

SLOW NONISOTHERMIC GAS FLOW IN A PLANE CHANNEL WITH POROUS WALLS

S. M A Y (WARSZAWA)

A slow flow of viscous and heat conducting gas in a plane channel with porous walls maintained at different temperatures is considered. The flow is generated by a uniform motion of a wall, and by uniform injection of gas at one wall and uniform removal at the other. The flow equations are simplified by neglecting compressibility of gas but not thermal expansion. In the general case, a solution is found in an integral form but for the special case of viscosity and heat conductivity coefficients proportional to a power of temperature, the closed form of solution is found. For large Peclet numbers corresponding to transversal gas motion, a large gradient layer appears at the wall where the gas is removed.

1. INTRODUCTION

One of the simplest solutions of Navier-Stokes equations is the well-known Couette solution describing a flow with a linear velocity profile. There are many interesting generalizations of the Couette solution in which compressibility of a fluid, thermal effects, and injection as well as removal of the fluid through porous walls are considered. Thermal effects in an inviscid fluid were considered by SCHLICHTING [1] and GROFF [2]. The compressible fluid flow was studied by ILLINGWORTH [3], TARAPOV [3], and MORGAN [5]. The temperature distribution in inviscid fluid flow was examined by ECKERT [6]. The more general case of flow of compressible, viscous, and heat conducting fluid in a channel with porous walls at different temperatures was analysed by BANSAL and JAIN [7]. However, due to the complex form of the equations involved, the method of successive approximations was applied and only the first approximation was found. In this paper a similar problem as in [7] is considered, but the assumption is made that the flow velocity is much lower than the velocity of sound. In such conditions the fluid is almost incompressible, and the only factor responsible for variations of density is the thermal expansion due to the temperature difference between the channel walls. The assumption of small velocity simplifies the equations and allows to obtain the solution of the problem in a closed form.

2. EQUATIONS

We consider a stationary flow of a viscous and heat conducting gas in a plane infinite channel whose porous walls 0 and 1 are at distance d apart (Fig. 1). The coordinates x, y and the corresponding velocity components u, v are shown in the figure. The indices 0 and 1 refer to the flow parameters at the corresponding walls. The flow is generated by a uniform

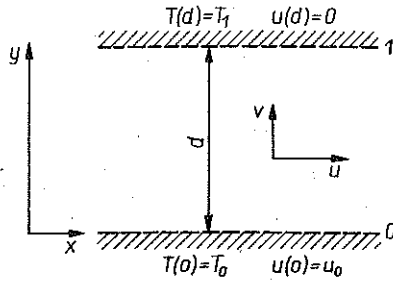


FIG. 1.

motion of the wall 0 in the x -direction with the velocity u_0 , and the gas injection at one wall and removal at the other, with the constant mass flux intensity $i = \rho_0 v_0 = \rho_1 v_1$; ρ being the gas density. The temperatures of both walls are different, and $T_1 > T_0$. It is assumed that the specific heat of gas is constant, and gas obeys the Clapeyron equation

$$(2.1) \quad p = R\rho T,$$

where p is the pressure and R — the gas constant. The coefficients of viscosity μ and heat conductivity λ are power functions of temperature with an exponent n

$$(2.2) \quad \frac{\mu}{\mu_0} = \frac{\lambda}{\lambda_0} = \left(\frac{T}{T_0} \right)^n.$$

The flow of gas is governed by the Navier-Stokes equations (2.3):

$$(2.3) \quad \begin{aligned} \rho v &= i, \\ \rho v \frac{du}{dy} &= \frac{d}{dy} \left(\mu \frac{du}{dy} \right), \\ \rho v \frac{dv}{dy} &= -\frac{dp}{dy} + \frac{4}{3} \frac{d}{dy} \left(\mu \frac{dv}{dy} \right), \\ \rho v c_p \frac{dT}{dy} &= v \frac{dp}{dy} + \frac{d}{dy} \left(\lambda \frac{dT}{dy} \right) + \mu \left[\left(\frac{du}{dy} \right)^2 + \frac{4}{3} \left(\frac{dv}{dy} \right)^2 \right]. \end{aligned}$$

The system (2.3) may be essentially simplified when the pressure variations are small. Two following assumptions about the pressure field are made:

