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A CLASS OF RIGID ANNULAR DISC INCLUSION PROBLEMS INVOLVING TRANSLATIONS AND ROTATIONS

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The paper considers the class of problems related to the annular inclusions located in an elastically supported layer. The layer is considered to be transversely isotropic and the inclusion is located on the plane of symmetry. The analysis is based on the potential functions referred to the cylindrical harmonics. The governing triple integral equations are solved by employing two approximate schemes: one focusses on the asymptotic expansion technique and the second focusses on the iterative series technique. The rotational and translational stiffnesses for the embedded annular disc inclusion, related to the displacement of the annulus in one of four ways: (i) translation and (ii) rotation in a direction normal to the plane face of the annulus; (iii) rotation and (iv) translation in a direction parallel to the face, are being investigated. The solutions of the special and limiting cases are also presented. Numerical calculations are carried out with some practical materials.

1. Introduction

When foreign inclusions exist in an elastic matrix, their presence leads to intensification of stress in their vicinity, which plays a dominant role in controlling the mechanical behaviour of the whole material [1]. The degree of this stress concentration depends on all sorts of influences, such as their size, the disparity in the elastic properties, the shapes or forms of the inclusions and the matrix, their existing locations and the sort of applied loads and others. The evaluation of the effect of inclusions on the strength of materials is important in engineering technology, especially in connection with brittle fracture, in geomechanics and in the study of multiphase composite materials. In the context of composite materials reinforced disc inclusions are used to strengthen non-metallic or metallic matrices or to increase the overall stiffness of a composite.

Most of the existing analyses [2–5] of inclusions have dealt with an infinite and isotropic medium. Therefore, their solutions do not satisfactorily clarify the mechanics of inclusions near or at the surfaces of the material. Inclusions problems in anisotropic media of infinite extent have been solved in some papers [6–9].

In this article we consider a series of axisymmetric and asymmetric problems related to the annular, rigid disc inclusion embedded in a bonded contact with a transversely isotropic, elastic layer, elastically supported at both surfaces, to clarify the effect of some factors mentioned above.

The axial, rotational and translational stiffnesses for the embedded annular disc inclusion are studied.

2. Basic equations

In the absence of body forces, the displacement equations of equilibrium in three dimensions may be written as follows:

$$
\begin{aligned}
&\left[\frac{1}{2}(c_{11}-c_{12})\nabla^2+c_{44}\frac{\partial^2}{\partial z^2}\right]u_r+\frac{\partial}{\partial r}\left[\frac{1}{2}(c_{11}+c_{12})\times\right.\\
&\qquad\left.\times\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_\theta}{r\partial\theta}\right)+(c_{13}+c_{44})\frac{\partial u_z}{\partial z}\right]-\\
&\qquad-\frac{1}{2}(c_{11}-c_{12})\frac{1}{r}\left(\frac{u_r}{r}+2\frac{\partial u_\theta}{r\partial\theta}\right)=0,\\
&\left[\frac{1}{2}(c_{11}-c_{12})\nabla^2+c_{44}\frac{\partial^2}{\partial z^2}\right]u_\theta+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{2}(c_{11}+c_{12})\times\right.\\
&\qquad\left.\times\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_\theta}{r\partial\theta}\right)+(c_{13}+c_{44})\frac{\partial u_z}{\partial z}\right]-\\
&\qquad-\frac{1}{2}(c_{11}-c_{12})\frac{1}{r}\left(\frac{u_\theta}{r}-2\frac{\partial u_r}{r\partial\theta}\right)=0,\\
&\left[c_{44}\nabla^2+c_{33}\frac{\partial^2}{\partial z^2}\right]u_z+(c_{13}+c_{44})\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_\theta}{r\partial\theta}\right)=0,
\end{aligned}
\tag{2.1}
$$

where c_{ij} are the elastic constants of a transversely isotropic solid body, the elastic displacements in cylindrical polar coordinates (r, θ, z) are denoted by u_r, u_θ, u_z and

$$
\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}, \tag{2.2}
$$

is the Laplace's operator referred to the polar coordinate system.

Introducing potential functions $\varphi_i(r, \theta, z)$, $(i = 1, 2, 3)$ [10] given by

$$
u_r=\frac{\partial}{\partial r}(k\varphi_1+\varphi_2)+\frac{1}{r}\frac{\partial}{\partial\theta}\varphi_3,
$$

$$u_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}(k\varphi_1+\varphi_2)-\frac{\partial}{\partial r}\varphi_3, \tag{2.3}$$

$$u_z = \frac{\partial}{\partial z}(\varphi_1+k\varphi_2),$$

the system of equations (2.1) is replaced by the following partial differential equations

$$\left(\nabla^2+s_i^{-2}\frac{\partial^2}{\partial z^2}\right)\varphi_i(r,\theta,z)=0 \qquad (i=1,2,3), \tag{2.4}$$

provided that s_1^2 and s_2^2 are the roots of the quadratic equation with respect to s^2

$$c_{33}c_{44}s^4-[c_{11}c_{33}-c_{13}(c_{13}+2c_{44})]s^2+c_{11}c_{44}=0, \tag{2.5}$$

while

$$s_3^2=(c_{11}-c_{12})/2c_{44}, \tag{2.6}$$

and k is the function of the elastic constants and the root s_1^2, namely

$$k=(c_{33}s_1^2-c_{44})/(c_{13}+c_{44}). \tag{2.7}$$

The components of the Cauchy stress tensor σ can be expressed in terms of the derivatives of the potentials $\varphi_i\ (r,\theta,z)$ as

$$\frac{\sigma_{rr}}{G_1} = -(k+1)\frac{\partial^2}{\partial z^2}(\varphi_1+\varphi_2)-2s_3^2\left[\left\{\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\}\times\right.$$
$$\left.\times(k\varphi_1+\varphi_2)-\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial\varphi_3}{\partial\theta}\right\}\right],$$

$$\frac{\sigma_{\theta\theta}}{G_1} = -(k+1)\frac{\partial^2}{\partial z^2}(\varphi_1+\varphi_2)-2s_3^2\left[\frac{\partial^2}{\partial r^2}(k\varphi_1+\varphi_2)+\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial\varphi_3}{\partial\theta}\right\}\right], \tag{2.8}$$

$$\frac{\sigma_{zz}}{G_1} = (k+1)\frac{\partial^2}{\partial z^2}(s_1^{-2}\varphi_1+s_2^{-2}\varphi_2);$$

$$\frac{\sigma_{r\theta}}{G_1} = 2s_3^2\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial\theta}\right\}(k\varphi_1+\varphi_2)-\frac{\partial^2\varphi_3}{\partial r^2}\right]-\frac{\partial^2\varphi_3}{\partial z^2},$$

$$\frac{\sigma_{rz}}{G_1} = (k+1)\frac{\partial^2}{\partial r\,\partial z}(\varphi_1+\varphi_2)+\frac{1}{r}\frac{\partial^2\varphi_3}{\partial\theta\,\partial z}, \tag{2.9}$$

$$\frac{\sigma_{\theta z}}{G_1} = (k+1)\frac{1}{r}\frac{\partial^2}{\partial\theta\,\partial z}(\varphi_1+\varphi_2)-\frac{\partial^2\varphi_3}{\partial r\,\partial z},$$

where $G_1=c_{44}$ is the shear modulus along the axis of symmetry of the material (z-axis) which has five components of the elastic stiffness c_{ij}.

3. The annular disc inclusion problem

We consider the problem of the annular rigid disc inclusion which is embedded in bonded contact with the elastic transversely isotropic layer of height $2h$. Inner and outer radii of the annulus are denoted by a and b, respectively (Fig. 1). The inclusion is located in the middle plane of the layer and is subjected to a system of forces and couples which causes: (i) a rigid body translation δ in the z-direction, (ii) a rigid body rotation Ω about the y-axis, (iii) a rigid body rotation ω about the z-axis and (iv) a rotation free lateral translation Δ in the x-direction.

