ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 34, 4, 445–456, 1986
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

DYNAMIC ANALYSIS OF HOIST CABLE USING TRIANGULAR SPACE-TIME ELEMENTS

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In the paper the space-time element method with triangular net was used in order to examine how to control a drum of hoist so that the load hung on the end a cable moves in a definite manner. Taking the boundary conditions in consideration this problem becomes a nonlinear geometrical one. It was shown, that the space-time element method with triangular net leads to an uncoupled equations system, also when used for nonlinear geometrical problems.

1. Formulation of the problem

The description of the motion of the cable (Fig. 1) coming from a hoist drum implies serious mathematical troubles. The increase of the length results not only from the drum rotation but from the elongation of the fibre as well. During the motion, transversal displacements coupled with longitudinal ones appear. From the technical point of view we are faced

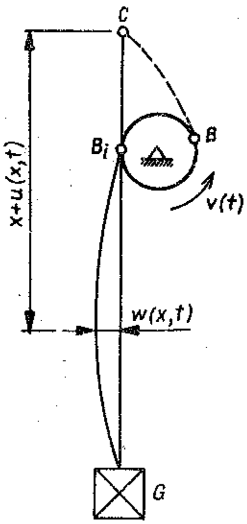

Fig. 1.

the problem of how to control a drum of hoist so that a load hung on the cable can move in a definite manner. The solution of this problem is the subject of the presented paper.

From the beginning of the movement the point B on the circumference of the drum is replaced to the position B_i. The position of the point B is not known prior to the solution of the problem. This means that the boundary condition is nonlinear. With the omission of the friction that occurs between the drum and the cable, we can present the distance $B-B_i$ as a strech-out so that the point C lying on the axis x is the reflection of the point B.

A mathematical description of the motion of the hoist cable consists of two coupled partial differential hyperbolic equations for the longitudinal and transversal displacements. The derivation and analysis of the equations are discussed in the book of SAVIN and GOROSHKO [1].

For the model of Fig. 1 the equations of motion are

$$\frac{\partial^2 u}{\partial t^2}-\frac{E}{\varrho}\frac{\partial^2 u}{\partial x^2}=g-\dot{v},$$

(1.1)

$$\frac{\partial^2 w}{\partial t^2}-\frac{E}{\varrho}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\right)=0,$$

where E — Young's modulus, ϱ — density, g — acceleration of gravity and $\dot{v}$ — time derivative of running speed of drum.

We can notice that the first equation comprises only the derivatives of longitudinal displacements; it can be solved independently of the transversal vibrations. The last ones are not discussed in the presented paper.

Let us consider two positions on the space-time plane (Fig. 2). In the former position ($t=0$) the cable has the length of $d+u_{st}$ where u_{st} is the static elongation of the fibres. The load G moves in a definite manner, for

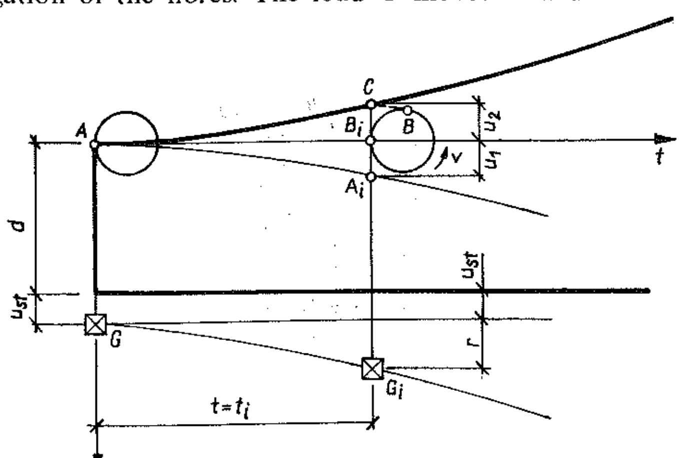

FIG. 2.

example the motion with uniform acceleration a. In the moment $t = t_i$ the load takes the position G_i. And so the line $G - G_i$ in the space-time is a given life-line of the load G. The point A in the course of motion is replaced to the position A_i, determined by the displacement u_i, which is unknown for the time being. The line $A - A_i$ is the life-line of the point A.

The curved line $A - C$ is the locus of the points C which have already been described. The position of these points is unknown and we indicate distance $C - B_i$ as the displacement u_2 which is measured from the point B.

