ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 34, 4, 531–542, 1986
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

A NOTE ON RECENT DEVELOPMENTS IN THE THEORY OF ELASTIC PLATES WITH MODERATE THICKNESS

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The paper refers to Reddy's concept (Int. J. Solids Struct., **20**, No 9/10, pp. 881–896, 1984) of a construction of the energy consistent theory for plates with moderate thickness; this theory is based on the kinematical hypothesis known from the monographs by Ambartsumian and Kączkowski. With the use of other independent physical quantities (we handle the averaged Reissner's rotations), the equations of motion and boundary conditions are derived. By means of simplifications of functionals, the governing equations and boundary conditions of the Reissner-type model (found by Kączkowski and then discovered again by Levinson) are arrived at. In the last section a proof is given that there is no simple generalization of the kinematical hypothesis used in the paper which would lead to an energy-consistent and physically correct theory of the Reissner class.

1. Introduction

The classical theory of thin plates does not account for the influence of transverse shear deformations on the resulting response of the plate. The Kirchoff's constraints being imposed on the plate behaviour result in essential errors if the plate is rather thick than thin, if the load are of a special type, or, if higher natural vibrations are examined. The shortcomings of the classical theory have inclined engineers to develop improved plate models in which the effects of transverse shear deformations could have been considered. After the original paper by Reissner [1] had been published, a great number of other (but similar) concepts were proposed by Hencky [2], Mindlin [3], Kroman [4], Panc [5], Ambartsumian [6], Vekua [7], see also [8], Kączkowski [9] and Jemielita [10]. A comprehensive survey of this subject has been given in the monograph [11].

Since it is very difficult at the present time to propose a new rational approach to this problem, not every paper devoted to thick plates can be treated as the original one; for example, the respectable author of [12] has unknowingly repeated the main ideas as well as the results given in [9, Sec. 3.5].

The present paper refers to the recent results by Reddy [13] who started from the kinematical hypothesis previously used in [9, 12]. This hypothesis enables to satisfy homogeneous boundary conditions at the top and the bottom faces of the plate. Thus an influence of σ_{33} stresses on the state of plate deformation is neglected.

In the commonly known models for moderately thick plates (see [11]), governing equations of motion are obtained from classical plate equations of motion (involving moments and transverse forces). These averaged equations are obtained by orthogonalization of the local equations of motion with weighed functions 1 and z along the plate thickness. In contrast to this method, the governing equations of Reddy's theory have been directly obtained from the Hamilton principle. These equations are energy-consistent but they are of higher order than classical ones.

In this paper we follow Reddy's approach: we start with the same hypotheses and insert them into the Hamilton variational principle. However, other physical quantities will be introduced. We show that it is convenient and reasonable to handle the Reissnerian, averaged rotations φ_α of plate cross-sections; hence we also define other stress resultants. The presented derivation makes it possible to reveal analogies between Reddy's equations and Reissner-type equations found in [9, 12] using the energy inconsistent method. It will be shown which components of the strain energy have been tacitly neglected (in the mentioned papers) in order to arrive at the Reissner--type plate model.

In a natural way the following question arises: is it possible to propose a new kinematical hypothesis which would lead, without any additional simplifications of the strain energy, to the well-known Reissner-type equations. We show that such a hypothesis does not exist. However, if one weakens the correctness criteria, namely if the shear stiffness H of the plate is treated as an arbitrary quantity $\left(\text{for isotropic plates it is usually required that } H = \frac{5}{6}\mu h,\ \mu\text{ — Lamé modulus}\right)$, then the energy-consistency criterion implies the well-known Hencky's kinematical hypothesis, [2], see also [11].

2. Basic assumptions. Notations

Consider a plate which occupies a domain $\Omega = \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right)$, where h stands for the plate thickness and ω is a mid-surface,being parametrized by the orthogonal, Cartesian coordinate system x_α, $\alpha = 1, 2$.; the axis $z = x_3$ is perpendicular to ω. A contour of ω is denoted by γ. The vectors $\mathbf{n} = (n_\alpha)$ and $\boldsymbol{\tau} = (\tau_\alpha)$, $\alpha = 1, 2$, stand for the unit vectors outwardly normal and tangent to γ, respectively. The top and bottom faces of the plate (the bounding surfaces) are denoted by $\Gamma_\pm = \omega \times \{\pm h/2\}$; a lateral surface of the plate $\Gamma_0 = \gamma \times (-h/2, h/2)$. The surfaces $\Gamma_\pm$ are subjected to normal loads $p_3^\pm$. For simplicity[1] we omit tangent loads. The lateral surface Γ_0 is loaded by tractions T_i, $i = 1, 2, 3$. Thus we have

