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NATURAL CONVECTION BETWEEN TWO NONUNIFORMLY HEATED COAXIAL CYLINDERS

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The problem of natural convection between two coaxial, horizontal cylinders is investigated. One of the cylinders is nonuniformly heated. To find the parameters of the flow, an approximate method is given. It is a generalization of the Mack's and Bishop's method. The method can be applied for not too high Rayleigh numbers and for symmetrically distributed temperature on cylinders, with respect to the vertical plane of symmetry. Two simple cases are discussed in detail. In the first case, the distribution of temperature on the outer cylinder has the form $\Theta = \cos\varphi$ (φ — angular coordinate) and the temperature of the inner cylinder is constant. In the second case, the temperature on the outer cylinder is given by $\Theta = 1 + a\cos\varphi$ and the inner cylinder is maintained at constant temperature.

1. INTRODUCTION

In all cases where the field of fluid density is nonuniform buoyancy forces are present. These forces initiate motion of the fluid, called natural convection. Due to its wide technological applications, the problem of natural convection between two coaxial horizontal cylinders and conjugate heat transfer, has been the subject of intensive research in recent years. Most studies were performed under the assumption of constant temperature of the cylinders. This includes the series of works ranging from experimental ones by BECKMANN [1] and KRAUSSOLD [2] to numerical solutions of high accuracy by KUEHN and GOLDSTEIN [3]. A comprehensive literature survey of the subject can be found in the paper by TSUE and TREMBLEY [4].

The assumption of isothermal boundaries is practically equivalent to the assumption of infinite thermal conductivity, but in the majority of applications the thermal conductivity is finite. The conjugate problem has been studied by Roten [5] who assumed finite thermal conductivity of the inner cylinder which was heated by a source placed on the cylinder's axis.

To solve the equations obtained from the Navier–Stokes system by applying the Boussinesq approximation, the perturbation method is usually used [5, 6, 7]. In this method, the solutions are expressed as power series of the Rayleigh number. The convergence of these series is granted by the assumption of the small average Nusselt number or small Rayleigh number.

In this paper the generalization of Bishop's method to the case of nonuniform distribution of temperature of one of the bounding cylinders is described and a discussion of possible patterns of flow is included. This partly makes up for the lack of information about the behaviour of a fluid bounded by nonuniformly heated surfaces.

2. Formulation of the problem and basic equation

Consider two-dimensional steady laminar convection in an annulus bounded by two concentric horizontal cylinders. The annulus is filled with a viscous Newtonian fluid which is set in motion by the difference in temperature of the cylinders. One of the cylinders is maintained at a uniform temperature $T_c = \text{const}$. The temperature T_0 of the other one is distributed symmetrically with respect to the horizontal axis of the cylinders, and has the form

$$T_0 = T_c + \Delta T \cdot \Theta(\varphi), \tag{2.1}$$

where φ is the angular coordinate of the polar coordinate system (see Fig. 1).

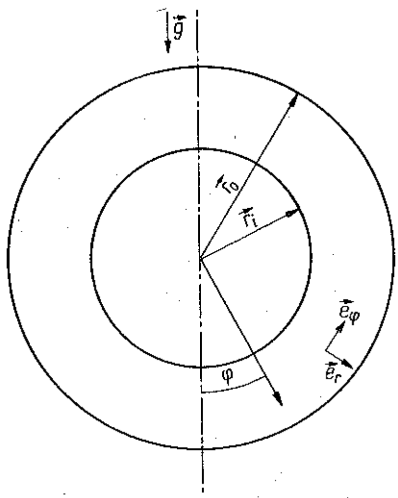

Fig. 1. Geometry of the problem and the coordinate system.

All fluid properties are taken to be constant, except for the density variation with temperature. Assuming a small difference in temperature between the cylinders in comparison with $1/\beta$, the state equation takes the following form:

$$\frac{\varrho}{\varrho_0} = 1 - \beta (T' - T_c'), \tag{2.2}$$

where T_c' = constant and β — volumetric coefficient of thermal expansion.