1) the amplitude of pressure variations is small as compared with the value of pressure

$$(2.4) \quad \frac{\Delta p}{p} \ll 1;$$

2) the relative variations of pressure are small with regard to the relative variations of temperature

$$(2.5) \quad \frac{\Delta p}{p} \ll \frac{\Delta T}{T}.$$

The assumption (2.4) is valid for slow flows when the flow velocity is much lower than the velocity of sound. Due to the inequality (2.5) the density variations are caused by nonisothermal effects but not by compressibility of gas. In such a case the density is inversely proportional to the temperature. The assumptions (2.4) and (2.5) make possible to neglect two terms in the energy equation: the term proportional to the pressure derivative and the term of viscous dissipation.

Let us consider the dimensionless parameters

$$(2.6) \quad Y = \frac{y}{d}, \quad U = \frac{u}{u_0}, \quad V = \frac{v}{v_0}, \quad \theta = \frac{T}{T_0},$$

$$M = \frac{\mu}{\mu_0}, \quad L = \frac{\lambda}{\lambda_0}, \quad P = \frac{p - p_0}{\rho_0 v_0^2}, \quad Pe = \frac{c_p i d}{\lambda_0}, \quad Pr = \frac{\mu_0 c_p}{\lambda_0},$$

where Pe and Pr are the Peclet number and the Prandtl number. Making use of the above simplifying assumptions and introducing the dimensionless parameters (2.6), we transform Eqs. (2.3) into the following form:

$$V = \theta,$$

$$(2.7) \quad Pe \frac{dU}{dY} = Pr \frac{d}{dy} \left(M \frac{dU}{dY} \right),$$

$$Pe \frac{dV}{dY} = -Pe \frac{dP}{dY} + \frac{4}{3} Pr \frac{d}{dY} \left(M \frac{dV}{dY} \right),$$

$$Pe \frac{d\theta}{dY} = \frac{d}{dY} \left(L \frac{d\theta}{dY} \right).$$

In Eqs. (2.7) the assumption (2.2) about the form of dissipative coefficients has not been employed as yet. The boundary conditions for Eqs. (2.7) are

$$(2.8) \quad \begin{aligned} U(0) &= 1, \\ U(1) &= 0, \\ \theta(0) &= 1, \\ \theta(1) &= \theta_1, \\ P(0) &= 0. \end{aligned}$$

3. SOLUTION

While discussing the solution of the system (2.7) it is convenient to introduce the Stanton number St , defined here as the ratio of the total energy flux Q to its convective component Q_{c0} at the wall 0

$$(3.1) \quad St = \frac{Q}{Q_{c0}} = \frac{Q_{h0} + Q_{c0}}{Q_{c0}},$$

where Q_h is the part of the energy flux due to the molecular heat conduction. Both components of energy flux at the wall 0 are given by Eqs. (3.2)

$$(3.2) \quad \begin{aligned} Q_{h0} &= -\frac{\lambda_0 T_0 \theta_{Y0}}{d}, \\ Q_{c0} &= c_p T_0 i, \end{aligned}$$

where θ_{Y0} is a dimensionless temperature derivative at the wall 0. The heat flux Q_{h0} is always negative because the wall 1 is assumed to have a higher temperature than the wall 0; however, the sign of Q_{c0} depends on the direction of transversal flow. For $i > 0$ there is $Q_c > 0$, hence the Stanton number (3.1) is less than 1

$$(3.3) \quad St < 1.$$

On the other side, for $i < 0$ there is $Q/Q_c > 1$, everywhere in the channel cross-section, and in particular at the wall 1. Hence the Stanton number is in this case larger than θ_1

$$(3.4) \quad St = \frac{Q}{Q_{c0}} = \frac{Q}{Q_{c1}} \theta_1 > \theta_1.$$

As a consequence of the inequalities (3.3) and (3.4) the Stanton number lies outside the interval $(1, \theta_1)$, whatever the mass flux may be.