By virtue of the symmetrical geometry of the annular inclusion this problem examines, completely, the generalized displacement of the inclusion. The embedded inclusion imposes certain symmetry properties in the displacements and stresses, with respect to the plane $z = 0$. We may therefore restrict the analysis to a layer $0 \leqslant z \leqslant h$ in which the plane $z = 0$ is subjected to appropriate mixed boundary conditions and the plane $z = h$ is elastically supported at all points. When a plate is elastically supported at all points (r, θ, h), the following conditions between deflection and normal stress, radial displacement and shearing stress, circumferential displacement and shearing stress must be satisfied there:

(3.1) $$\sigma_{zz} = -k_n u_z, \quad \sigma_{zr} = -k_t u_r, \quad \sigma_{z\theta} = -k_t u_\theta,$$

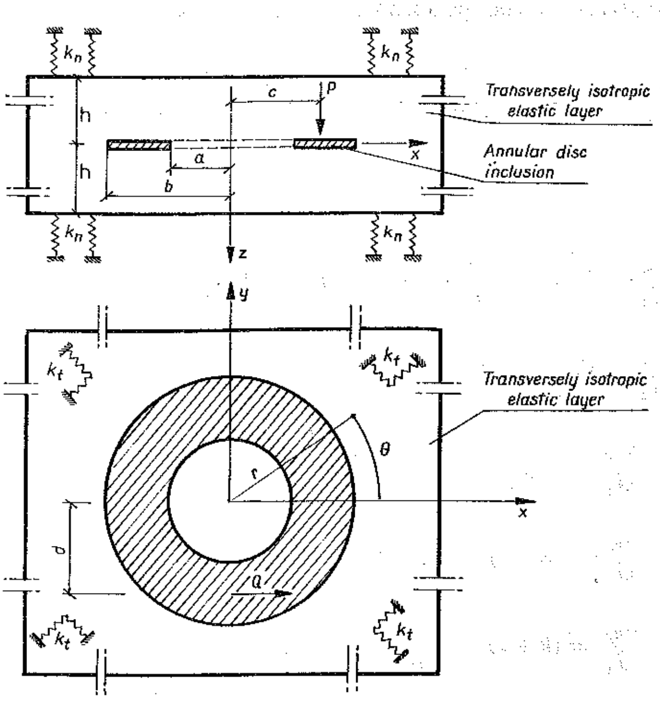

Fig. 1. Geometry of the annular disc inclusion and the resultant forces.

where k_n and k_t denote the spring stiffnesses for the lateral deflection and the displacements in the radial and circumferential directions. When $k_n = \infty$ and $k_t = 0$, the plate is supported by a rigid bases along which it may slide without friction; when both the stiffnesses are equal to infinity, the plate is clamped at all points, etc.

The relevant boundary conditions are summarized:

(i) For the rigid body translation in the z-direction

$$u_r(r, 0^+) = 0, \quad r \geqslant 0; \tag{3.2}$$

$$u_z(r, 0^+) = \delta, \quad a \leqslant r \leqslant b; \tag{3.3}$$

$$\sigma_{zz}(r, 0^+) = 0, \quad 0 \leqslant r < a, \quad b < r; \tag{3.4}$$

$$\begin{aligned} \sigma_{zz}(r, h) &= -k_n u_z(r, h), \quad r \geqslant 0; \\ \sigma_{zr}(r, h) &= -k_t u_r(r, h), \quad r \geqslant 0. \end{aligned} \tag{3.5}$$

(ii) For the rigid body rotation about the y-axis

$$\begin{aligned} u_r(r, \theta, 0^+) &= 0, \quad r \geqslant 0, \quad 0 \leqslant \theta \leqslant 2\pi; \\ u_\theta(r, \theta, 0^+) &= 0, \quad r \geqslant 0, \quad 0 \leqslant \theta \leqslant 2\pi; \end{aligned} \tag{3.6}$$

$$u_z(r, \theta, 0^+) = \Omega r \cos\theta, \quad a \leqslant r \leqslant b, \quad 0 \leqslant \theta \leqslant 2\pi; \tag{3.7}$$

$$\sigma_{zz}(r, \theta, 0^+) = 0, \quad 0 \leqslant r < a, \quad b < r, \quad 0 \leqslant \theta \leqslant 2\pi; \tag{3.8}$$

$$\begin{aligned} \sigma_{zz}(r, \theta, h) &= -k_n u_z(r, \theta, h), \\ \sigma_{zr}(r, \theta, h) &= -k_t u_r(r, \theta, h), \\ \sigma_{z\theta}(r, \theta, h) &= -k_t u_\theta(r, \theta, h), \quad r \geqslant 0, \quad 0 \leqslant \theta \leqslant 2\pi. \end{aligned} \tag{3.9}$$

(iii) For the rigid body rotation about the z-axis

$$u_\theta(r, 0^+) = \omega r, \quad a \leqslant r \leqslant b; \tag{3.10}$$

$$\sigma_{z\theta}(r, 0^+) = 0, \quad 0 \leqslant r < a, \quad b < r; \tag{3.11}$$

$$\sigma_{z\theta}(r, h) = -k_t u_\theta(r, h), \quad r \geqslant 0. \tag{3.12}$$

(iv) For the rigid body translation along the x-direction

$$u_z(r, \theta, 0^+) = 0, \quad r \geqslant 0, \quad 0 \leqslant \theta \leqslant 2\pi; \tag{3.13}$$

$$\sigma_{rz}\sin\theta + \sigma_{\theta z}\cos\theta = 0, \quad r \geqslant 0, \quad 0 \leqslant \theta \leqslant 2\pi; \tag{3.14}$$

$$\begin{aligned} u_r(r, \theta, 0^+) &= \Delta\cos\theta, \\ u_\theta(r, \theta, 0^+) &= -\Delta\sin\theta, \quad a \leqslant r \leqslant b, \quad 0 \leqslant \theta \leqslant 2\pi; \end{aligned} \tag{3.15}$$

$$\sigma_{rz}\cos\theta - \sigma_{\theta z}\sin\theta = 0, \quad 0 \leqslant r < a, \quad b < r, \quad 0 \leqslant \theta \leqslant 2\pi, \tag{3.16}$$

$$\begin{aligned} \sigma_{zz}(r,\theta,h) &= -k_n u_z(r,\theta,h),\\ \sigma_{z\theta}(r,\theta,h) &= -k_t u_\theta(r,\theta,h), \qquad r\geqslant 0, \quad 0\leqslant\theta\leqslant 2\pi.\\ \sigma_{zr}(r,\theta,h) &= -k_t u_r(r,\theta,h), \end{aligned} \tag{3.17}$$

The boundary conditions (3.14) and (3.16) relate to the traction vectors which act on the plane $z=0^+$ along the y and x direction respectively.

In addition there are the equilibrium equations of the inclusion:

$$\begin{aligned} P &= -4\pi \int_a^b r\sigma_{zz}(r,0^+)\,dr,\\ M_y &= -2\int_0^{2\pi}\int_a^b r^2\sigma_{zz}(r,\theta,0^+)\cos\theta\,dr\,d\theta,\\ M_z &= 4\pi\int_a^b r^2\sigma_{z\theta}(r,0^+)\,dr,\\ Q &= 2\int_0^{2\pi}\int_a^b r\left[\sigma_{zr}(r,\theta,0^+)\cos\theta-\sigma_{z\theta}(r,\theta,0^+)\sin\theta\right]dr\,d\theta, \end{aligned} \tag{3.18}$$

where the resulting forces and the moments, acting on the annulus, are denoted by P, Q, M_y and M_z, respectively.

The modes of deformation of the inclusion considered in this paper can be induced by the iteraction of the composite and the electromagnetic field.

4. Triple integral equation formulation

The displacement functions $\varphi_i(r,\theta,z)$ that represent the appropriate symmetries of the displacement and stress states, which are bounded at infinity, are selected in the following representations by using the techniques of the Hankel transform as:

(i) For the rigid body translation in the z-direction

$$\varphi_i(r,z) = -\frac{(-k)^{i-1}}{G_1(k^2-1)}\int_0^\infty \xi^{-1}\left[A_i^{(1)}(\xi)\,\mathrm{sh}\,s_i\xi z + A_i^{(2)}(\xi)\,\mathrm{ch}\,s_i\xi z\right]J_0(\xi r)\,d\xi, \qquad i=1,2, \tag{4.1}$$

$$\varphi_3(r,z)=0.$$

(ii) For the rigid body rotation about the y-axis

$$\varphi_i(r,\theta,z) = -\frac{(-k)^{i-1}}{G_1(k^2-1)}\cos\theta\int_0^\infty \xi^{-1}\left[B_i^{(1)}(\xi)\,\mathrm{sh}\,s_i\xi z+\right. \tag{4.2}$$

$$+B_i^{(2)}(\xi)\,\mathrm{ch}\, s_i\, \xi z]\, J_1(\xi r)\, d\xi, \qquad i=1,2.$$

$$\varphi_3(r,\theta,z)=0.$$

(iii) For the rigid body rotation about the z-axis

$$\varphi_i(r,z)=0, \qquad i=1,2$$

(4.3) $$\varphi_3(r,z)=-\frac{1}{G_{\mathrm{arg}}}\int_0^\infty \xi^{-1}\,[C^{(1)}(\xi)\,\mathrm{sh}\, s_3\,\xi z+C^{(2)}(\xi)\,\mathrm{ch}\, s_3\,\xi z]\, J_0(\xi r)\, d\xi,$$

where G_{arg} denotes the average shear modulus, namely $G_{\mathrm{arg}}=\sqrt{GG_1}$; G is the shear modulus in the isotropic-plane.