By using Eq. (2.2) and the Boussinesq approximation, the set of the continuity, momentum and energy equations can be transformed to the following system [6]:

$$\nabla^4 \psi = R_a \cdot L(T) + \frac{1}{p_r} \frac{1}{r} \frac{\partial (\nabla^2 \psi, \psi)}{\partial (r, \varphi)}, \tag{2.3}$$

$$\nabla^2 T = \frac{1}{r} \frac{\partial (T, \psi)}{\partial (r, \varphi)}, \tag{2.4}$$

where

$$L(T) = \sin \varphi \frac{\partial T}{\partial r} + \frac{\cos \varphi}{r} \frac{\partial T}{\partial \varphi},$$

$$\frac{\partial (T, \psi)}{\partial (r, \varphi)} = \frac{\partial T}{\partial r} \cdot \frac{\partial \psi}{\partial \varphi} - \frac{\partial T}{\partial \varphi} \cdot \frac{\partial \psi}{\partial r},$$

and the operators ∇^4 and ∇^2 are given by $\nabla^4 = \nabla^2 (\nabla^2)$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \varphi^2},$$

T, ψ denotes the dimensionless temperature of the fluid and the dimensionless stream function respectively.

All dimensionless quantities are obtained by the following transformations:

$$r = \frac{r'}{r_i}, \quad \psi = \frac{\psi'}{\nu}, \quad T = \frac{T' - T_c'}{\Delta T},$$

r_i — the radius of the inner cylinder, ν — the kinematic viscosity, $R_a = \frac{g \beta \Delta T r_i^3}{\nu \alpha}$ — the Rayleigh number, $\Pr = \nu/\alpha$ — the Prandtl number, g — the acceleration of gravity, α — the thermal diffusivity.

For the case of rigit cylinders and due to the symmetry of the problem, the boundary conditions can be expressed as

$$\psi = \frac{\partial \psi}{\partial r} = 0 \quad \text{at} \quad r = 1, R,$$

$$\psi = \frac{\partial^2 \psi}{\partial \varphi^2} = 0 \quad \text{at} \quad \varphi = 0, \pi,$$

$$T=0 \quad \text{at} \quad r=1,$$
$$T=\Theta(\varphi) \quad \text{at} \quad r=R,$$

for constant temperature of the inner cylinder, or $T=\Theta$ at $r=1$, $T=0$ at $r=R$ for constant temperature of the outer cylinder. In both cases, additionally, $\partial T/\partial\varphi=0$ at $\varphi=0,\ \pi$ holds, $R=r_0/r_i$ — being the radius ratio of the cylinders.

In the next section the case of constant temperature of the inner cylinder will be considered. The nonlinear system (2.3), (2.4), will be solved for small values of the Rayleigh number and for such boundary conditions, where the function θ can be expressed in the form of finite Fourier series of cosines, $\Theta=\sum_{k=0}^{N} a_k \cos k\varphi$. Naturally, the obtained solution will be approximate.

3. The method of solution

For a small value of the Rayleigh number R_a, one can express the solutions of the system (2.3), (2.4) in the form of power series of Ra, such that:

$$T=\sum_{m=0}^{\infty} \mathrm{R}_{\mathrm{a}}^{\mathrm{m}}\, T_m, \tag{3.1}$$

$$\psi=\sum_{m=1}^{\infty} \mathrm{R}_{\mathrm{a}}^{\mathrm{m}}\, \psi_m. \tag{3.2}$$

Substituting (3.1) and (3.2) into equations (2.3), (2.4) and equating terms of the same order in R_a one readily obtains the following hierarchy of linear inhomogeneous equations:

$$\nabla^2 T_0=0, \tag{3.3}$$

$$\nabla^4 \psi_1=L(T_0), \tag{3.4}$$

$$\nabla^2 T_1=\frac{1}{r}\frac{\partial(T_0,\psi_1)}{\partial(r,\varphi)}, \tag{3.5}$$

$$\nabla^4 \psi_2=L(T_1)+\frac{1}{\mathrm{Pr}}\frac{1}{r}\frac{\partial(\nabla^2\psi_1,\psi_1)}{\partial(r,\varphi)}, \tag{3.6}$$

$$\nabla^2 T_2=\frac{1}{r}\frac{\partial(T_0,\psi_2)}{\partial(r,\varphi)}+\frac{1}{r}\frac{\partial(T_1,\psi_1)}{\partial(r,\varphi)}.$$