$$(2.1)\qquad \begin{aligned} &\sigma_{33}(x_\sigma, \pm h/2) = p_3^\pm, \quad \sigma_{\alpha 3}(x_\sigma, \pm h/2) = 0 \quad \text{on} \quad \Gamma_\pm, \\ &\sigma_{i\alpha}(s, z)\, n_\alpha = T_i(s, z), \quad i = 1, 2, 3, \quad \alpha, \sigma = 1, 2 \quad \text{on} \quad \Gamma_0. \end{aligned}$$

As the antiplane state of stress is considered, the boundary condition $(2.1)_1$ can be substituted by

$$(2.2)\qquad \sigma_{33}(x_\sigma, \pm h/2) = \pm p_3/2 \quad \text{where} \quad p_3 = p_3^+ + p_3^-.$$

Moreover, the body forces are omitted. Throughout the work the following convention is used: Greek indices take the values 1, 2; Latin indices have the range 1, 2, 3.

3. A new formulation of Reddy's model

As it has been mentioned in the Introduction, we start with reexamination of Reddy's approach [13]. The aim is to derive governing equations and natural boundary conditions in terms of Reisnerian quantities.

The following constraints are imposed:

(i) kinematical assumptions (for the fixed moment of time)

$$(3.1)\qquad u_\alpha(x_i) = -zw(x_\sigma)_{,\alpha} + z\left(1 - \frac{4}{3}\cdot\frac{z^2}{h^2}\right)\theta_\alpha(x_\sigma), \qquad u_3(x_i) = w(x_\sigma),$$

(ii) stress hypothesis: stress-strain relations in the plate obey the plane-stress assumption, hence the physical equations read

$$(3.2)\qquad \begin{aligned} &\sigma_{\alpha\beta} = A_{\alpha\beta\gamma\delta}\,\gamma_{\gamma\delta}(\mathbf{u}), \quad \sigma_{\alpha 3} = 2C_{\alpha 3\gamma 3}\cdot\gamma_{\gamma 3}(\mathbf{u}), \quad \sigma_{33} = 0, \\ &A_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - C_{\alpha\beta 33}\cdot C_{33\gamma\delta}\cdot(C_{3333})^{-1}, \end{aligned}$$

where

$$(3.3)\qquad \gamma_{ij}(\mathbf{u}) = u_{(i,j)}.$$

Elastic moduli of the plate material has been denoted by C_{ijkl}.

It is worth mentioning here that the hypothesis (i) was an inspiration for Jemielita [10] who found its generalization which enables to get rid of the statical hypothesis (3.2).

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(1) For the problem of how to generalize the kinematical hypothesis (3.1) to the case of non zero tangent loads p_α, see [9, 10].

The strains associated with the displacement field u_i given by Eq. (3.1) are

(3.4)
$$\gamma_{\alpha\beta}(\mathbf{u}) = z\cdot\varkappa_{\alpha\beta}(w)+z\left(1-\frac{4}{3}\left(\frac{z}{h}\right)^2\right)\gamma_{\alpha\beta}(\boldsymbol{\theta}),$$
$$\gamma_{\alpha 3}(\mathbf{u}) = \frac{1}{2}\left(1-4\left(\frac{z}{h}\right)^2\right)\theta_\alpha,\qquad \gamma_{33}(\mathbf{u})=0,$$

where $\varkappa_{\alpha\beta}(w) = -w_{,\alpha\beta}$. The components of stress read

(3.5)
$$\sigma_{\alpha\beta} = z\cdot A_{\alpha\beta\delta\mu}\cdot\varkappa_{\sigma\mu}(w)+z\cdot\left(1-\frac{4}{3}\frac{z^2}{h^2}\right)A_{\alpha\beta\sigma\mu}\,\gamma_{\sigma\mu}(\boldsymbol{\theta}),$$
$$\sigma_{\alpha 3} = C_{\alpha 3\gamma 3}\cdot\left(1-4\frac{z^2}{h^2}\right)\theta_\gamma,\qquad \sigma_{33}=0.$$

It is easy to note the boundary conditions $\sigma_{\alpha 3}(\pm h/2)=0$, were satisfied exactly. However, the boundary conditions $\sigma_{33}(\pm h/2) = p_3^{\pm}$ are, in fact, violated.