Let us take the space-time area limited in Fig. 1 by the thick lines as a base for our analysis. On the boundary $x = d$ the displacements are known and equal with $r + u_{st}$ where r describes the motion of the load G. The boundary $t = 0$ is the line of Cauchy's initial conditions. Finally the unknown displacements on the curved line are equal to u_2. This initial boundary problem in the space-time domain belongs to the class of mixed problems composed of Cauchy's, Picard's and Darboux' basic problems and is described in KRZYŻAŃSKI'S book [2].

2. THE TRIANGULAR NET OF THE SPACE-TIME ELEMENTS

We solve the initial boundary problem as previously formulated by using the space-time finite element method with the triangular net of elements. In this case it can be convenient to introduce a metric space by multiplying the time by some scale-speed. This matter has been described in the author's paper [3]. In the wave problem this speed may be equal to the velocity of wave propagation. In the case of a longitudinal wave this velocity can be described as follows:

$$s = \sqrt{\frac{E}{\varrho}}. \tag{2.1}$$

The space-time domain has been divided into finite elements which are conformable to the characteristic lines where possible (Fig. 3). If the space-

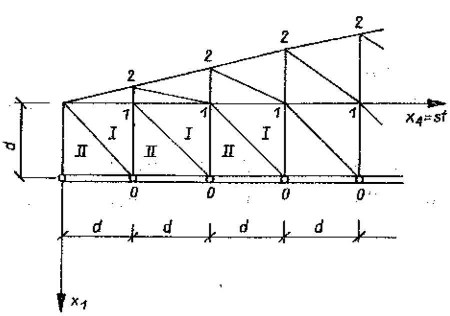

FIG. 3.

-time domain is scaled like this, then the characteristic lines are coordinate bisectrixes in the formula (2.1).

Remark that along the line $x = d$, besides the boundaries of the triangular elements, the linear mass elements appear as related to mass as

$$m = \frac{G}{g}, \tag{2.2}$$

where G is gravity of the mass.

If we introduce a metric space, then we can determine space-time stiffness matrices and load vectors for all elements like this in a standard finite element method.

The shape function for the mass element (Fig. 4) is

$$\mathbf{N} = [N_k, N_n], \tag{2.3}$$

where

$$N_i = \frac{1}{2}(1+\tau_i \tau), \qquad i = k, n,$$

$$\tau = 2\,\frac{x_4 - x_{4s}}{d}.$$

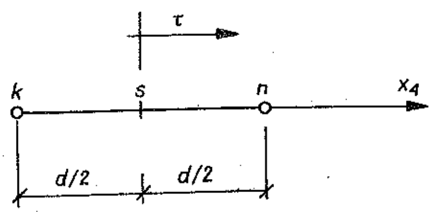

FIG. 4.

The strains vector has only one element

$$\varepsilon = \frac{du}{dx_4}. \tag{2.4}$$

In this connection the strains matrix is

$$\mathbf{B} = [B_k, B_n], \tag{2.5}$$

where

$$B_i = \frac{dN_i}{dx_4} = \frac{2}{d}\,\frac{dN_i}{d\tau} = \frac{\tau_i}{d}, \qquad i = k, n.$$

In this case the constitutive matrix contains only the mass, which is multiplied by s^2 because we introduce the metric space-time.

We also have

$$E = -ms^2. \tag{2.6}$$

The space-time stiffness matrix looks like

$$\mathbb{K} = \begin{bmatrix} K_{kk} & K_{kn} \\ K_{nk} & K_{nn} \end{bmatrix}. \tag{2.7}$$

We may describe the elements of this matrix as follows:

$$K_{ij} = \int_{-1}^{1} B_i^T EB_j \frac{d}{2} d\tau = -\frac{\tau_i \tau_j}{d} ms^2 \quad (i, j = k, n). \tag{2.8}$$

Substituting Eq. (2.2) to the formula (2.7), we finally obtain

$$\mathbb{K} = \frac{Gs^2}{dg} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{2.9}$$

The shape functions for the triangle (Fig. 5) can be written in the form

$$\mathbf{N} = \frac{1}{2\Delta} [1, x_1, x_4] \, \mathbf{W}, \tag{2.10}$$

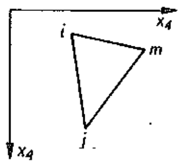

FIG. 5.

where Δ is the area of the triangle and the matrix $\mathbf{W}$ is composed of node coordinates:

$$\mathbf{W} = \begin{bmatrix} a_i & a_j & a_m \\ b_i & b_j & b_m \\ c_i & c_j & c_m \end{bmatrix}. \tag{2.11}$$

The notation is defined as follows:

$$\begin{aligned} a_i &= x_1^j x_4^m - x_1^m x_4^j, \\ b_i &= x_4^j - x_4^m, \\ c_i &= x_1^m - x_1^j, \end{aligned} \tag{2.12}$$

and the other can be obtained by a cyclical change of indices.