To obtain the solution of the system (2.7) in an integral form, one does not need to specify the functions $L(\theta)$ and $M(\theta)$. The energy equation (2.7)₄

can be integrated independently of the other equations (2.7). Upon the first integration one gets

$$(3.5) \quad \frac{d\theta}{dY} = \frac{\text{Pe}(\theta-1) + \theta_{Y0}}{L(\theta)},$$

where the dimensionless temperature derivative θ_{Y0} is a constant of integration. From Eqs. (3.1), (3.2) and (3.5) the temperature derivatives θ_{Y0} , θ_{Y1} at both walls can be expressed by the Stanton number

$$(3.6) \quad \begin{aligned} \theta_{Y0} &= \text{Pe}(1 - \text{St}), \\ \theta_{Y1} &= \frac{\text{Pe}(\theta_1 - \text{St})}{L(\theta_1)}. \end{aligned}$$

To solve the system (2.7) it is convenient to consider θ as an independent variable, and Y , U , P as unknown functions. Instead of Eqs. (2.7), the system (3.7) of linear equations is then obtained (for $\text{Pe} \neq 0$):

$$(3.7) \quad \begin{aligned} \text{Pe} \frac{dY}{d\theta} &= \frac{L(\theta)}{\theta - \text{St}}, \\ \frac{dU}{d\theta} &= \text{Pr} \frac{d}{d\theta} \left[\frac{M(\theta)}{L(\theta)} (\theta - \text{St}) \frac{dU}{d\theta} \right], \\ \frac{dP}{d\theta} &= \frac{4}{3} \text{Pr} \frac{d}{d\theta} \left[\frac{M(\theta)}{L(\theta)} (\theta - \text{St}) \right] - 1, \end{aligned}$$

in which the energy equation is already once integrated. The solution of Eqs. (3.7) for arbitrary $L(\theta)$ and $M(\theta)$ is

$$(3.8) \quad \begin{aligned} Y &= \frac{1}{\text{Pe}} \int_1^\theta \frac{L(\theta) d\theta}{\theta - \text{St}}, \\ U &= \exp \left[\frac{1}{\text{Pr}} \int_1^\theta \frac{L(\theta)}{M(\theta)} \frac{d\theta}{\theta - \text{St}} \right], \\ P &= \left(\frac{4}{3} \frac{M(\theta)}{L(\theta)} \text{Pr} - 1 \right) (\theta - 1) + \frac{4}{3} \text{Pr} (1 - \text{St}) \left(1 - \frac{M(\theta)}{L(\theta)} \right). \end{aligned}$$

If Eq. (2.2) is valid, then $L(\theta) = M(\theta) = \theta^n$. For this special case the solution (3.8) becomes

$$Y = \frac{I}{\text{Pe}},$$

$$(3.9) \quad U = \frac{|\theta - \text{St}|^{1/\text{Pr}} - |\theta_1 - \text{St}|^{1/\text{Pr}}}{|1 - \text{St}|^{1/\text{Pr}} - |\theta_1 - \text{St}|^{1/\text{Pr}}},$$

$$P = \left(\frac{4}{3} \text{Pr} - 1 \right) (\theta - 1),$$

where

$$(3.10) \quad I = \int_1^\theta \frac{\bar{\theta}^n d\bar{\theta}}{\bar{\theta} - \text{St}}.$$

Integrating by parts, one can express integral I by an integral K with the denominators exponent larger by one

$$(3.11) \quad I = \frac{1}{n} \left(\frac{\theta^{n+1}}{\theta - \text{St}} - \frac{1}{1 - \text{St}} \right) + \frac{\text{St}}{n} K,$$

$$(3.12) \quad K = \int_1^\theta \frac{\bar{\theta}^n d\bar{\theta}}{(\bar{\theta} - \text{St})^2}.$$

An integral like K can be expressed by the hypergeometric functions if its denominators exponent is larger than $n+1$ and the Stanton number lies outside the interval of integration. The second condition is fulfilled in our problem, while the first is generally valid for gases. If it is not, integrating by parts can be performed over again until the denominator's exponent becomes larger than $n+1$. The corresponding formulas for K are (see [9, 10])

$$(3.13) \quad K = \frac{1}{1-n} \left[{}_2F_1 \left(2, 1-n; 2-n; \text{St} \right) - \theta^{n-1} {}_2F_1 \left(2, 1-n; 2-n; \frac{\text{St}}{\theta} \right) \right],$$

for $\text{St} < 1$ and

$$(3.14) \quad K = \frac{1}{(1+n)\text{St}^2} \left[\theta^{1+n} {}_2F_1 \left(2, 1+n; 2+n; \frac{\theta}{\text{St}} \right) - {}_2F_1 \left(2, 1+n; 2+n; \frac{1}{\text{St}} \right) \right],$$

for $\text{St} > \theta_1$ and $\text{St} < 0$. For negative St the formulas (3.13) and (3.14) are equivalent, each being an analytical continuation of the other (see [8]). For rational n the integral I can be expressed by elementary functions, although in most cases the corresponding formulas are rather complicated [9]. However, for some values of n simple formulas for I are obtained. For example, for $n = 1$ one gets

$$(3.15) \quad I = \theta - 1 + \text{St} \ln \frac{\theta - \text{St}}{1 - \text{St}}.$$