The corresponding boundary conditions are

(3.8) $$\psi_m = \frac{\partial \psi_m}{\partial r} = 0 \quad \text{at} \quad r = 1, R,$$

(3.9) $$\psi_m = \frac{\partial^2 \psi_m}{\partial \varphi^2} = 0 \quad \text{at} \quad \varphi = 0, \pi,$$

(3.10) $$T_0 = 0 \quad \text{at} \quad r = 1,$$

(3.11) $$T_0 = \Theta \quad \text{at} \quad r = \mathrm{R},$$

(3.12) $$\frac{\partial T_m}{\partial \varphi} = 0 \quad \text{at} \quad \varphi = 0, \pi \quad \text{and} \quad T_m = 0 \quad \text{at} \quad r = 1, R, \quad m \geqslant 1.$$

Using the method separation of variables, one can solve the system (3.3)—(3.7) assuming the following form of the solutions:

(3.13) $$T_m = \sum_k T_{m,k}(r) \cos k\varphi, \quad m = 0, 1, 2, \ldots,$$

(3.14) $$\psi_m = \sum_k \psi_{m,k}(r) \sin k\varphi, \quad m = 1, 2, 3, \ldots,$$

where $T_{m,k}$ and $\psi_{m,k}$ satisfy the ordinary differential equations:

(3.15) $$D_k T_{m,k} = f_{m,k}(r),$$

(3.16) $$D_k^2 \psi_{m,k} = g_{m,k}(r),$$

$$D_k = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{k^2}{r^2},$$

$f_{m,k}$ — known functions defined by earlier found $T_l\,(l < m)$, and $\psi_l\,(l \leqslant m)$, $g_{m,k}$ — defined by T_l, $\psi_l\,(l < m)$.

The functions $T_{m,k}$ and $\psi_{m,k}$ must satisfy the following boundary conditions:

(3.17) $$T_{0,k}(1) = 0, \quad T_{0,k}(R) = a_k,$$

(3.18) $$T_{m,k}(1) = T_{m,k}(R) = 0 \quad \text{for} \quad m > 0,$$

(3.19) $$\psi_{m,k}(1) = \psi_{m,k}(R) = 0,$$

(3.20) $$\frac{d\psi_{m,k}(1)}{dr} = \frac{d\psi_{m,k}(R)}{dr} = 0.$$

The general solution $G(r)$ of Eq. (3.15) has the form

$$G(r; f_{m,k}, c_1, c_2) = \frac{1}{2k}\left(r^k \int \frac{f_{m,k}(r)}{r^{k-1}}\, dr - \frac{1}{r^k}\int f_{m,k}(r)\, r^{k+1}\, dr\right) + \frac{c_1}{r^k} + c_2 r^k, \quad \text{for} \quad k \geqslant 0, \tag{3.21}$$

and

$$G(r; f_{m,k}, c_1, c_2) = -\int r \ln r . f_{m,k}(r)\, dr + \ln r \int r . f_{m,k}(r)\, dr + c_1 \ln r + c_2, \quad \text{for} \quad k = 0. \tag{3.22}$$

The general solution for $\psi_{m,k}$ can be obtained by using twice the formula (3.21). The functions $\psi_{m,k}$, $T_{m,k}$ satisfying the boundary conditions (3.17)—(3.20) generally depend on r, R and Pr. Particularly the functions $\psi_{1,k}$, $T_{0,k}$, $T_{1,k}$ depend on r and R only.

As mentioned above, the distribution of temperature on one cylinder must have the form

$$H = \Theta(\varphi) = \sum_{n=0}^{N} a_n \cos n\varphi .$$

(In another case our method can not be used). The simplest but still having practical interest case is

$$\Theta = \cos \varphi \quad \text{or} \quad \Theta = 1 + a \cos \varphi .$$

In the next sections we restrict our discussion to those two cases.