Let us define the averaged rotations of the cross-sections of the plate by means of Reissner's formula [11]:

(3.6)
$$\varphi_\alpha(x_\sigma) = \frac{12}{h^3}\int\limits_{-h/2}^{h/2} z\cdot u_\alpha(x_\sigma, z)\,dz.$$

By performing z-integration, we arrive at the relations

(3.7)
$$\varphi_\alpha = -w_{,\alpha}+\frac{4}{5}\theta_\alpha,$$

well-known from [9, 10].

The quantities (φ_α, w) are basic unknowns in our project. Reddy used the quantities $(\theta_\alpha - w_{,\alpha}, w)$. The angles $\theta_\alpha - w_{,\alpha}$ stand for rotations of the cross-sections measured at the mid-surface of the plate, while the quantities φ_α describe the averaged rotations, see [11] and [10, p. 492].

In order to derive equations of motion, let us start from Hamilton's principle

(3.8)
$$\int\limits_{t_1}^{t_2}(\delta W-\delta E_k-\delta L)\,dt = 0,\qquad t_2>t_1,$$

where $\delta(\cdot)$ means variation with respect to (φ_α, w) being treated as independent functions. It is imposed that $\delta\varphi_\alpha(x_\sigma, t_\mu) = \delta w(x_\sigma, t_\mu) = 0$; moreover, $\delta\varphi_\alpha$ and δw are assumed to satisfy the same boundary conditions as φ_α and w.

The strain energy, the kinetic energy and the potential of external loads are denoted by W, E_k and L. Simple calculation gives

$$\int_{t1}^{t_2} \delta E_k\, dt = \int_{t1}^{t_2} \int_{\omega} [(-I\ddot{w} + I_2 \nabla^2 \ddot{w} - I_4 \operatorname{div} \ddot{\varphi})\, \delta w + (-I_3 \ddot{\varphi}_\alpha + I_4 \ddot{w}_{,\alpha})\, \delta\varphi_\alpha]\, dx_1\, dx_2\, dt + \int_{t1}^{t_2} \int_{\gamma} (-I_2 \ddot{w}_{,n} + I_4 \ddot{\varphi}_n)\, \delta w\, ds\, dt,$$

where $w_{,n} = w_{,\alpha} n_\alpha$, $\varphi_n = \varphi_\alpha n_\alpha$; $(\ddot{\ }) = \partial^2(\cdot)/\partial t^2$ and

$$I = \int_{-h/2}^{h/2} \varrho(z)\, dz, \quad I_2 = \int_{-h/2}^{h/2} \varrho(z)\, z^2 \left(\frac{1}{4} - \frac{5}{3}\frac{z^2}{h^2}\right)^2 dz,$$

$$I_3 = \frac{25}{16} \int_{-h/2}^{h/2} \varrho(z)\, z^2 \left(1 - \frac{4}{3}\frac{z^2}{h^2}\right)^2 dz,$$

$$I_4 = \frac{5}{4} \int_{-h/2}^{h/2} \varrho(z)\, z^2 \left(\frac{1}{4} - \frac{5}{3}\frac{z^2}{h^2}\right)\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right) dz.$$

The variation of the strain energy can be written in the form

$$\delta W = \int_{\omega} [(M_{\alpha\beta} - \mathfrak{M}_{\alpha\beta})\, \varkappa_{\alpha\beta}(\delta w) + \mathfrak{M}_{\alpha\beta}\, \gamma_{\alpha\beta}(\delta\varphi) + W_\alpha (\delta w_{,\alpha} + \delta\varphi_\alpha)]\, dx_1\, dx_2,$$

where

$$M_{\alpha\beta}(x_\sigma) = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}(x_\sigma, z)\, z\, dz, \quad \mathfrak{M}_{\alpha\beta}(x_\sigma) = \frac{5}{4} \int_{-h/2}^{h/2} z\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right) \times \sigma_{\alpha\beta}(x_\sigma, z)\, dz,$$

(3.9)

$$W_\alpha(x_\sigma) = \frac{5}{4} \int_{-h/2}^{h/2} \left(1 - 4\frac{z^2}{h^2}\right) \sigma_{\alpha 3}(x_\sigma, z)\, dz.$$