(iv) For the rigid body translation along the x-direction

(4.4)
$$\varphi_i(r,\theta,z)=\frac{(-k)^{2-i}}{G_1(k^2-1)\,s_i}\cos\theta\int_0^\infty \xi^{-1}\,[D_i^{(1)}(\xi)\,\mathrm{sh}\, s_i\,\xi z+$$
$$+D_i^{(2)}(\xi)\,\mathrm{ch}\, s_i\,\xi z]\, J_1(\xi r) d\xi, \qquad i=1,2,$$

$$\varphi_3(r,\theta,z)=-\frac{1}{G_{\mathrm{arg}}}\sin\theta\int_0^\infty \xi^{-1}\,[F^{(1)}(\xi)\,\mathrm{sh}\, s_3\,\xi z+$$
$$+F^{(2)}(\xi)\,\mathrm{ch}\, s_3\,\xi z]\, J_1(\xi r)\, d\xi.$$

In the above expressions $J_n(\xi r)$ is the Bessel function of the first kind and nth order and four unknowns $A_i^{(i)}(\xi)$ in Eqs. (4.1), four unknowns $B_i^{(i)}(\xi)$ in Eqs. (4.2), two unknowns $C^{(i)}(\xi)$ in Eqs. (4.3) and six unknowns $D_i^{(i)}(\xi)$ and $F^{(i)}(\xi)$ in Eqs. (4.4) are functions of ξ. By deriving the stresses and displacements from the displacement functions (4.1)–(4.4) and relationships (2.3), (2.8) and (2.9) and making use of the conditions (3.2), (3.5); (3.6), (3.9); (3.12); (3.13), (3.14) and (3.17) respectively, we obtain the homogeneous linear equations for the unknow functions in each problem. Their solution may be written:

(i) For translation in the z-direction

(4.5)
$$A_1^{(2)}(\xi)=A_2^{(2)}(\xi)=t_1(\xi),$$
$$A_i^{(1)}(\xi)=-t_1(\xi)\sum_{n=0}^{3}\frac{\varkappa_n\, a_n^{(i)}(\xi h)}{\varkappa_n\, m_n(\xi h)}, \qquad i=1,2.$$

(ii) For rotation about the y-axis

(4.6) $$B_i^{(i)}(\xi)\, t_1(\xi)=A_i^{(i)}(\xi)\, t_2(\xi), \qquad i=1,2.$$

(iii) For rotation about the z-axis

$$C^{(1)}(\xi) = -t_3(\xi),$$
$$C^{(2)}(\xi) = [1-H_3(\xi h)]\, t_3(\xi). \tag{4.7}$$

(iv) For translation along the x-direction

$$D_1^{(1)}(\xi) = D_2^{(1)}(\xi) = F^{(1)}(\xi) = -t_4(\xi),$$
$$F^{(2)}(\xi) = [1-H_3(\xi h)]\, t_4(\xi), \tag{4.8}$$
$$D_i^{(2)}(\xi) = t_4(\xi) \sum_{n=0}^{3} \frac{\varkappa_n d_n^{(i)}(\xi h)}{\varkappa_n \bar{m}_n(\xi h)} \quad i = 1, 2.$$

Hence, $t_i(\xi)$, $i = 1, 2, 3, 4$ is the only unknown in each problem, respectively, which can be found from mixed boundary conditions. In above expressions $a_n^{(i)}(\xi h)$, $m_n(\xi h)$, $d_n^{(i)}(\xi h)$, $\bar{m}_n(\xi h)$ and $H_3(\xi h)$ are functions involving material and geometrical parameters and $\varkappa$, $\varkappa_n$ describe the relative rigidity of foundation to layer. Their full expressions are given as:

$$a_0^{(1)}(x) = d_0^{(2)}(x) = \operatorname{ch} \alpha x - \beta^{-1}(\alpha \operatorname{ch} \beta x - 2ks_2),$$
$$a_0^{(2)}(x) = d_0^{(1)}(x) = \operatorname{ch} \alpha x + \beta^{-1}(\alpha \operatorname{ch} \beta x - 2k^{-1} s_1),$$
$$a_1^{(1)}(x) = a_2^{(2)}(x) = d_1^{(2)}(x) = d_2^{(1)}(x) = x^{-1}(\operatorname{sh} \alpha x + \operatorname{sh} \beta x),$$
$$a_1^{(2)}(x) = a_2^{(1)}(x) = d_1^{(1)}(x) = d_2^{(2)}(x) = x^{-1}(\operatorname{sh} \alpha x - \operatorname{sh} \beta x), \tag{4.9}$$
$$a_3^{(1)}(x) = d_3^{(2)}(x) = x^{-2}\beta^{-1}[(k^2 s_2 - s_1) \operatorname{ch} \alpha x + (k^2 s_2 + s_1) \operatorname{ch} \beta x - 2k^2 s_2],$$
$$a_3^{(2)}(x) = d_3^{(1)}(x) = x^{-2}\beta^{-1}[(k^2 s_2 - s_1) \operatorname{ch} \alpha x - (k^2 s_2 + s_1) \operatorname{ch} \beta x + 2s_1],$$

$$\{m_0(x), \bar{m}_0(x)\} = \operatorname{sh} \alpha x \mp \alpha\beta^{-1} \operatorname{sh} \beta x,$$
$$\{m_1(x), \bar{m}_1(x)\} = x^{-1}(\operatorname{ch} \alpha x \pm \operatorname{ch} \beta x), \tag{4.10}$$
$$\{m_2(x), \bar{m}_2(x)\} = x^{-1}(\operatorname{ch} \alpha x \mp \operatorname{ch} \beta x),$$
$$\{m_3(x), \bar{m}_3(x)\} = x^{-2}\beta^{-1}[k^2 s_2 - s_1) \operatorname{sh} \alpha x \pm (k^2 s_2 + s_1) \operatorname{sh} \beta x],$$

$$\alpha = s_1 + s_2, \quad \beta = s_1 - s_2, \tag{4.11}$$

$$H_3(x) = -2(x-\varkappa)[e^{2s_3 x}(x+\varkappa) - (x-\varkappa)]^{-1}; \quad x = \xi h, \tag{4.12}$$

$$\varkappa = k_t h G_{\text{arg}}^{-1}, \quad \varkappa_0 = 1, \quad \varkappa_1 = k_n h (G_1 C_0)^{-1},$$
$$\varkappa_2 = k_t h (G_1 C_0 s_1 s_2)^{-1}, \quad \varkappa_3 = k_n k_t h^2 [G_1(k+1)]^{-2}, \tag{4.13}$$

where

$$C_0 = (k+1)(k-1)^{-1}(s_2^{-1} - s_1^{-1}), \tag{4.14}$$

is the real-valued function of the material parameters s_1, s_2 and k. There must be imposed the remaining condition specified on $z=0^+$, inside and outside the annulus. This requires the only unknown $t_i(\xi)$, in each problem, to satisfy a set of triple integral equations.

We find for the following problems:

(i) Translation in the z-direction.

Using the displacement potentials (4.1) it is found that

$$\begin{aligned} \sigma_{zz}(r,0^+) &= -\frac{1}{k-1}\int_0^\infty \xi\,[A_1^{(2)}(\xi)-kA_2^{(2)}(\xi)]\,J_0(\xi r)\,d\xi,\\ u_z(r,0^+) &= -\frac{1}{G_1(k^2-1)}\int_0^\infty [s_1 A_1^{(1)}(\xi)-k^2 s_2 A_2^{(1)}(\xi)]\,J_0(\xi r)\,d\xi. \end{aligned} \tag{4.15}$$

Substituting the constants (4.5) into Eqs. (4.15), the normal stress and displacement are related to one function $t_1(\xi)$ as follows

$$\begin{aligned} \sigma_{zz}(r,0^+) &= \int_0^\infty \xi t_1(\xi)\,J_0(\xi r)\,d\xi,\\ u_z(r,0^+) &= -\frac{k^2 s_2-s_1}{G_1(k^2-1)}\int_0^\infty t_1(\xi)\,[1-H_1(\xi h)]\,J_0(\xi r)\,d\xi, \end{aligned} \tag{4.16}$$

where the function $H_1(x)$, $x=\xi h$, is defined as

$$H_1(x) = 1-\sum_{n=0}^{3}\frac{\varkappa_n h_n(x)}{\varkappa_n m_n(x)}, \tag{4.17}$$

with $h_n(x)$ being defined as

$$\begin{aligned} h_0(x) &= \operatorname{ch}\alpha x+[(k^2 s_2-s_1)\beta]^{-1}[(k^2 s_2+s_1)\alpha\operatorname{ch}\beta x-4ks_1 s_2],\\ h_1(x) &= x^{-1}[\operatorname{sh}\alpha x-(k^2 s_2+s_1)(k^2 s_2-s_1)^{-1}\operatorname{sh}\beta x],\\ h_2(x) &= x^{-1}[\operatorname{sh}\alpha x+(k^2 s_2+s_1)(k^2 s_2-s_1)^{-1}\operatorname{sh}\beta x],\\ h_3(x) &= x^{-2}[(k^2 s_2-s_1)\beta]^{-1}[(k^2 s_2-s_1)^2\operatorname{ch}\alpha x-\\ &\qquad -(k^2 s_2+s_1)^2\operatorname{ch}\beta x+4k^2 s_1 s_2]. \end{aligned} \tag{4.18}$$