4. Negligence of higher order terms in the solution

Before discussing the result obtained from the solutions it should be noted that these expansions in power series of R_a do not converge uniformly with respect to R. In fact, it can be checked numerically that when R tends to infinity, they will diverge even for very small Rayleigh numbers. For finite values of the radius ratio R, one can except that the series converge in a certain range of the Rayleigh numbers. For given boundary conditions it was observed that the ratio of $j+1$-th term in either the expansion (3.1) or (3.2) to the j-th term is approximately independent of j, thus, if this ratio is less than unity, then the series converge. On the basis of this observation the upper limit of the Rayleigh number R_A for which the series solutions converge is defined:

$$R_a = \sup \left[\left| \frac{\psi_j(r, \varphi)}{\psi_{j+1}(r, \varphi)} \right|, \left| \frac{T_j(r, \varphi)}{T_{j+1}(r, \varphi)} \right| \right], \quad 1 < r < R, \quad 0 < \varphi < \pi.$$

Using the first two terms of expansions for T and the values of R_A for radius ratios from 1.2 to 2.8 and the Prandtl numbers $\mathrm{Pr} = 0.02$, $\mathrm{Pr} = 0.7$,

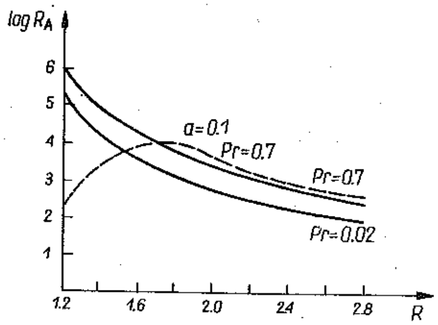

FIG. 2. R_A in dependence of R; ——— for case $\Theta = \cos\varphi$, — — — — — for case $\Theta = 1 + a\cos\varphi$.

numerical calculation has been done and the results are presented in Fig. 2. For the case $\Theta = \cos\varphi$, R_A is a decreasing function of R and when $\Theta = 1 + a\cos\varphi$, R_A increases rapidly with increasing R to maximum value, then decreases with further increase in R. Similar results were also obtained by Mack and Bishop for the problem of uniformly-heated cylinders. It may be shown from the form of the expansions (3.1) and (3.2), and from the above considerations that the validity of series solution depends on the ratio R_a/R_A. For $R_a/R_A = 0.4$ the ratio of the third term to the first one in the expansions (3.1) and (3.2) is about 0.19. It indicates that the earliest term neglected in our two-term solutions is roughly 19 percent of the first term. Thus, when the condition $R_a/R_A = 0.4$ is satisfied, one can reasonably expect that the truncated solution

(4.1) $$T = T_0 + R_a T_1,$$

(4.2) $$\psi = R_a \psi_1 + R_a^2 \psi_2$$

will approximate the solution to within several percent.

5. RESULTS AND DISCUSSION

The series expansions (4.1) and (4.2) have been evaluated numerically for various values of the parameters R, R_a, Pr and a. It was found necessary to perform the numerical calculations with double precision.

For the case $\Theta = \cos\varphi$, it can be shown that

$$T(\varphi) = -T(\pi-\varphi)$$

and

$$\psi(\varphi) = -\psi(\pi-\varphi).$$

As a consequence of the above and due to the symmetry of the problem with respect to the vertical axis, it is convenient to represent the numerical results in a single graph with the streamline pattern in the right part of the cavity and the isothermals in the left one. In the second case $\Theta = 1+a\cos\varphi$ the streamline pattern will be represented in the right half of the cavity and the isothermals in the left one. Figure 3 presents the streamlines and