The virtual work of external loads $p_3^{\pm}$ reads

$$\delta L_1 = \int_{\omega} p_3\, \delta w\, dx_1\, dx_2.$$

The virtual work of the tractions T_i subjected to Γ_0 is

$$\delta L_2 = \int_{-h/2}^{h/2} \int_{\gamma} (T_\alpha\, \delta u_\alpha + T_3\, \delta w)\, ds\, dz,$$

and can be reduced to the boundary integral

$$\delta L_2 = \int_\gamma \left[(-\tilde{M}_n + \tilde{\mathfrak{M}}_n)\, \delta w_{,n} + \left((\tilde{M}_\tau - \tilde{\mathfrak{M}}_\tau)_{,\tau} + \tilde{Q}\right) \delta w + \tilde{\mathfrak{M}}_n\, \delta\varphi_n + \right.$$
$$\left. + \tilde{\mathfrak{M}}_\tau\, \delta\varphi_\varphi\right] ds + \sum_i [\![\tilde{\mathfrak{M}}_\tau - \tilde{M}_\tau]\!]_i\, \delta w\,(s_i),$$

where s_i are points in which $m\,(s) = (\tilde{\mathfrak{M}}_\tau - \tilde{M}_\tau)\,(s)$ can have jumps

$$[\![m]\!]_i = m\,(s_i+0) - m\,(s_i-0).$$

Moreover,

$$\tilde{M}_n = \tilde{M}_\alpha\, n_\alpha, \quad \tilde{M}_\tau = \tilde{M}_\alpha\, \tau_\alpha, \quad \tilde{\mathfrak{M}}_n = \tilde{\mathfrak{M}}_\alpha\, n_\alpha, \quad \tilde{\mathfrak{M}}_\tau = \tilde{\mathfrak{M}}_\alpha\, \tau_\alpha,$$

where

$$\tilde{M}_\alpha = \int_{-h/2}^{h/2} T_\alpha\, z\, dz, \quad \tilde{\mathfrak{M}}_\alpha = \frac{5}{4} \int_{-h/2}^{h/2} z\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right) T_\alpha\, dz, \quad \tilde{Q} = \int_{-h/2}^{h/2} T_3\, dz.$$

Take notice that if $T_\alpha = zT_\alpha^0$; $T_\alpha^0 = \text{const}$, then $\tilde{\mathfrak{M}}_\alpha = h^3\, T_\alpha^0/12 = \tilde{M}_\alpha$. Thus the differences between $\tilde{\mathfrak{M}}_\alpha$ and $\tilde{M}_\alpha$ are essential if only T_α are nonlinearly distributed along the plate thickness.

The virtual work of external loads equals $\delta L = \delta L_1 + \delta L_2$. After using Hamilton's principle (3.8), one arrives at:

equations of motion

$$\begin{gathered} -\mathfrak{M}_{\alpha\beta,\beta} + W_\alpha = -I_3\, \ddot{\varphi}_\alpha + I_4\, \ddot{w}_{,\alpha}, \\ (-M_{\alpha\beta} + \mathfrak{M}_{\alpha\beta})_{,\alpha\beta} - W_{\alpha,\alpha} = p_3 - I\ddot{w} + I_2\, \nabla^2 \ddot{w} - I_4\, \mathrm{div}\, \ddot{\boldsymbol{\varphi}}, \end{gathered} \tag{3.10}$$

and boundary conditions

$$\begin{aligned} V_n &= \tilde{V}_n & &\text{or} & w &= 0, \\ \mathfrak{M}_n - M_n &= \tilde{\mathfrak{M}}_n - \tilde{M}_n & &\text{or} & w_{,n} &= 0, \\ \mathfrak{M}_n &= \tilde{\mathfrak{M}}_n & &\text{or} & \varphi_n &= 0, \\ \mathfrak{M}_\tau &= \tilde{\mathfrak{M}}_\tau & &\text{or} & \varphi_\tau &= 0, \\ [\![\mathfrak{M}_\tau - M_\tau]\!]_i &= [\![\tilde{\mathfrak{M}}_\tau - \tilde{M}_\tau]\!]_i & &\text{or} & w\,(s_i) &= 0; \end{aligned}$$

where

$$\begin{aligned} & V_n = (M_\tau - \mathfrak{M}_\tau)_{,\tau} + (M_{\alpha\beta} - \mathfrak{M}_{\alpha\beta})_{,\beta}\, n_\alpha + W_n + I_2\, \ddot{w}_{,n} - I_4\, \ddot{\varphi}_n, \\ & \tilde{V}_n = \tilde{Q} + (\tilde{M}_\tau - \tilde{\mathfrak{M}}_\tau)_{,\tau}, \\ & M_n = M_{\alpha\beta}\, n_\alpha\, n_\beta, \quad M_\tau = M_{\alpha\beta}\, n_\alpha\, \tau_\beta, \quad W_n = W_\alpha\, n_\alpha, \\ & \mathfrak{M}_n = \mathfrak{M}_{\alpha\beta}\, n_\alpha\, n_\beta, \quad \mathfrak{M}_\tau = \mathfrak{M}_{\alpha\beta}\, n_\alpha\, \tau_\beta. \end{aligned}$$