The triple integral equations of the problem are

$$\begin{aligned} &\int_0^\infty t_1(\xi)\,[1-H_1(\xi h)]\,J_0(\xi r)\,d\xi = -C_1 G_1\,\delta, \qquad a\leqslant r\leqslant b,\\ &\int_0^\infty \xi t_1(\xi)\,J_0(\xi r)d\xi = 0, \qquad 0\leqslant r<a, \quad b<r, \end{aligned} \tag{4.19}$$

where C_1 depends on the material parameters, namely

(4.20) $$C_1 = \frac{k^2-1}{k^2 s_2 - s_1}.$$

(ii) Rotation about the y-axis

The expressions for the normal stress and displacement are

(4.21)
$$\sigma_{zz}(r,\theta,0^+) = \cos\theta \int_0^\infty \xi t_2(\xi) J_1(\xi r)\, d\xi,$$
$$u_z(r,\theta,0^+) = -\frac{k^2 s_2 - s_1}{G_1(k^2-1)} \cos\theta \int_0^\infty \xi t_2(\xi)[1-H_2(\xi h)] J_1(\xi r)\, d\xi,$$

where $H_2(\xi h) = H_1(\xi h)$. The unknown function $t_2(\xi)$ can be found from the triple integral equations.

(4.22)
$$\int_0^\infty t_2(\xi)[1-H_2(\xi h)] J_1(\xi r)\, d\xi = -C_2 G_1 \Omega r, \qquad a \leqslant r \leqslant b,$$
$$\int_0^\infty \xi t_2(\xi) J_1(\xi r)\, d\xi = 0, \qquad 0 \leqslant r < a, \qquad b < r,$$

with $C_2 = C_1$ being defined as in Eq. (4.20).

(iii) Rotation about the z-axis normal to the inclusion.

In the middle plane of the layer, where the inclusion exists, we have:

(4.23)
$$\sigma_{z\theta}(r,0^+) = \int_0^\infty \xi t_3(\xi) J_1(\xi r)\, d\xi,$$
$$u_\theta(r,0^+) = -\frac{1}{G_{\text{arg}}} \int_0^\infty t_3(\xi)[1-H_3(\xi h)] J_1(\xi r)\, d\xi,$$

so that the triple integral equations are

(4.24)
$$\int_0^\infty t_3(\xi)[1-H_3(\xi h)] J_1(\xi r)\, d\xi = -C_3 G_1 \omega r, \qquad a \leqslant r \leqslant b,$$
$$\int_0^\infty \xi t_3(\xi) J_1(\xi r)\, d\xi = 0, \qquad 0 \leqslant r < a, \qquad b < r,$$

where $C_3 = s_3$ and $C_3 G_1 = G_{\text{arg}}$; with $H_3(\xi h)$ being defined by Eq. (4.12).

(iv) Translation along the x-direction

The boundary conditions (3.13), (3.14) and (3.17) are automatically satisfied under the relations (4.8) and the potentials (4.4) lead to

$$(4.25)\quad \begin{aligned} &(\sigma_{rz}\cos\theta-\sigma_{\theta z}\sin\theta)_{(r,\theta,0^+)}=\int\limits_0^\infty \xi t_4(\xi)\,J_0(\xi r)\,d\xi,\\ &G_1\left(\frac{u_r}{\cos\theta}\mp\frac{u_\theta}{\sin\theta}\right)_{(r,\theta,0^+)}=\mp\int\limits_0^\infty t_4(\xi)\,\{(C_1 s_1 s_2)^{-1}[1-\bar H_4(\xi h)\pm\\ &\qquad\pm C_3^{-1}[1-H_3(\xi h)]\}\,\{J_0(\xi r),J_2(\xi r)\}\,d\xi,\end{aligned}$$

where the function $\bar H_4(x)$, $x=\xi h$, is defined as

$$(4.26)\qquad \bar H_4(x)=1-\sum_{n=0}^{3}\frac{\varkappa_n h_n(x)}{\varkappa_n \bar m_n(x)},$$

with $\bar m_n(x)$ and $h_n(x)$ being defined as in Eqs. (4.10) and (4.18) respectively, and $\varkappa_n$ as in Eqs. (4.13).

The mixed boundary conditions (3.15) and (3.16) reduce to set of triple integral equations for an unknown function $t_4(\xi)$

$$(4.27)\quad \begin{aligned} &\int\limits_0^\infty t_4(\xi)\,[1-H_4(\xi h)]\,J_0(\xi r)\,d\xi=-C_4 G_1 \Delta, \qquad a\leqslant r\leqslant b,\\ &\int\limits_0^\infty \xi t_4(\xi)\,J_0(\xi r)\,d\xi=0, \qquad 0\leqslant r<a, \qquad b<r,\end{aligned}$$

where

$$(4.28)\qquad H_4(\xi h)=C_5\bar H_4(\xi h)+(1-C_5)\,H_3(\xi h),$$

and

$$(4.29)\quad C_4=2\,\frac{(k^2-1)\,s_1 s_2 s_3}{k^2 s_2(s_1+s_3)-s_1(s_2+s_3)},\qquad C_5=\frac{(k^2 s_2-s_1)\,s_3}{k^2 s_2(s_1+s_3)-s_1(s_2+s_3)},$$

are the boundary function and the material paraдeters, respectively. It can be shown immediately from Eq. $(4.25)_2$, that the second boundary condition, which corresponds to Eqs. (3.15), is automatically satisfied, since if the r.h.s. of Eq. $(4.27)_1$ is constant, then

$$(4.30)\quad \int\limits_0^\infty t_4(\xi)\,\{(C_1 s_1 s_2)^{-1}[1-\bar H_4(\xi h)]-C_3^{-1}[1-H_3(\xi h)]\}\,J_2(\xi r)\,d\xi\equiv 0,$$

in the interval $a\leqslant r\leqslant b$.

For convenience, we define the nth order Hankel operator as follows:

$$(4.31)\qquad \mathscr{H}_n[f(\xi);r]=\int\limits_0^\infty \xi f(\xi)\,J_n(\xi r)\,d\xi.$$

The mixed boundary conditions reduce to sets of triple integral equations for the unknown functions $t_i(\xi)$ $(i=1,2,3,4)$.

Two types of the triple integral equations are obtained for the translation of the inclusion in the z-direction and along the x-axis we have:

$$\mathscr{H}_0\,[t_i(\xi); r] = 0, \quad 0 \leqslant r < a, \quad b < r, \tag{4.32}$$

$$\mathscr{H}_0\,[\xi^{-1} t_i(\xi)\,[1-H_i(\xi h)]; r] = -C_i\, G_1\, \{\delta \text{ or } \Delta\}, \quad a \leqslant r \leqslant b, \tag{4.33}$$

where $i = 1$ and 4. Both sets of the rotation problems are also similar:

$$\mathscr{H}_1\,[t_i(\xi); r] = 0, \quad 0 \leqslant r < a, \quad b < r, \tag{4.34}$$

$$\mathscr{H}_1\,[\xi^{-1} t_i(\xi)\,[1-H_i(\xi h)]; r] = -C_i\, G_1\, r\, \{\Omega \text{ or } \omega\}; \quad a \leqslant r \leqslant b, \tag{4.35}$$

where $i = 2$ and 3.

This completes the formulation of the annular disc inclusion problem for the elastically supported transversely isotropic layer.

The sets of the triple integral equations

$$\begin{aligned} &\mathscr{H}_n\,[t(\xi); r] = 0, \quad 0 \leqslant r < a, \quad b < r, \\ &\mathscr{H}_n\,[\xi^{-1} t(\xi)\,[1-H(\xi h)]; r] = f(r), \quad a \leqslant r \leqslant b, \end{aligned} \tag{4.36}$$

can be solved by employing a variety of approximate techniques. Detailed expositions of these methods are given by WILLIAMS [11], COOKE [12], TRANTER [13], COLLINS [14] and JAIN and KANWAL [15]. Complete accounts of these methods are also given by SNEDDON [16] and KANWAL [17].

Two types of the solutions are presented below. In one method considering the singularities of the distribution of the contact stresses at the inner and outer edges of the inclusion, we use a series representation and reduce the problem to the one of finding the solution of an infinite system of algebraic equations [19–21]. The second method of solution is based on an iterative scheme proposed by GUBENKO and MOSSAKOVSKII [18] and WILLIAMS [11].