The Stanton number is an important overall characteristic of the solution. It is a function of the Peclet number Pe , the viscosity exponent n , and the temperature ratio θ_1 . The implicit formula for St may be obtained by inserting $Y = 1$, $\theta = \theta_1$ to (3.9)₁

$$(3.16) \quad I(\theta_1, n, St) = Pe.$$

The solution (3.9) is valid for $Pe \neq 0$. In the opposite case $Pe = 0$ the solution can be obtained directly from Eqs. (2.7). It has a simple known form (3.17)

$$(3.17) \quad \begin{aligned} Y &= \frac{\theta^{1+n} - 1}{\theta_1^{1+n} - 1}, \\ U &= \frac{\theta - \theta_1}{1 - \theta_1}, \\ P &= 0. \end{aligned}$$

4. ANALYSIS OF THE CASE $n = Pr = 1$

For the sake of simplicity, a more detailed analysis of the solution (3.9) is given for the special case $n = Pr = 1$. For real gases the values of n and Pr differ not very much from 1, and one may expect that these results should be representative at least in a qualitative sense. Besides n and Pr the solution (3.9) depends on 2 other independent parameters. We choose Pe and $\Delta\theta = \theta_1 - 1$ as these parameters. Then St is given by the implicit relation (3.16) shown graphically in Fig. 2. The Peclet number is a dimensionless measure of an injected mass flux. It is positive if gas is injected at the cold wall, and negative in the reverse case. For $Pe \rightarrow -\infty$ $St \rightarrow 1$, and for $Pe \rightarrow \infty$ $St \rightarrow \theta_1$, i.e. the heat flux at the wall where gas is removed tends to 0, and the whole energy is there transmitted by convective way. On the other side, for $Pe \rightarrow 0$ when the flow pattern is very close to that with impermeable walls, the convective component of the energy flux tends to 0, and consequently the Stanton number tends to $\pm\infty$. There is a particular intermediate case $Pe = \Delta\theta$. In this case $St = 0$, so that the convective energy flux and the heat flux counterbalance each to another, and the resulting energy flux is 0.

To examine the temperature profiles across the channel, let us introduce, besides θ , another dimensionless temperature t , varying in the interval (0, 1):

$$t = \frac{T - T_0}{T_1 - T_0} = \frac{\theta - 1}{\Delta\theta}.$$

The temperature profiles corresponding to some values of $\Delta\theta$ and Pe are