The strains vector has two elements:

$$\boldsymbol{\varepsilon} = \{\varepsilon_{11}, \gamma_{14}\}, \tag{2.13}$$

and depends on the displacements by the relation

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_4} \end{bmatrix} u = \partial u. \tag{2.14}$$

The constitutive matrix may be expressed as

$$\mathbf{E} = \lceil EA, \ -\varrho A s^2 \rfloor. \tag{2.15}$$

We transform the above formula by substituting Eq. (2.1).

We then obtain

$$\mathbf{E} = \lceil EA \ 1, \ -1 \rfloor. \tag{2.16}$$

The strains matrix can be given the form

$$\mathbf{B} = \partial \mathbf{N} = \frac{1}{2\Delta} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{W} = \frac{1}{2\Delta} \mathbf{FW}. \tag{2.17}$$

Let us remark that the matrix **B** is independent of any coordinates, therefore the space-time stiffness matrix can be described as

$$\mathbf{K} = \mathbf{B}^T \mathbf{E} \mathbf{B} \Delta = \frac{EA}{4\Delta} \mathbf{W}^T \mathbf{F}^T \mathbf{LFW}. \tag{2.18}$$

Substituting Eqs. (2.11), (2.12), (2.16) and (2.17) to the above formula, we find

$$\mathbf{K} = \frac{EA}{4\Delta} \begin{bmatrix} b_i^2 - c_i^2 & b_i b_j - c_i c_j & b_i b_m - c_i c_m \\ b_i b_j - c_i c_j & b_j^2 - c_j^2 & b_j b_m - c_j c_m \\ b_i b_m - c_i c_m & b_j b_m - c_j c_m & b_m^2 - c_m^2 \end{bmatrix}. \tag{2.19}$$

There are two types of triangular elements (Fig. 3). For type I we have

$$\begin{aligned} &b_i = -d, \qquad b_j = d, \qquad b_m = 0, \\ &c_i = 0, \qquad c_j = -d, \qquad c_m = d, \\ &\Delta = \frac{1}{2} d^2, \end{aligned} \tag{2.20}$$

and Eq. (2.19) has the form

$$K = \frac{EA}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}. \tag{2.21}$$

The space-time stiffness matrix for the triangle II can be written as

$$K = \frac{EA}{2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}. \tag{2.22}$$

Now let us determine the loads in the net nodes. For the bottom points of the cable we can write the following formula (Fig. 6):

$$Q = Q_G + Q_\varrho = Gd + 3 \frac{gA_\varrho}{3} \frac{d^2}{2}. \tag{2.23}$$

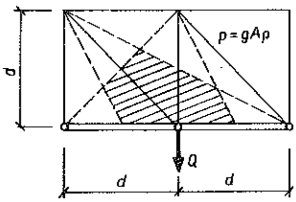

FIG. 6.

If we take into account Eq. (2.1), we can transform the above formula to

(2.24)
$$Q = Gd + \frac{EAd^2 g}{2s^2}.$$

The static elongation of the cable takes the form

(2.25)
$$u_{st} = \frac{Q}{EA} = \frac{Gd}{EA} + \frac{gd^2}{2s^2} = u_G + u_\varrho.$$

We may compose the dynamic equilibrium equations for the nodes which are on the line $x = d$.

A part of the element net with the described stiffness is presented in Fig. 7. If r indicates the known displacements of the load G, the uniform

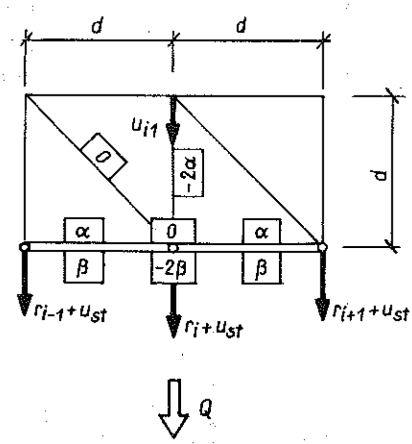

FIG. 7.

accelerated motion leads to the relation

(2.26)
$$r_i = \frac{a}{2} t_i^2 = \frac{ad^2}{2s^2} i^2.$$

All displacements of the load are increased in the quantity of static elongation of the cable described by the formula (2.25).