For completeness one should add appropriate initial conditions. Inserting Eqs. (3.5) into Eqs. (3.9), the constitutive equations are found:

$$M_{\alpha\beta} = \frac{h^3}{12} A_{\alpha\beta\sigma\mu} \gamma_{\sigma\mu}(\varphi),$$

(3.11) $$\mathfrak{M}_{\alpha\beta} = \kappa_1 M_{\alpha\beta} - \frac{h^3}{12} \kappa_2 A_{\alpha\beta\sigma\mu} \varkappa_{\sigma\mu}(w),$$

$$W_\alpha = \frac{5}{6} hC_{\alpha 3\gamma 3}(w_{,\gamma} + \varphi_\gamma),$$

where $\kappa_1 = 85/84$, $\kappa_2 = -1+\kappa_1 = 1/84$. The point of introducing these coefficients will be made clear further. In the isotropic case the relations (3.11) simplify as follows:

$$M_{\alpha\beta} = D\,[(1-\nu)\,\gamma_{\alpha\beta}(\boldsymbol{\varphi}) + \nu\gamma_{\sigma\sigma}(\boldsymbol{\varphi})\,\delta_{\alpha\beta}],$$

$$\mathfrak{M}_{\alpha\beta} = \kappa_1 M_{\alpha\beta} - \kappa_2 D\,[(1-\nu)\,\varkappa_{\alpha\beta}(w) + \nu\varkappa_{\sigma\sigma}(w)\,\delta_{\alpha\beta}],$$

$$W_\alpha = H\,(w_{,\alpha} + \varphi_\alpha), \qquad H = \frac{5}{6}\mu h, \qquad D = \frac{\mu h^3}{6(1-\nu)}.$$

The shearing stiffness of the isotropic plate and its bending stiffness has been denoted by H and D. In the isotropic case the equations of motion (3.10) expressed in terms of φ_α, w are

$$-\frac{1}{2}\kappa_1 D\,[(1+\nu)\operatorname{div}\boldsymbol{\varphi}_{,\alpha} + (1-\nu)\nabla^2\varphi_\alpha] - \kappa_2 D\nabla^2 w_{,\alpha} + H\,(w_{,\alpha}+\varphi_\alpha) = $$
$$= -I_3\ddot{\varphi}_\alpha + I_4\ddot{w}_{,\alpha},$$

(3.12) $$\kappa_2 D\,[\nabla^2\operatorname{div}\boldsymbol{\varphi} + \nabla^4 w] - H\,(\nabla^2 w + \operatorname{div}\boldsymbol{\varphi}) = p_3 - I\ddot{w} + I_2\nabla^2\ddot{w} - I_4\operatorname{div}\ddot{\boldsymbol{\varphi}}.$$

On performing simple rearrangements the following (partly decoupling) system can be found:

$$D\nabla^2\psi = -p_3 + I\ddot{w} - (I_2+I_4)\nabla^2\ddot{w} + (I_3+I_4)\ddot{\psi},$$

$$\nabla^2\chi - \frac{10}{\kappa_1 h^2}\chi = \frac{2}{\kappa_1}\frac{I_3}{D(1-\nu)}\ddot{\chi},$$

(3.13) $$D(-\kappa_2 l^2\nabla^6 + \nabla^4)\,w = (1-\kappa_1 l^2\nabla^2)\,p_3 + [-I + (I_2+I_4+\kappa_1 l^2 I)\nabla^2 +$$
$$-l^2(\kappa_2 I_4 + \kappa_1 I_2)\nabla^4]\,\ddot{w} + [-(I_3+I_4) + l^2(\kappa_2 I_3 + \kappa_1 I_4)\nabla^2]\,\ddot{\psi},$$

where $l^2 = D/H$, and

$$\psi = \operatorname{div}\boldsymbol{\varphi}, \qquad \chi = \varepsilon_{\alpha\beta}\,\varphi_{\alpha,\beta} = \varphi_{1,2} - \varphi_{2,1}.$$