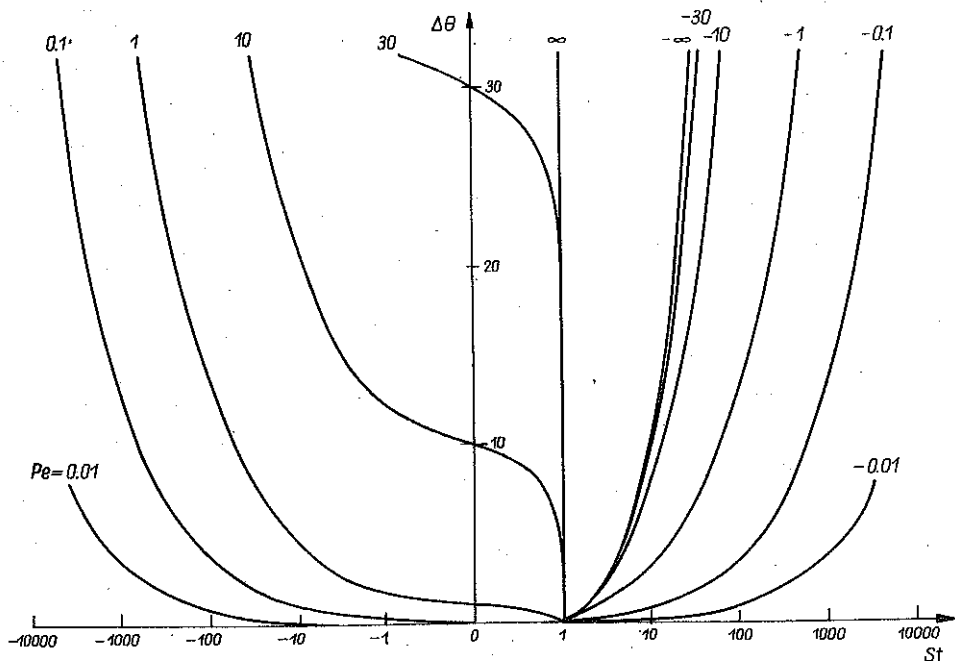


FIG. 2.

shown in Fig. 3: In Fig. 3a the asymptotic case $\Delta\theta \rightarrow 0$ is presented. In this case the formula (3.8)₁ takes a simple form (for $Pe \neq 0$)

$$(4.1) \quad t = \frac{e^{Pe Y} - 1}{e^{Pe} - 1},$$

given by ECKERT [6]. For $Pe = 0$ there is a simple heat conduction, and the temperature profile for small temperature variations is, of course, linear. The positive and negative values of Pe correspond to gas motion against and accordingly to the temperature gradient respectively. In this case the temperature profiles are symmetrical with respect to the change of direction of transversal flow: the formula (4.1) preserves validity upon transformation

$$\begin{aligned} Pe &\rightarrow -Pe, \\ Y &\rightarrow 1 - Y, \\ t &\rightarrow 1 - t. \end{aligned}$$

This symmetry, however, vanishes for finite $\Delta\theta$ (Fig. 3b, c). The larger $\Delta\theta$ is, the more pronounced the influence of transversal flow direction on the shape of profile.

In the specific case of flow with no energy exchange between walls, when the convective and the heat conductive components of energy flux

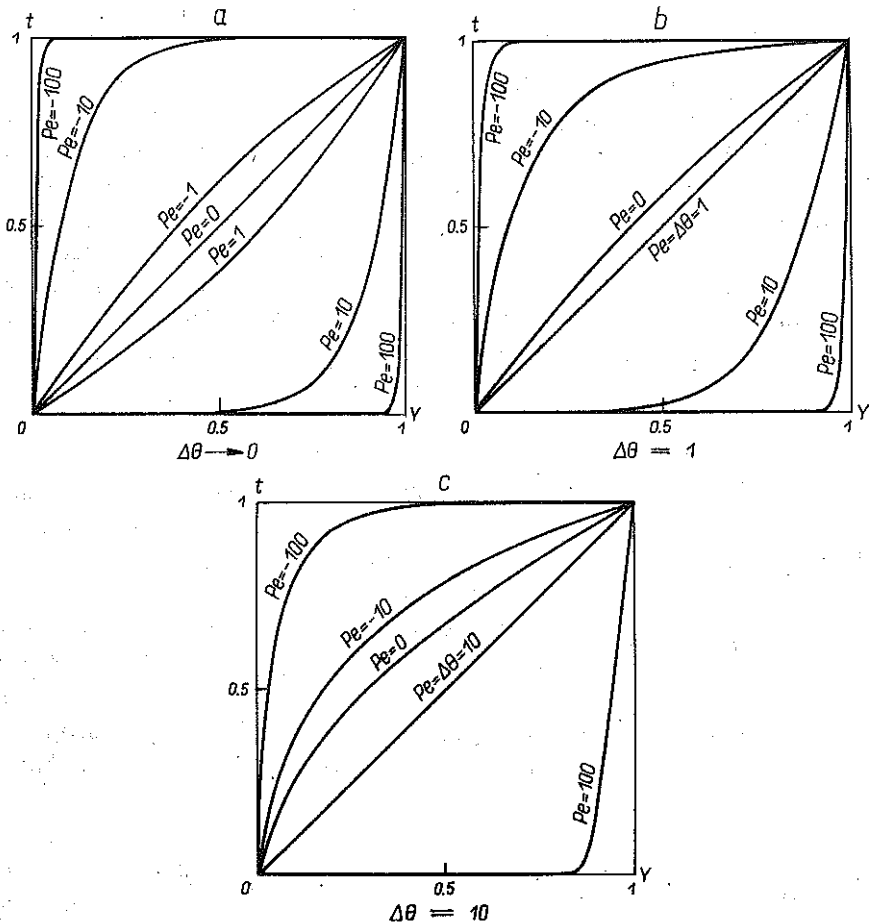


FIG. 3.