Other notations in Fig. 7 are as follows:

(2.27)
$$\alpha = \frac{EA}{2}, \quad \beta = \frac{Gs^2}{dg}.$$

The dynamic equilibrium equation for the i point can be described as

$$(2.28)\qquad -2\alpha u_{i1}+\alpha(2u_{st}+r_{i+1}+r_{i-1})-2\beta(r_i+u_{st})+ +\beta(2u_{st}+r_{i+1}+r_{i-1})=Q.$$

Substituting Eqs. (2.26) and (2.27) to Eq. (2.28), we obtain

$$(2.29)\qquad u_{i1}=\frac{ad^2}{2s^2}(i^2+\lambda+1),$$

where

$$(2.30)\qquad \lambda=\frac{2Gs^2}{EAdg}.$$

The above formula enables us to determine the displacements on the line $x=0$ independently of the displacements of other points of this line.

The displacements along the curve boundary can be determined basing on dynamic equilibrium conditions for the points of the line $x=0$. These equations are different from the previous ones because the dimensions of the new triangular elements are not known. However, these dimensions have to be equal to the displacements of the curve boundary. The manner of generating suitable equilibrium conditions is presented in Fig. 8.

This condition can be written in the following form:

$$(2.31)\qquad \alpha\,[u_{i+1,1}(1+\tilde{K}^{\mathrm{I}}_{23})+u_{i-1,1}(1+\tilde{K}^{\mathrm{III}}_{23})-2(r_i+u_{st})+ +u_{i1}(\tilde{K}^{\mathrm{I}}_{22}+\tilde{K}^{\mathrm{II}}_{22}+\tilde{K}^{\mathrm{III}}_{33})+u_{i-1,2}(\tilde{K}^{\mathrm{II}}_{12}+\tilde{K}^{\mathrm{III}}_{13})+ +u_{i2}(\tilde{K}^{\mathrm{I}}_{12}+\tilde{K}^{\mathrm{II}}_{23})]=Q_\varrho+\frac{1}{6}gA\varrho d(2u_{i2}+u_{i-2,2}),$$

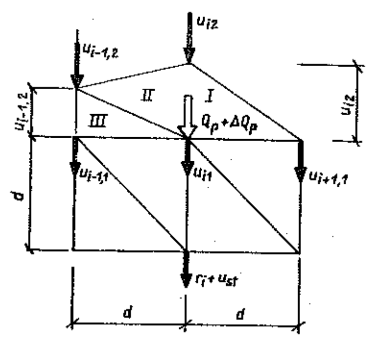

FIG. 8.

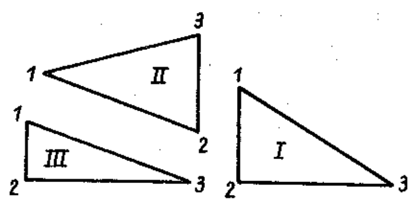

FIG. 9.

where

$$\alpha \tilde{K}^s_{ik} = K^s_{ik}, \quad s = \mathrm{I}, \mathrm{II}, \mathrm{III},$$

and node numbers in the triangles I, II, III begin from the left bottom nodes and change leftwards (Fig. 9).

The space-time stiffness matrices according to Eq. (2.19) have the form

$$\tilde{K}^{\mathrm{I}}_{23} = \frac{u_{i2}}{d}, \quad \tilde{K}^{\mathrm{I}}_{12} = -\frac{d}{u_{i2}}, \quad \tilde{K}^{\mathrm{I}}_{22} = \frac{d^2 - u_{i2}^2}{du_{i2}},$$

$$\tilde{K}^{\mathrm{II}}_{22} = \frac{d^2 - (u_{i2} - u_{i-1,2})^2}{du_{i2}}, \quad \tilde{K}^{\mathrm{II}}_{12} = \frac{u_{i2} - u_{i-1,2}}{d},$$

$$\tilde{K}^{\mathrm{II}}_{23} = -\frac{d^2 + u_{i-1,2}(u_{i2} - u_{i-1,2})}{du_{i2}}, \tag{2.32}$$

$$\tilde{K}^{\mathrm{III}}_{23} = -\tilde{K}^{\mathrm{III}}_{33} = \frac{u_{i-1,2}}{d}, \quad \tilde{K}^{\mathrm{III}}_{13} = 0,$$