Let us assume $\varrho(x_\sigma, z) = \varrho(x_\sigma)$, then

$$I = \varrho h, \qquad I_2 = -\frac{19}{24}\kappa_2\,\varrho h^3, \qquad (I_3, I_4) = \frac{1}{12}\varrho h^3(\kappa_1, -\kappa_2),$$

and Eq. $(3.13)_2$ takes the form

$$\nabla^2 \chi - \frac{168}{17h^2} \chi = \frac{\varrho}{\mu} \ddot{\chi},$$

somewhat different from the analogous equation following from the Reissner model, see [9]: instead of the coefficient 168/17 we then have 10. This fact suggests that by assuming $\kappa_1 = 1$ $(\kappa_2 = 0)$ the equations found in this section ought to constitute a Reissner-type model.

4. Avenue from Reddy to Reissner-type model

The hypotheses (i, ii) do not account for effects due to σ_{33} stresses and therefore the formulae obtained in the preceding section cannot imply the equations found by Reissner [1], see also [11]. However, it is possible to arrive at Kączkowski's model [9], (see also [12]) in which the σ_{33} effect has been neglected. The slight differences between Reissner's and Kączkowski's equations occur in the right-hand sides concerning external loads, hence the model proposed in [9, 12] can be treated as the Reissner-type theory.

Consider the case $\varrho(x_\sigma, z) = \varrho(x_\sigma)$. Assume the following simplifications:

$$\kappa_1 = 1, \quad \kappa_2 = 0, \quad \mathfrak{M}_\alpha = \tilde{M}_\alpha. \tag{4.1}$$

Then the set of equations (3.10) reduces to the form

$$\begin{gathered} -\mathfrak{M}_{\alpha\beta,\beta} + W_\alpha = -\frac{1}{12} \varrho h^3 \ddot{\varphi}_\alpha, \\ -W_{\alpha,\alpha} = p_3 - \varrho h \ddot{w}, \end{gathered} \tag{4.2}$$

whereas Eqs. (3.12) can be rewritten as follows:

$$\begin{gathered} -\frac{1}{2} D \left((1+\nu) \operatorname{div} \boldsymbol{\varphi}_{,\alpha} + (1-\nu) \nabla^2 \varphi_\alpha\right) + H (w_{,\alpha} + \varphi_\alpha) = -\frac{1}{12} \varrho h^3 \ddot{\varphi}_\alpha, \\ -H (\nabla^2 w + \operatorname{div} \boldsymbol{\varphi}) = p_3 - \varrho h \ddot{w}. \end{gathered} \tag{4.3}$$

Moreover, Eqs. (3.13) simplify to the form

$$\begin{gathered} D \nabla^2 \psi = -p_3 + \varrho h \ddot{w} + \frac{1}{12} \varrho h^3 \ddot{\psi}, \\ \nabla^2 \chi - \frac{10}{h} \chi = \frac{\varrho}{\mu} \ddot{\chi}, \\ \left(\nabla^2 - \frac{6}{5} \frac{\varrho}{\mu} \frac{\partial^2}{\partial t^2}\right) \left(D \nabla^2 - \frac{1}{12} \varrho h^3 \frac{\partial^2}{\partial t^2}\right) w + \varrho h \frac{\partial^2 w}{\partial t^2} = \end{gathered} \tag{4.4}$$

$$= \left(1 - \frac{h^2}{5(1-\nu)} \nabla^2 + \frac{\varrho h^2}{10\mu} \frac{\partial^2}{\partial t^2}\right) p_3,$$

well known from the cited above works by Kączkowski and Levinson. The simplification $(4.1)_1$ decreases the order of the system of equations of motion (3.10) and reduces the number of boundary conditions. Once we have $\mathfrak{M}_{\alpha\beta} = M_{\alpha\beta}$, the boundary conditions read as follows:

$$W_n = \tilde{Q} \quad \text{or} \quad w = 0,$$
$$\mathfrak{M}_n = \tilde{\mathfrak{M}}_n \quad \text{or} \quad \varphi_n = 0,$$
$$\mathfrak{M}_\tau = \tilde{\mathfrak{M}}_\tau \quad \text{or} \quad \varphi_\tau = 0.$$