counterbalance one another ($St = 0$ or $Pe = \Delta\theta$), the viscosity and heat conductivity coefficients depend linearly on Y (as it follows from Eqs. (3.7₁) and (2.2)). If $n = 1$, this linear dependence is also valid for the temperature. On the other side, for intensive transversal flow ($|Pe| \gg 1$) the temperature gradient becomes very nonuniform across the channel. In the greater part of the channel cross-section the temperature slightly differs from the temperature of oncoming gas, and almost the whole temperature jump is realized in a thin temperature layer which appears at the wall where gas is removed.

The velocity profiles are simply related with the temperature profiles. Both velocity components are linear functions of temperature: $V = 1 + \Delta\theta t$, and $U = 1 - t$ (for $Pr = 1$). The conclusions drawn for the temperature profiles may be easily extended to the velocity profiles. In particular, the

layer of large temperature gradients is accompanied by the analogous layer for the both velocity components.

The pressure field is described by Eq. (3.8)₃ or by Eq. (3.9)₃ if $M = L$. In the last case the dimensionless pressure P is proportional to the temperature with the coefficient of proportionality depending on Pr . Some interesting consequences follow from this formula. In the inviscid case, the gas that moves in direction of increasing pressure decelerates, while the gas moving in direction of decreasing pressure accelerates. However, this feature not necessarily occurs for viscous gas motion when the viscous strain may play an important role. In particular, from the solution (3.9) it follows that gas injected at the cold wall moves in the direction of increasing pressure with deceleration for $Pr < 3/4$, and with acceleration for $Pr > 3/4$. On the other side, gas injected at the hot wall moves in a direction of decreasing pressure with acceleration for $Pr < 3/4$, and with deceleration for $Pr > 3/4$. For $Pr = 3/4$ the pressure remains constant in the flow field. The situation is associated with the fact that the transversal component of gas motion is fully determined by the energy equation (2.7)₄ (because $V = \theta$). For the known temperature field, the momentum equation (2.7)₃ determines the pressure field. In Eq. (2.7)₃ the pressure derivative is an algebraic sum of inertial and viscous terms. Due to the energy equation (2.7)₄, both these terms have the same structure and differ only by a constant factor depending on Pr . The signs of both terms are opposite; for $Pr > 3/4$ the viscous term prevails, and the qualitative flow pattern becomes opposite to that for inviscid gas.

In the assumed flow model a decisive role is played by the thermal effects which predominate over the dynamic effects. It does not exclude, however, the asymptotic case $\Delta\theta \rightarrow 0$. In this case the pressure variations tend to 0, too, and the inequalities (2.4) and (2.5) are fulfilled. In fact from Eqs. (3.9) it follows that Eqs. (2.4) and (2.5) are equivalent to

$$\frac{\Delta p}{p} = \left(\frac{4}{3} Pr - 1\right) \Delta\theta \frac{v_0^2}{RT_0} \ll 1,$$

$$\frac{T}{p} \frac{\Delta p}{\Delta T} = \left(\frac{4}{3} Pr - 1\right) \theta \frac{v_0^2}{RT_0} \ll 1.$$

Both these inequalities are satisfied if injection velocities are much less than the velocity of sound (if Pr and θ are of order 1).

5. FINAL REMARKS

One-dimensional gas flow in a channel with porous walls maintained at different temperatures was considered. Gas motion is generated by a

constant motion of one wall within its plane, and by uniform injection of gas at one wall and uniform removing of equal intensity at the other. To simplify the problem, the flow velocity is assumed very small with respect to the velocity of sound, and consequently compressibility of gas is neglected so that the variations of gas density are only due to thermal expansion. For such conditions the energy equation can be integrated independently of the momentum equations. Because the small pressure variations are neglected in the equation of state, the temperature remains the unique factor determining the density and, due to the continuity equation, the transversal component of velocity. The longitudinal component of velocity is determined as a function of temperature from the corresponding momentum equation. The other momentum equation determines the small pressure variations, proportional to the square of Mach number which may be considered as a small parameter of the problem.