5. THE SERIES SOLUTION

It is convenient to make the variables and parameters dimensionless so that the contact interval is $\lambda \leqslant \varrho \leqslant 1$ and corresponds to $0 \leqslant \varphi \leqslant \pi$, such that $\varrho = \lambda$ is equal to $\varphi = 0$ and $\varrho = 1$ is $\varphi = \pi$. Write

$$\begin{aligned} &r = \varrho b, \quad x = \xi b, \quad \lambda = a/b, \quad \eta = h/b, \\ &2\varrho^2 = 1+\lambda^2-(1-\lambda^2)\cos\varphi. \end{aligned} \tag{5.1}$$

We assume series expansions with unknown parameters $a_0, a_1, \ldots$ for the functions $t_i(x)$ in Eqs. (4.32), (4.33) and $b_0, b_1, \ldots$ for the functions $t_i(x)$ in Eqs. (4.34) and (4.35) as

(5.2) $$t_i(x) = -C_i G_1 \{\delta \text{ or } \Delta\} b \sum_{n=0}^{\infty} a_n Z_n(x) \quad (i = 1, 4),$$

(5.3) $$t_i(x) = C_i G_1 \{\Omega \text{ or } \omega\} \frac{1+\lambda^2}{2} b^2 \sum_{n=0}^{\infty} b_n \frac{\partial Z_n(x)}{\partial x} \quad (i = 2, 3),$$

where

(5.4) $$Z_n(x) = J_n\left[\frac{x}{2}(1-\lambda)\right] J_n\left[\frac{x}{2}(1+\lambda)\right].$$

The series representations (5.2) and (5.3) satisfy two of the three equations exactly, namely equations for stresses (4.32) and (4.34) respectively, according to the identities

(5.5)
$$\int_0^{\infty} xJ_0(x\varrho) Z_n(x)\,dx = \begin{cases} 0, & 0 \leqslant \varrho < \lambda, \quad 1 < \varrho, \\ \dfrac{4}{\pi(1-\lambda^2)} \dfrac{\cos n\varphi}{\sin\varphi}, & \lambda < \varrho < 1, \end{cases}$$

$$\int_0^{\infty} xJ_1(x\varrho) \frac{\partial Z_n(x)}{\partial x}\,dx = \begin{cases} 0, & 0 \leqslant \varrho < \lambda, \quad 1 < \varrho, \\ -\dfrac{4\varrho}{\pi(1-\lambda^2)} \dfrac{\cos n\varphi}{\sin\varphi}, & \lambda < \varrho < 1, \end{cases}$$

while the third equations, for displacements, lead to infinite systems of simultaneous equations with respect to the coefficients a_n and b_n introduced in the representations [20, 21]

(5.6) $$\sum_{n=0}^{\infty} a_n \int_0^{\infty} [1-H_i(x\eta)]\, Z_m(x)\, Z_n(x)\,dx = \delta_{0m} \quad (i = 1, 4),$$

(5.7) $$\sum_{n=0}^{\infty} b_n \int_0^{\infty} [1-H_i(x\eta)] \frac{\partial}{\partial x}[Z_m(x)] \frac{\partial}{\partial x}[Z_n(x)]\,dx = $$
$$= \delta_{0m} - \frac{1}{2}\frac{1-\lambda^2}{1+\lambda^2}\delta_{1m} \quad (i = 2, 3),$$

where $m = 0, 1, 2, \ldots$ and δ_{0m}, δ_{1m} are Kronecker's deltas. The matrix in the system (5.6) is different in both translational problems, because of the dissimilar forms of the boundary functions $H_1(x\eta)$ and $H_4(x\eta)$, so that the parameters a_n are different in both problems. A similar situation exists in the system (5.7). Using a_n and b_n and the results (5.2) and (5.3) all components of displacement and stress can be found. In particular, when the appropriate expression (5.2) or (5.3) is substituted into the first equations (4.16), (4.21), (4.23) and (4.25) and the results into equations (3.18) we find the series solutions for contact stresses and the load-displacement relationships for the following problems:

(i) Translation normal to the plane

$$\sigma_{zz}(\varrho b, 0^+) = -\frac{4G_1 C_1 \delta}{\pi(1-\lambda^2) b} \frac{1}{\sin\varphi} \sum_{n=0}^{\infty} a_n \cos n\varphi; \tag{5.8}$$

$$\lambda < \varrho < 1, \quad 0 < \varphi < \pi,$$

$$P = 4\pi b G_1 C_1 a_0. \tag{5.9}$$

(ii) Rotation about an axis parallel to the plane

$$\sigma_{zz}(\varrho b, \theta, 0^+) = -\frac{2G_1 C_2 \Omega(1+\lambda^2)}{\pi(1-\lambda^2)} \frac{\varrho\cos\theta}{\sin\varphi} \sum_{n=0}^{\infty} b_n \cos n\varphi, \tag{5.10}$$

$$\lambda < \varrho < 1, \quad 0 < \varphi < \pi,$$

$$M_y = \frac{1}{2}\pi\Omega b^3 (1+\lambda^2)^2 G_1 C_2 \left(b_0 - \frac{1}{2}\frac{1-\lambda^2}{1+\lambda^2} b_1\right). \tag{5.11}$$

(iii) Rotation about an axis normal to the plane

$$\sigma_{z\theta}(b\varrho, 0^+) = -\frac{2\omega G_{\text{arg}}(1+\lambda^2)}{\pi(1-\lambda^2)} \frac{\varrho}{\sin\varphi} \sum_{n=0}^{\infty} b_n \cos n\varphi, \tag{5.12}$$

$$\lambda < \varrho < 1, \quad 0 < \varphi < \pi,$$

$$M_z = \pi\omega G_{\text{arg}} b^3 (1+\lambda^2)^2 \left(b_0 - \frac{1}{2}\frac{1-\lambda^2}{1+\lambda^2} b_1\right). \tag{5.13}$$

(iv) Translation along the plane

$$\begin{Bmatrix} \sigma_{zr}(\varrho b, \theta, 0^+) \\ \sigma_{z\theta}(\varrho b, \theta, 0^+) \end{Bmatrix} = \frac{4G_1 C_4 \Delta}{\pi(1-\lambda^2) b} \begin{Bmatrix} -\cos\theta \\ \sin\theta \end{Bmatrix} \frac{1}{\sin\varphi} \sum_{n=0}^{\infty} a_n \cos n\varphi, \tag{5.14}$$

$$\lambda < \varrho < 1, \quad 0 < \varphi < \pi,$$

$$Q = 4\pi\Delta b G_1 C_4 a_0. \tag{5.15}$$

The compliances of the annular inclusion, i.e. the ratio of a static displacement to a static force, are determined by means of the first parameter a_0 in the set a_n made by solution of system of algebraic equations (5.6) for translational problems and by means of the first and second parameters b_0, b_1, which are the solutions of the system (5.7) for rotational problems.

The contact stresses are obtained in the form of the Fourier cosine series; are continuous at all inner points of the contact area and have a square root singularity at the ends of the contact region.

6. Iterative solution of the integral equation

To solve the triple integral equations (4.32), (4.33) and (4.34), (4.35), note that

$$t_i(\xi) = -G_1 C_i \{\delta \text{ or } \Omega \text{ or } \omega \text{ or } \Delta\} \int_a^b r g_i(r) J_n(\xi r)\, dr \qquad (n = 0, 1) \tag{6.1}$$

satisfies (4.32) and (4.34) identically and determines the contact stresses as proportional to the unknown function $g_i(r)$

$$\sigma_{i\,\text{contact}} \sim g_i(r), \qquad a < r < b. \tag{6.2}$$

Substituting (6.1) into (4.33) or (4.35), we obtain the standard Fredholm integral equation with respect to the function $g_i(r)$

$$\int_a^b g(u) K(u, r)\, du = f(r), \qquad a \leqslant r \leqslant b, \tag{6.3}$$

with the kernel

$$K(u, r) = u \int_0^\infty [1 - H_i(\xi h)] J_n(\xi u) J_n(\xi r)\, d\xi \qquad (n = 0, 1), \tag{6.4}$$

and the right-hand side $f(r) = 1$ for $n = 0$ and $i = 1, 4$; $f(r) = r$ for $n = 1$ and $i = 2, 3$, respectively. The Fredholm integral equation of the first kind (6.3) can be reduced to the solution of a system of four Fredholm integral equations of the second kind, with respect to four unknown functions, which determine $g(r)$. An iterative solution of these integral equations, which focusses on the asymptotic expansion technique, has been obtained for a small non-dimensional geometric parameter $\lambda = a/b$ and a thick layer, i.e. for a large value of parameter $\eta = h/b$.