5. Nonexistence of the energy-consistent Reissner-type models

The set of equations (4.2) and the relevant boundary conditions (4.5) can be arrived at by two manners: a) starting from the hypotheses (i, ii) making use of Hamilton's principle and performing appropriate simplifications (4.1) of functionals. b) by substituting the following constitutive equations:

$$M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}\, z\, dz = \frac{h^3}{12} A_{\alpha\beta\sigma\mu}\, \gamma_{\sigma\mu}(\boldsymbol{\varphi}),$$

$$Q_\alpha = \int_{-h/2}^{h/2} \sigma_{\alpha 3}\, dz = \frac{5}{6} h C_{\alpha 3\gamma 3} (w_{,\gamma} + \varphi_\gamma),$$

into classical equations of the plate motion

$$-M_{\alpha\beta,\beta} + Q_\alpha = \frac{1}{12} \varrho h^3 \ddot{\varphi}_\alpha,$$

$$-Q_{\alpha,\alpha} = p_3 - \varrho h \ddot{w},$$

following from the orthogonalization of the local equations of motion along the thickness of the plate with weighed functions 1 and z. Thus a vital question now arises: can one generalize the hypothesis (3.1) so that the equalities $\mathfrak{M}_{\alpha\beta} = M_{\alpha\beta}$ yield its simple consequence? Moreover, it is worth considering whether the correct definition of the shearing stiffnesses $H_{\alpha\beta} = 5/6\, C_{\alpha 3\beta 3}\, h$ could then be held. In fact, it is not our aim to introduce correction factors to the definition of the H tensor (as it is necessary in the Mindlin [3] and Hencky [2] approach).

Let us substitute the hypothesis (i) by

$$u_\alpha(x_i) = -zw(x_\beta)_{,\alpha} + k(z)\,\theta_\alpha(x_\beta),\qquad u_3(x_i) = w(x_\beta), \tag{5.1}$$

where $k(\cdot)$ is an unknown function. Let us introduce a new quantity $\boldsymbol{\beta} = \boldsymbol{\theta}/a$, $a \in R_+$ is arbitrary (in the Reissner model $a = 5/4$). The stresses associated with the field (5.1) are

$$\sigma_{\alpha\beta} = zA_{\alpha\beta\sigma\mu}\,\varkappa_{\sigma\mu}(w) + ak(z)\,A_{\alpha\beta\sigma\mu}\,\gamma_{\sigma\mu}(\boldsymbol{\beta}),$$

$$\sigma_{\alpha 3} = aC_{\alpha 3\gamma 3}\,k'(z)\,\beta_\gamma, \qquad \sigma_{33} = 0.$$

The variation of the strain energy equals

$$\delta W = \int_\omega [M_{\alpha\beta}\,\varkappa_{\alpha\beta}(\delta w) + \mathfrak{M}_{\alpha\beta}\,\gamma_{\alpha\beta}(\delta\boldsymbol{\beta}) + W_\alpha\,\delta\beta_\alpha]\,dx_1\,dx_2,$$

where

$$M_{\alpha\beta} = \int_{-h/2}^{h/2} z\sigma_{\alpha\beta}\,dz = \frac{h^3}{12}A_{\alpha\beta\sigma\mu}\,\varkappa_{\sigma\mu}(w) + bA_{\alpha\beta\sigma\mu}\,\gamma_{\sigma\mu}(\boldsymbol{\beta}),$$

$$\mathfrak{M}_{\alpha\beta} = a\int_{-h/2}^{h/2} k(z)\,\sigma_{\alpha\beta}\,dz = bA_{\alpha\beta\sigma\mu}\,\varkappa_{\sigma\mu}(w) + cA_{\alpha\beta\sigma\mu}\,\gamma_{\sigma\mu}(\boldsymbol{\beta}),$$

$$W_\alpha = a\int_{-h/2}^{h/2} k'(z)\,\sigma_{\alpha 3}\,dz = dC_{\alpha 3\sigma 3}\,\beta_\sigma,$$

$$b = a\int_{-h/2}^{h/2} k(z)\,z\,dz, \qquad c = a^2\int_{-h/2}^{h/2}(k(z))^2\,dz, \qquad d = a^2\int_{-h/2}^{h/2}(k'(z))^2\,dz.$$

Let $\varphi_\alpha = \beta_\alpha - w_{,\alpha}$. The expression δW assumes the form

$$\delta W = \int_\omega [(M_{\alpha\beta} - \mathfrak{M}_{\alpha\beta})\,\varkappa_{\alpha\beta}(\delta w) + \mathfrak{M}_{\alpha\beta}\,\gamma_{\alpha\beta}(\delta\boldsymbol{\varphi}) + W_\alpha(\delta w_{,\alpha} + \delta\varphi_\alpha)]\,dx_1\,dx_2.$$