For the viscosity and heat conductivity coefficients given as power functions of temperature, the temperature can be expressed by hypergeometric functions, and for rational power exponent — by elementary functions. For intensive transversal flow ($Pe \gg 1$) the flow parameters vary strongly nonuniformly across the channel, thus producing a layer of large gradients at the wall where gas is removed.

REFERENCES

1. H. SCHLICHTING, *Einige Exakte Lösungen für die Temperaturverteilung in Einer Laminaren Stromung*, ZAMM, **31**, 3, 78–83, 1951.
2. H. M. GROFF, *On viscous heating*, J. Aero. Sci., **23**, 4, 395–396, 1956.
3. C. R. ILLINGWORTH, *Some solutions of the equations of flow of a viscous compressible fluid*, Cambridge Phil. Soc. Proc., **46**, 3, 469–478, 1950.
4. И. Е. ТАРАПОВ, *Решение задачи о движении вязкого газа между двумя движущимися параллельными пластинами с теплоотдачей*, Прикл. Мат. и Мех., **19**, 3, 325–330, 1955.
5. A. J. A. MORGAN, *On the Couette flow of a compressible, viscous, heat conducting, perfect gas*, J. Aero. Sci., **24**, 4, 315–316, 1957.
6. E. R. G. ECKERT, *Heat and mass transfer*, New York 1959.
7. J. L. BANSAL, N. C. JAIN, *On the plain Couette flow of a viscous compressible fluid with transpiration cooling*, Proc. Ind. Acad. Sci., **80**, A, 1, 1–16, 1974.
8. H. ВАТЕМАН, А. ЕРДЕЛЫИ, *Higher transcendental functions*, **1**, New York 1953.
9. И. С. ГРАДСТЕЙН, И. М. РЫЖИК, *Таблицы интегралов, сумм, рядов и производений*, Москва 1962.
10. А. П. ПРУДНИКОВ, Ю. А. БРЫЧКОВ, О. И. МАРИЧЕВ, *Интегралы и ряды. Элементарные функции*, Москва 1981.

STRESZCZENIE

POWOLNY NIEIZOTERMICZNY PRZEPLYW GAZU W KANALE PŁASKIM
O ŚCIANKACH POROWATYCH

Rozważany jest powolny przepływ gazu lepkiego i przewodzącego ciepło w kanale płaskim o ściankach porowatych z różnicą temperatur między ściankami. Ruch gazu generowany jest przez jednostajny ruch ścianki a także przez jednorodny nadmuch wzdłuż jednej ze ścianek i jednorodne odsysanie wzdłuż ścianki przeciwległej. Pomijając wskutek małej prędkości ściśliwość gazu, ale uwzględniając zmienność gęstości wywołaną nieizotermicznością przepływu, można uprościć równania i uzyskać rozwiązanie w kwadraturach. Przyjmując potęgową zależność od temperatury współczynników lepkości i przewodnictwa ciepła uzyskano rozwiązanie w jawnej postaci. Dla dużych wartości liczby Pecleta, odpowiadającej ruchowi poprzecznemu, stwierdzono przy ściance z odsysaniem występowanie warstwy dużych gradientów prędkości i temperatury.

РЕЗЮМЕ

МЕДЛЕННОЕ НЕИЗОТЕРМИЧЕСКОЕ ТЕЧЕНИЕ ГАЗА
В ПЛОСКОМ КАНАЛЕ С ПОРИСТЫМИ СТЕНКАМ

Рассматривается медленное течение вязкого и теплопроводящего газа в плоском канале с пористыми стенками с разницей температур между стенками. Движение газа генерируется равномерным движением стенки, а также однородной наддувкой вдоль одной из стенок и однородным отсосом вдоль противоположающей стенки. Пренебрегая, вследствие малой скорости, сжимаемостью газа, но учитывая переменную плотность, вызванную неизоотермичностью течения, можно упростить уравнения и получить решение в квадратурах. Принимая степенную зависимость коэффициентов вязкости и теплопроводности от температуры, получено решение в явном виде. Для больших значений числа Пекле, отвечающего поперечному движению, констатировано, при стенке с отсосом, что выступает слой больших градиентов скорости и температуры.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

Received February 7, 1985.