0*}:

1. $J_* = 80$ A, $B_{0*} = 2.71$ T,
2. $J_* = 100$ A, $B_{0*} = 2.16$ T,
3. $J_* = 200$ A, $B_{0*} = 1.08$ T.

In the above mentioned cases the density of critical current is less than the admissible density $j_0 \sim 5 \times 10^7$ A/m^2, below which the Joule thermal effects are small.

In conclusion we note that the current-carrying cantilevered rod is statically stable. Such non-self-adjoint boundary value problems have to be investigated by means of the dynamic stability method analogous to the nonconservative problem of a cantilevered rod subjected to a follower force [2].

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STRESZCZENIE

MAGNETOSPRĘŻYSTA STATECZNOŚĆ PRĘTA PRZEWODZĄCEGO PRĄD I UMIESZCZONEGO W ZEWNĘTRZNYM POLU MAGNETYCZNYM

Przeprowadzono analizę teoretyczną wpływu podłużnego pola magnetycznego na stateczność pręta przewodzącego prąd. Z ogólnych nieliniowych równań statyki pręta otrzymano zlinearyzowany układ równań stateczności. Wykazano, że dla niemal wszystkich zazwyczaj stosowanych warunków brzegowych odpowiedni układ równań jest samosprzężony. Stosując statyczne podejście do problemu stateczności wyznaczono krytyczną wartość siły Ampèra, powyżej której pręt staje się niestateczny.

РЕЗЮМЕ

МАГНИТОУПРУГАЯ УСТОЙЧИВОСТЬ СТЕРЖНЯ ПРОВОДЯЩЕГО ТОК И ПОМЕЩЕННОГО ВНУТРЬ МАГНИТНОГО ПОЛЯ

Проведено теоретическое исследование вопроса влияния внешнего продольного магнитного поля на устойчивость токонесущего стержня. На основе общих неличейных уравнений статического равновесия получена линейная система уравнений устойчивохси. Доказано, что для почти всех обычных граничных условий полученная система уравнений является самосопряженной. Исходя из статического подхода к задачам устойчивости получены критические значения сил Ампера, при которых стрежень пространственно неустойчив.

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