It is required that: $M_{\alpha\beta} = \mathfrak{M}_{\alpha\beta}$ and $W_\alpha = \dfrac{5}{6}hC_{\alpha 3\beta 3}\,\beta_\beta$ hence the following equations should be satisfied:

$$b = h^3/12, \qquad c = h^3/12, \qquad d = 5h/6. \tag{5.2}$$

Let us write: $ak(z) = z + f(z)$, where $f(\cdot)$ is an unknown function. It is easy to ascertain that the conditions $(5.2)_{1,2}$ imply

$$\int_{-h/2}^{h/2}(f(z))^2\,dz = 0,$$

hence $f(z) = 0$ for every $z \in (-h/2, h/2)$, so $ak(z) = z$. Thus we have

$$d = h \neq 5/6h.$$

Ww see that regardless of the choice of the parameter a the conditions (5.2) cannot be simultaneously fulfilled. If the first two conditions $(5.2)_{1,2}$ are satisfied, then the hypothesis (5.1) will assume the form of HENCKY'S [2] assumption:

$$u_\alpha = z\varphi_\alpha, \qquad u_3 = w.$$

This is the only hypothesis which ensures $\mathbf{M} = \mathfrak{N}$.

6. Concluding remarks

The hypothesis (3.1) can be found in the monographs by AMBARTSUMIAN [6] and KĄCZKOWSKI [9]. As to the present author's knowledge of the literature in this subject, two attempts for improving the Reissner-type model [9] implied by this hypothesis have been undertaken. The first one was due to JEMIELITA [10] who put forward its generalization which enables to formulate a new model devoid of the fundamental contradiction: between the stress assumption about the plane stress state and the apparent presence of vertical loads p_3. Another generalization has been proposed by REDDY [13], who has found equations being energy consistent with the field (3.1). One can find it queer only twenty years after the publication of the books [6, 9] that a self-evident substitution of the hypothesis (3.1) into a variational principle of virtual work was made. Perhaps the reason was that the writers did not want to increase the order of governing plate equations. In the present computer age a finite element analysis of Reddy's model does not seem to be more complex than similar analysis of the Kirchhoff theory. Both models demand the same C^1-continuity class of approximation.

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STRESZCZENIE

NOTA O NOWYCH OSIĄGNIĘCIACH W TEORII PŁYT SPRĘŻYSTYCH ŚREDNIEJ GRUBOŚCI

Praca nawiązuje do pomysłu Reddy'ego (Int. Solids Struct., 20, No. 9/10, 1984, str. 881–896) konstrukcji energetycznie konsekwentnej teorii płyt średniej grubości opartej na hipotezie kinematycznej znanej z monografii Ambarcumjana i Kączkowskiego. Przy użyciu innych wielkości niezależnych (operujemy uśrednionymi kątami obrotu Reissnera) formułujemy równania ruchu i warunki brzegowe. Drogą uproszczeń funkcjonałów znajdujemy równania i warunki brzegowe modelu typu Reissnera znalezione przez Kączkowskiego a następnie odkryte na nowo przez Levinsona. W ostatnim punkcie pracy dowodzimy, że nie istnieje proste uogólnienie stosowanej w tej pracy hipotezy kinematycznej prowadzące do spójnej energetycznie i fizycznie poprawnej teorii płyt typu Reissnera.

РЕЗЮМЕ

ЗАМЕТКА О НОВЫХ ДОСТИЖЕНИЯХ В ТЕОРИИ УПРУГИХ ПЛИТ СРЕДНЕЙ ТОЛЩИНЫ

Работа навязывает к идеи Редди (Int. J. Solids Struct. **20**, № 9/10, 1984, стр. 881–896) построения энергетически последовательной теории плит средней толщины, опирающейся на кинематическую гипотезу, известную из монографий Амбарцумяна и Кончковского. При использовании других независимых величин (оперируем усредненными углами вращения Рейсснера) формулируем уравнения движения и граничные условия. Путем упрощений функционалов находим уравнения и граничные условия модели типа Рейсснера, найденные Кончковским, а затем вновь открытые Левинсоном. В последнем пункте работы доказываем, что не существует простое обобщение, применяемой в этой работе кинематической гипотезы, приводящие к связной энергетически и физически правильной теории плит типа Рейсснера.

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Received November 6, 1985