Substituting the formulas (2.32) to Eq. (2.31), we finally obtain the equilibrium condition in the form of a quadratic equation with the unknown displacement u_{i2}:

$$\left(u_{i+1,1} - 2u_{i1} - \frac{2}{3} u_\varrho\right) u_{i2}^2 + \Big[d\,(u_{i+1,1} + u_{i-1,1}) - 2d\,(u_{st} + u_\varrho + r_i + d) + u_{i-1,2}\left(u_{i-1,1} + u_{i1} - \frac{1}{3} u_\varrho\right)\Big] u_{i2} + u_{i1}\,(2d^2 - u_{i-1,2}^2) = 0. \tag{2.33}$$

The above equation is the nonlinear recurrence equation because its factors also contain the displacements on the curve boundary, except for the displacements on the line $x = 0$. They are determined in the previous point. The triangular net enables us to solve only single nonlinear equations instead

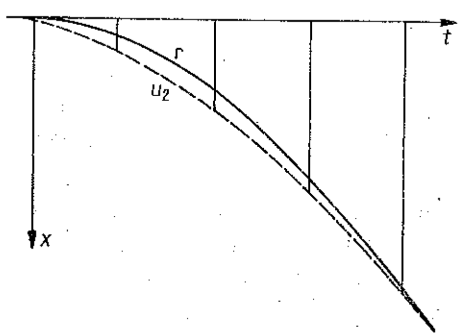

FIG. 10.

Table 1.

i	r_i	u_{i2}
10	0.45	0.57
20	1.80	1.91
30	4.04	4.15
40	7.19	7.30
50	11.23	11.34
60	16.17	16.28
70	22.00	22.11
80	28.75	28.85
90	36.37	36.47
100	44.91	45.01
110	54.35	54.45
120	64.68	64.78
130	75.91	76.01
140	88.03	88.13
150	101.07	101.16
160	114.99	115.08
164	119.34	119.43

of the whole system. The results of this evaluation are presented in Table 1. The original length of the cable is d and at the end of the calculation — $2d$.

In Fig. 10 we may see the essential character of the displacements u_2 which should be applied on the drum of the hoist so that the load G can move in the manner described by r. In Fig. 10 the scale is not observed for the displacements u_2 because the differences $u_2 - r$ are slight as is shown in Table 1.

The sample problem is solved by the following data:

$$E = 10^5 \text{ MPa}, \quad G = 100 \text{ kN}, \quad d = 120 \text{ m},$$
$$A = 8 \text{ cm}^2, \quad \varrho = 7800 \text{ kg/m}^3, \quad a = 8 \text{ m/s}^2.$$

3. Conclusions

The discretization of space-time using finite elements makes possible the direct step from the partial differential equations which describe the motion of matter continuum to algebraic equations. The efficiency of the algorithm depends in a great measure on the form of the algebraic equations matrix.

The triangular element net is very convenient because it leads to a lower triangular matrix of the equations system which does not require any inversion. It was Oden [4] who first noticed this property for the longitudinal vibrations of a straight bar. Other experiments which use these nets were described in the work of Kączkowski [5] and the author's paper [6].

In the presented treatise the triangular net was used to the analysis of a geometrical nonlinear problem. Nonlinearity of the system of equations ensues from the property of boundary conditions. The most interesting conclusion of the presented analysis is that the use of the triangular net leads to an uncoupled system of equations both in linear and nonlinear problems.

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Streszczenie

ANALIZA DYNAMICZNA LINY WYCIĄGOWEJ PRZY UŻYCIU TRÓJKĄTNYCH ELEMENTÓW CZASOPRZESTRZENNYCH

W pracy zastosowano metodę elementów czasoprzestrzennych, z siatką trójkątną, do zbadania jak należy sterować bębnem wciągarki dźwigu, aby ciężar zawieszony na końcu liny wykonywał znany ruch. Tak sformułowany problem jest geometrycznie nieliniowy ze względu na warunki brzegowe. Wykazano, że metoda czasoprzestrzennych elementów z trójkątną siatką prowadzi do rozprzęgnięcia układu równań także w zagadnieniach geometrycznie nieliniowych.

Резюме

ДИНАМИЧЕСКИЙ АНАЛИЗ КАНАТАВ КРАНОВОЙ ЛЕБЕДКИ ТРЕУГОЛЬНЫМИ ВРЕМЕНИ-ПРОСТРАНСТВЕННЫМИ ЭЛЕМЕНТАМИ

В работе применен метод времени-пространственных элементов с треугольной сеткой исследования, как следует управлять барабаном крановой лебедки, чтобы груз, подвешенный на конце каната, совершал известное движение. Так сформулированная задача является геометрически нелинейной из-за граничных условий. Показано, что метод времени-пространственных элементов с треугольной сеткой приводит к распряжению системы уравнений также в задачах геометрически нелинейных.

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Received August 2, 1985