In both translational problems and in both rotational problems the solutions $g_i(b\varrho)$ assume the form [22, 23], respectively:

$$g_i(b\varrho) = \frac{2}{\pi} \left\{ \left(1 - \frac{2}{\pi\eta} I_0 + \frac{4}{\pi^2\eta^2} I_0^2\right) \times \right.$$
$$\times \left[\frac{1}{\sqrt{1-\varrho^2}} + \frac{2}{\pi}\left(\frac{\lambda}{\sqrt{\varrho^2-\lambda^2}} - \arcsin\frac{\lambda}{\varrho}\right)\right] -$$
$$- \frac{\lambda^2}{72\pi} I_2 \left[3\left(\frac{\varrho}{\lambda}\right)^2 \arcsin\frac{\lambda}{\varrho} - \frac{3\varrho^2 - \lambda^2}{\lambda\sqrt{\varrho^2-\lambda^2}}\right] - \frac{2\lambda^3}{3\pi\varrho^3} \times$$
$$\times \left[1 - \frac{2}{\pi}\left(\frac{\varrho^2}{\sqrt{1-\varrho^2}} + \arcsin\varrho\right)\right] - \frac{2}{3\pi\eta^3} \frac{1}{\sqrt{1-\varrho^2}} \times \tag{6.5}$$

(6.5) [cont.]
$$\times\left[\frac{1-3\varrho^2}{24}I_2+\frac{12}{\pi^2}I_0^3\right]+O(\lambda^7)\Big\},\quad \lambda<\varrho<1 \quad (i=1,4),$$

(6.6)
$$g_i(b\varrho)=\frac{4}{\pi}\Big\{\frac{\varrho}{\sqrt{1-\varrho^2}}\left[1-\frac{2}{3\pi\eta^3}I_1+\frac{2}{9\pi\eta^5}\left(\varrho^2-\frac{1}{5}\right)I_2+\right.$$
$$\left.+\frac{4}{9\pi^2\eta^6}I_1^2+\frac{32}{45\pi^3\eta^3}\lambda^5\left(1+\frac{4}{7}\lambda^2\right)I_1\right]+$$
$$+\frac{8\lambda^5}{45\pi^2}\left[\left(1+\frac{2}{7}\lambda^2\right)\frac{1}{\varrho^3}\left(\frac{3-\varrho^2}{\sqrt{1-\varrho^2}}-\frac{3}{\varrho}\arcsin\varrho\right)+\right.$$
$$\left.+\frac{3\lambda^2}{14\varrho^5}\left(\frac{15-5\varrho^2-2\varrho^4}{\sqrt{1-\varrho^2}}-\frac{15}{\varrho}\arcsin\varrho\right)\right]+$$
$$+\frac{2\lambda}{3\pi}\left(\frac{3\varrho^2-\lambda^2}{\varrho\sqrt{\varrho^2-\lambda^2}}-3\frac{\varrho}{\lambda}\arcsin\frac{\lambda}{\varrho}\right)\times$$
$$\times\left(1-\frac{2}{3\pi\eta^3}I_1-\frac{2}{45\pi\eta^5}I_2+\frac{4}{9\pi^2\eta^6}I_1^2\right)+$$
$$+\frac{\lambda\varrho^2}{15\pi}\left(\frac{15\varrho^4-5\lambda^2\varrho^2-2\lambda^4}{\varrho^3\sqrt{\varrho^2-\lambda^2}}-15\frac{\varrho}{\lambda}\arcsin\frac{\lambda}{\varrho}\right)\times$$
$$\times\left(1-\frac{2}{3\pi\eta^3}I_1+\frac{98}{45\pi\eta^5}I_2\right)-\frac{4}{135\pi\eta^5}\lambda^6I_0\times$$
$$\times\left(\frac{3\varrho^2-\lambda^2}{\varrho\sqrt{\varrho^2-\lambda^2}}+3\frac{\varrho}{\lambda}\arccos\frac{\lambda}{\varrho}\right)+O(\lambda^9)\Big\},\quad \lambda<\varrho<1 \quad (i=2,3),$$

where $O(\lambda^n)$ is the Landau symbol and

(6.7)
$$I_n=-\int_0^\infty x^{2n}H(x)\,dx \quad (n=0,1,2).$$

The improper integrals (6.7) involved in the expressions (6.5) and (6.6) are difficult to evaluate analytically, but these can be evaluated either numerically or approximately. The functions $H_i(x)$ tend exponentially to zero as x tends to infinity, are continuous for any $x\in(0,\infty)$ and are bounded as x tends to zero.

The contact stresses are proportional to $g_i(b\varrho)$, namely:

(6.8)
$$\sigma_{zz}(b\varrho,0^+)=-G_1C_1\delta b^{-1}g_1(b\varrho),\quad \lambda<\varrho<1;$$

(6.9)
$$\begin{Bmatrix}\sigma_{zr}(b\varrho,\theta,0^+)\\ \sigma_{z\theta}(b\varrho,\theta,0^+)\end{Bmatrix}=G_1C_4\Delta b^{-1}\begin{Bmatrix}-\cos\theta\\ \sin\theta\end{Bmatrix}g_4(b\varrho),\quad \lambda<\varrho<1;$$

(6.10)
$$\sigma_{zz}(b\varrho,\theta,0^+)=-G_1C_2\Omega\cos\theta g_2(b\varrho),\quad \lambda<\varrho<1;$$

$$\sigma_{z\theta}(b\varrho, 0^+) = G_{\text{arg}}\, \omega g_3(b\varrho), \quad \lambda < \varrho < 1; \tag{6.11}$$

for the translation of the inclusion in the z-direction and along the x-axis and when the rigid rotation is about the y-axis and z-axis, respectively. In all the problems there are stress singularities at the boundaries of the contact region. Integrating Eqs. (6.8)–(6.11) respectively to Eqs. (3.18), the load-displacement relationships are found to be:

$$\left\{\frac{P}{\delta b G_1 C_1}; \frac{Q}{\Delta b G_1 C_4}\right\} = 8\left\{1 - \frac{4\lambda^3}{3\pi^2} - \frac{8\lambda^5}{15\pi^2} - \frac{2}{\pi\eta} I_0 \left[\left(1 - \frac{2}{\pi\eta} I_0\right)\left(1 + \frac{4}{\pi^2\eta^2} I_0^2\right) - \frac{8\lambda^3}{3\pi^2}\right] + \frac{1}{36\pi\eta^3}\left(1 - \frac{4}{\pi\eta} I_0\right) I_2 + O(\lambda^7)\right\}; \tag{6.12}$$

$$\left\{\frac{M_y}{\Omega b^3 G_1 C_2}; \frac{M_z}{2\omega b^3 G_{\text{arg}}}\right\} = \frac{16}{3}\left\{1 - \frac{16\lambda^5}{15\pi^2} - \frac{64\lambda^7}{105\pi^2} - \frac{2}{3\pi\eta^3}\left[\left(1 - \frac{2\lambda^5}{5\pi} + \frac{\lambda^7}{2\pi}\right) I_1 - \frac{1}{15\eta^2}\left(1 + \frac{9\lambda^4}{4} - \frac{\lambda^6}{16}\right) I_2 + \frac{2}{3\pi\eta^3} \times \left(1 - \frac{3\lambda^4}{4} - \frac{\lambda^6}{16}\right) I_1^2 + \frac{\lambda^5}{40\eta^2} I_0\right] + O(\lambda^9)\right\}, \quad \lambda = \frac{a}{b}, \quad \eta = \frac{h}{b}, \tag{6.13}$$

for translational and rotational problems, respectively. The r.h.s. in Eqs. (6.12) is different in both translational problems, because of the dissimilar values of the integrals I_0, I_2 in those problems. A similar situation exists in Eqs. (6.13). The ratios Q/Δ and M_z/ω are of practical interest to engineering applications and determine the translational and rotational stiffnesses for the embedded annular disc inclusion. The ratios P/δ and My/Ω, obtained under artificial assumption of a rigid thin inclusion for the bending loads, are interest from a theoretical viewpoint, although they represent the upper bounds of the axial stiffnesses of the elastic inclusion.

7. Special and limiting cases

7.1. *An infinite medium*

In this case the parameter η tends to infinity and all the functions $H_i(x\eta)$ vanish. The normal and tangential translational and rotational stiffnesses for the embedded annular disc inclusion are given by the expressions

$$\text{(7.1)} \qquad \frac{P}{\delta b G_1 C_1} = \frac{Q}{\Delta b G_1 C_4} = 4\pi a_0,$$

$$= 8\left[1 - \frac{4\lambda^3}{3\pi^2} - \frac{8\lambda^5}{15\pi^2} + O(\lambda^7)\right],$$

$$\text{(7.2)} \qquad \frac{M_y}{\Omega b^3 G_1 C_2} = \frac{M_z}{2\omega b^3 G_{\text{arg}}} = \frac{\pi}{2}(1+\lambda^2)^2\left(b_0 - \frac{1}{2}\frac{1-\lambda^2}{1+\lambda^2}b_1\right),$$

$$= \frac{16}{3}\left[1 - \frac{16\lambda^5}{15\pi^2} - \frac{64\lambda^7}{105\pi^2} + O(\lambda^9)\right],$$

as determined from the two schemes, where a_0 is the first parameter in the set a_n made by the solution of infinite set of the equations

$$\text{(7.3)} \qquad \sum_{n=0}^{\infty} a_n \int_0^{\infty} Z_m(x)\, Z_n(x)\, dx = \delta_{0m} \qquad (m = 0, 1, 2, \ldots),$$

while b_0 and b_1 are the first and second parameter in the set b_n made by solution of the equations

$$\text{(7.4)} \qquad \sum_{n=0}^{\infty} b_n \int_0^{\infty} \frac{\partial}{\partial x}[Z_m(x)] \frac{\partial}{\partial x}[Z_n(x)]\, dx = \delta_{0m} - \frac{1}{2}\frac{1-\lambda^2}{1+\lambda^2}\delta_{1m}$$

$$(m = 0, 1, 2, \ldots).$$

For the disc inclusion problem ($\lambda = 0$) in an infinite medium from the above mentioned results we obtain the exact solutions, namely:

$$a_n = -\frac{4}{\pi(4n^2-1)(1+\delta_{0n})}, \qquad b_n = 4a_n \qquad (n = 0, 1, 2, \ldots),$$

$$t_i(x) = -C_i G_1 \{\delta \text{ or } \Delta\}\, b\, \frac{2}{\pi}\frac{\sin x}{x} \qquad (i = 1, 4),$$

$$\text{(7.5)} \qquad t_i(x) = C_i G_1 \{\Omega \text{ or } \omega\}\, b^2 \frac{4}{\pi}\frac{d}{dx}\left(\frac{\sin x}{x}\right) \qquad (i = 2, 3),$$

$$g_i(b\varrho) = \frac{2}{\pi}\frac{H(1-\varrho)}{\sqrt{1-\varrho^2}} \qquad (i = 1, 4),$$

$$g_i(b\varrho) = \frac{4}{\pi}\frac{\varrho H(1-\varrho)}{\sqrt{1-\varrho^2}} \qquad (i = 2, 3),$$

where $H(1-\varrho)$ is the Heaviside's function. Both mathematical methods used here give the exact solutions. We find for the following problems:

(i) Translation normal to the plane

$$\sigma_{zz}(\varrho b, 0^+) = -\frac{2}{\pi} G_1 C_1 \frac{\delta}{b} \frac{H(1-\varrho)}{\sqrt{1-\varrho^2}},$$

(7.6) $$u_z(\varrho b, 0^+) = \delta \left\{1 - \left[1 - \frac{2}{\pi} \arcsin\left(\frac{1}{\varrho}\right)\right] H(\varrho-1)\right\}.$$

$$P = 8\delta b G_1 C_1.$$

(ii) Rotation about an axis parallel to the plane

$$\sigma_{zz}(\varrho b, \theta, 0^+) = -\frac{4}{\pi} G_1 C_2 \Omega \cos\theta \frac{\varrho H(1-\varrho)}{\sqrt{1-\varrho^2}},$$

(7.7) $$u_z(\varrho b, \theta, 0^+) = \Omega \varrho b \cos\theta \left\{1 - \left[1 - \frac{2}{\pi}\left(\arcsin\left(\frac{1}{\varrho}\right) - \frac{1}{\varrho^2}\sqrt{\varrho^2-1}\right)\right] H(\varrho-1)\right\},$$

$$M_y = \frac{16}{3} \Omega b^3 G_1 C_2.$$

(iii) Rotation about an axis normal to the plane

$$\sigma_{z\theta}(\varrho b, 0^+) = -\frac{4}{\pi} G_{\text{arg}} \omega \frac{\varrho H(1-\varrho)}{\sqrt{1-\varrho^2}},$$

(7.8) $$u_\theta(\varrho b, 0^+) = \omega \varrho b \left\{1 - \left[1 - \frac{2}{\pi}\left(\arcsin\left(\frac{1}{\varrho}\right) - \frac{1}{\varrho^2}\sqrt{\varrho^2-1}\right)\right] H(\varrho-1)\right\},$$

$$M_z = \frac{32}{3} \omega b^3 G_{\text{arg}}.$$

(iv) Translation along the plane

$$\begin{Bmatrix} \sigma_{zr}(\varrho b, \theta, 0^+) \\ \sigma_{z\theta}(\varrho b, \theta, 0^+) \end{Bmatrix} = \begin{Bmatrix} -\cos\theta \\ \sin\theta \end{Bmatrix} \frac{2}{\pi} G_1 C_4 \frac{\Delta}{b} \frac{H(1-\varrho)}{\sqrt{1-\varrho^2}},$$

$$\sigma_{zz}(\varrho b, \theta, 0^+) = \frac{2}{\pi} G_1 C_6 \frac{\Delta}{b} \frac{H(\varrho-1)}{\varrho\sqrt{\varrho^2-1}} \cos\theta,$$

$$(7.9)\qquad \begin{Bmatrix} u_r(\varrho b, \theta, 0^+) \\ u_\theta(\varrho b, \theta, 0^+) \end{Bmatrix} = \begin{Bmatrix} \cos\theta \\ -\sin\theta \end{Bmatrix} \Delta \left\{ 1 - \left[1 \mp C_7 \frac{2}{\pi} \frac{\sqrt{\varrho^2-1}}{\varrho^2} - \frac{2}{\pi} \arcsin\left(\frac{1}{\varrho}\right) \right] H(\varrho - 1) \right\},$$

$$Q = 8\Delta b G_1 C_4,$$

$$C_6 = 2\frac{(k+1)(ks_2-s_1)s_3}{k^2 s_2(s_1+s_3)-s_1(s_2+s_3)}, \qquad C_7 = \frac{k^2 s_2(s_1-s_3)-s_1(s_2-s_3)}{k^2 s_2(s_1+s_3)-s_1(s_2+s_3)},$$

where $H(1-\varrho)$ and $H(\varrho-1)$ are the step functions.

7.2. *A transversely isotropic medium with an array of annular disc inclusion*

An infinite row of parallel, periodic annular disc inclusions in the elastic transversely isotropic medium under combinated loads can also be considered on the basis of the above mentioned results. If equal and opposite loads are applied to the two neighbouring inclusions located in the planes $z = 2sh$ the problems are symmetrical or asymmetrical with respect to the planes $z = (2s+1)h$, $(s = 0, \pm 1, \pm 2, ...)$. Then, taking $\varkappa_1 \to \infty$ and $\varkappa_2 = 0$ in Eq. (4.17), we have

$$(7.10)\qquad H_1(x) = H_2(x) = [\operatorname{ch}\beta x + (k^2 s_2 + s_1)(k^2 s_2 - s_1)^{-1} \operatorname{sh}\beta x + e^{-\alpha x}] \times [\operatorname{ch}\alpha x + \operatorname{ch}\beta x]^{-1}.$$

Substituting these functions into the appropriate expressions corresponding to translation and rotation normal to the plane $z = 0$ we obtain the solutions of an array of the annular disc inclusions in the elastic medium located in the planes $z = 2sh$.

Similarly, taking $\varkappa = \infty$ in Eq. (4.12), we get

$$(7.11)\qquad H_3(x) = 2[1 + e^{2s_3 x}]^{-1},$$

and the solution for the corresponding rotation problem about the z-axis.

On the other hand, if we take $\varkappa_2 \to \infty$, $\varkappa \to \infty$ and $\varkappa_1 = 0$ in Eqs. (4.12) and (4.26) then the function in Eq. (4.28) takes the form

$$(7.12)\qquad H_4(x) = C_5 [\operatorname{ch}\beta x + (k^2 s_2 + s_1)(k^2 s_2 - s_1)^{-1} \operatorname{sh}\beta x + e^{-\alpha x}] [\operatorname{ch}\alpha x + \operatorname{ch}\beta x]^{-1} + 2(1 - C_5)[1 + e^{2s_3 x}]^{-1},$$

and determines the solution for the rigid body translation along the x-direction, when two of the array inclusion in the elastic medium translate in the opposite directions but parallel to its planes.

7.3. *Isotropic medium*

All the results obtained in this paper can also be applied for completely isotropic bodies, provided that $s_1 \to 1$, $s_2 = 1$, i.e. $\alpha = 2$, $\beta \to 0$ and $s_3 \to 1$, $k \to 1$. By the evaluation of the limits by means of the de L'Hospital's rule in the above mentioned expressions and making use of the limiting identities

$$(7.13) \qquad \frac{dk(s_1)}{ds_1} = 2(1-\nu), \qquad \frac{ds_3(s_1)}{ds_1} = \frac{1-\nu}{2(1-2\nu)}, \quad \lim_{\beta \to 0} \frac{\operatorname{sh} \beta x}{\beta} = x,$$

we can obtain without any difficulties all the functions $m_n(x)$, $\bar{m}_n(x)$, $h_n(x)$, which in the isotropic case are independent of the material properties of the medium, the relative rigidities of foundation to layer $\varkappa$ and $\varkappa_n$ $(n = 1, 2, 3)$ and the material parameters C_i $(i = 0, 1, 2, 3, 4, 5, 6, 7)$, which assume the values

$$(7.14) \qquad \begin{gathered} C_0 = \frac{1}{1-\nu}, \quad C_1 = C_2 = 4\,\frac{1-\nu}{3-4\nu}, \quad C_3 = 1, \quad C_4 = 8\,\frac{1-\nu}{7-8\nu}, \\ C_5 = \frac{3-4\nu}{7-8\nu}, \quad C_6 = 4\,\frac{1-2\nu}{7-8\nu}, \quad C_7 = \frac{1}{7-8\nu}, \end{gathered}$$

when the layer is an isotropic medium with the Poisson's ratio ν.

7.4. *Related problems*

The presented results may be applied to some bonded contact problems with the annulus and layer elastically supported on the lower surface, provided that outside the contact region elastic layer is bonded to a rigid diaphragm for translation and rotation normal to the plane and is stress free for rotation about an axis perpendicular to the plane. In these cases the load-displacement relationships may be obtained immediately, by multiplying the right-hand sides of obtained results by one half.

8. NUMERICAL RESULTS

The translational and rotational stiffnesses: $P/b\delta$, $Q/b\Delta$, $M_y/b^3\,\Omega$ and $M_z/b^3\,\omega$ will be presented numerically in this section for the three cases of $\lambda = 0.25$, $\lambda = 0.50$, $\lambda = 0.75$ and for five different materials such as cadmium (denoted symbolically as C) and magnesium (M) crystals, fiber-reinforced composite materials with the fiber direction normal to the inclusion, E glass-epoxy (E G–E), graphite-epoxy (G–E) and comparative isotropic material (I). The values of the tensorial elastic constants of the materials used in the numerical computations are shown in Table 1. The results for an infinite medium are shown in Table 2 and Figs. 2 and 3. From these numerical results it is seen that:

Table 1. Values of the elastic constants c_{ij} in units of 10^4 M Pa

	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}
Cadmium	11.00	4.04	3.83	4.69	1.56
Magnesium	5.97	2.62	2.17	6.17	1.64
E glass-epoxy	1.51	0.61	0.52	4.68	0.47
Graphite-epoxy	0.82	0.26	0.32	8.68	0.41
Isotropy ($\nu = 0.30$)	3.50	1.50	1.50	3.50	1.00

Table 2. Values of roots of equations (7.3) (a_0) and (7.4) (b_0 and b_1).

	$\lambda = 0.25$	$\lambda = 0.50$	$\lambda = 0.75$
a_0	0.635 380 5	0.624 670 2	0.586 628 3
b_0	2.345 493 3	1.856 885 9	1.267 678 3
b_1	−1.502 717 0	−1.025 632 9	−0.489 980 0

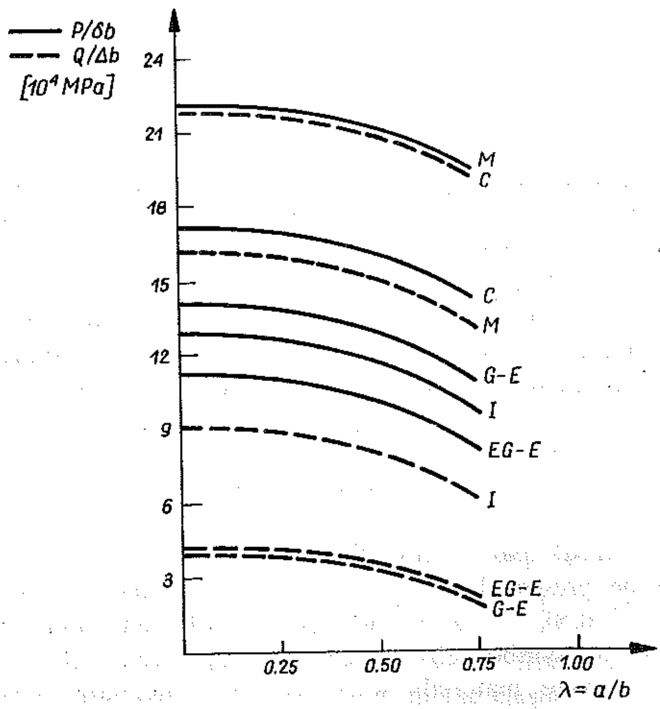

FIG. 2. The force-displacement relationships for the embedded annular inclusion in different materials.

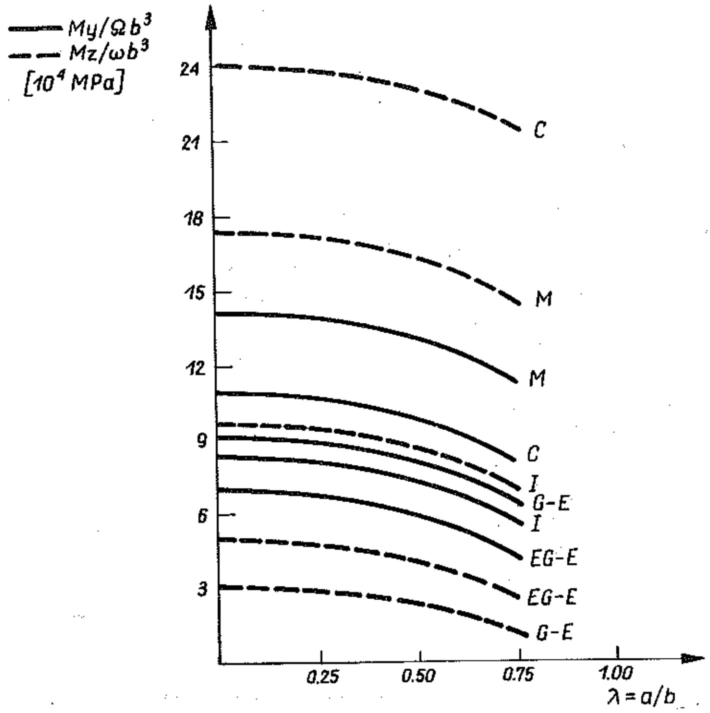

FIG. 3. The moment-rotation relationships for the embedded annular inclusion in different materials.

(a) The isotropic values lie between those of typical glass-epoxy and graphite-epoxy composites for translation and rotation normal to the inclusion.

(b) The stiffnesses for translation and rotation of the annulus in the direction normal to the fibres of the composite materials are considerably smaller than those in the fiber direction.

(c) Almost all stiffnesses for metallic substances C and M are considerably larger than those for the other presented materials. Here the effect of material dissimilarity is also apparent.

(d) The stiffness of the system is nearly constant when $a/b < 0.25$, decreases slowly when $0.25 < a/b < 0.5$ and quickly when $a/b > 0.5$.

The stiffnesses in all cases $\lambda = 0.25$, 0.50, 0.75 agreed almost exactly in the approximate and series solutions.

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Streszczenie

ZAGADNIENIE SZTYWNEGO PIERŚCIENIOWEGO WTRĄCENIA Z UWZGLĘDNIENIEM PRZESUNIĘCIA I OBROTU

Rozpatrzono zagadnienie sztywnego wtrącenia o kształcie pierścienia, umieszczonego w płaszczyźnie środkowej warstwy poprzecznie izotropowej i połączonego z nią. Warstwa jest podparta sprężyście na obu płaszczyznach. Do analizy użyto potencjały przemieszczenia.

W wyniku działania na pierścień układu sił i momentów doznaje on przesunięcia i obrotu w kierunku prostopadłym do jego podstawy oraz obrotu i przesunięcia w kierunku równoległym. Potrójne równania całkowe zagadnień rozwiązano za pomocą dwóch metod; iteracyjnej, opartej na technice asymptotycznego rozkładu względem małego parametru — stosunku promieni pierścienia i analitycznej, prowadzącej do rozwiązania nieskończonego układu równań algebraicznych liniowych. Określono sztywności pierścienia umieszczonego w sprężystej warstwie odpowiadające analizowanym czterem przypadkom przemieszczeń i obrotów. Pokazano także rozwiązania przypadków szczególnych i granicznych. Wyniki obliczeń numerycznych odniesione są do pewnych praktycznych materiałów.

Резюме

ЗАДАЧА ЖЕСТКОГО КОЛЬЦЕВОГО ВКЛЮЧЕНИЯ С УЧЕТОМ ПЕРЕМЕЩЕНИЯ И ВРАЩЕНИЯ

Рассмотрена задача жесткого включения о форме кольца, помещенного в срединной плоскости поперечно изотропного слоя, и соединенного с ним. Слой опирается упруго на обоих плоскостях. Для анализа использованы потенциалы перемещения. В результате действия на кольцо системы сил и моментов, оно испытывает перемещение и вращение в направлении перпендикулярном к его основе, а также вращение и перемещение в параллельном направлении. Тройные интегральные уравнения задач решены при помощи двух методов: итерационного, опирающегося на технику асимптотического разложения по отношению к малому параметру — отношения раддиусов кольца и аналитического, приводящего к решению бесконечной системы линейных алгебраических уравнений. Определены жесткости кольца, помещенного в упругом слое, отвечающие анализированным четырам случаям перемещений и вращений. Показаны тоже решения частных и предельных случаев. Результаты численных расчетов отнесены к некоторым практическим материалам.

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