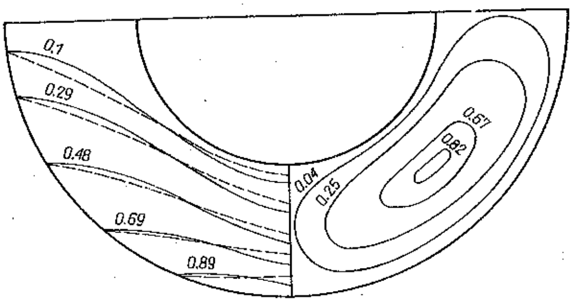

FIG. 3. Streamlines and isothermals for $R_a = 1500$, $\mathrm{Pr} = 0.7$, $R = 1.85$; ——— isothermals for pure conduction ($T = T_0$).

the isothermals for the case with higher perturbation of temperature on the outer cylinder $\Theta = \cos\varphi$ for $R_a = 1500$, $\mathrm{Pr} = 0.7$ and $R = 1.85$. It is observed that the flow pattern forms a single cell of the "crescent-eddy" type. The flow is upward along the outer cylinder and downward along the inner cylinder. The centre of the eddy, where ψ has its maximum value, is in the lower half of the flow region. A change of the Rayleigh number causes a little fluctuation of the centre of the eddy between the radius $\varphi = 45°$ and $\varphi = 35°$. Multicellular flow is not observed in this case. A change in the Prandtl number and the Rayleigh number has very little qualitative effect upon the streamlines. For comparison, the isothermals for pure conduction ($T = T_0$) and for the case when $R_a = 1500$, $\mathrm{Pr} = 0.7$ and $R = 1.85$ are shown on the left half of Fig. 3. It is seen that the flow region is divided into two parts. At a typical location, in the right part $T > T_0$ and in the left one $T < T_0$. This indicates that the ratio of heat transfer due to convection is directed from right to left in the flow region.

Figures 4a and 4b show the streamline and isothermals for the case $\Theta = 1+a\cos\varphi$, $R = 1.85$ and few sets of R_a, Pr, a. For $a = 0.1$, $\mathrm{Pr} = 0.7$, $R = 1.85$, $R_a = 3000$, as it is shown in Fig. 4a, the flow has the form of a

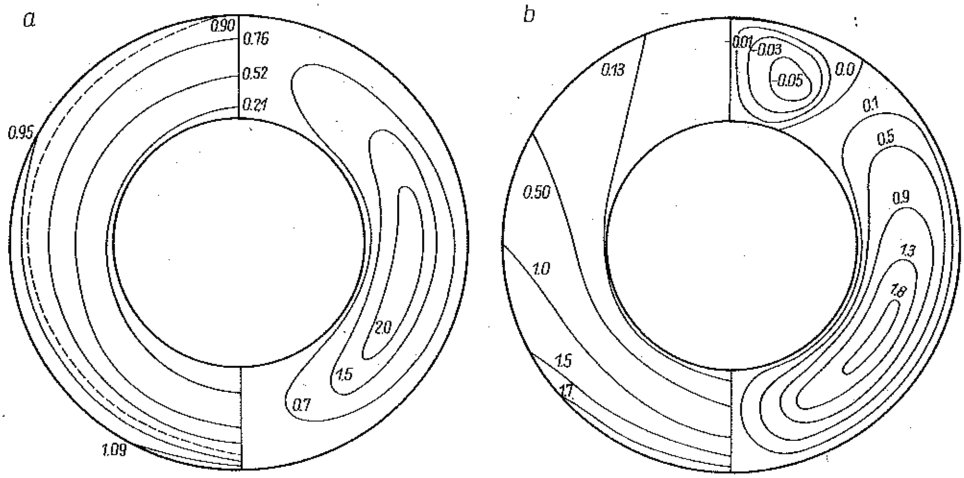

FIG. 4. Streamlines and isothermals a) for $a = 0.1$, $R_a = 3000$, $\mathrm{Pr} = 0.7$, $R = 1.85$; b) for $a = 1.0$, $R_a = 1500$, $\mathrm{Pr} = 0.7$, $R = 1.85$.

single cell. The centre of the cell is placed below the horizontal plane of symmetry and its location does not depend very much on the Rayleigh number. A similar situation takes place for all sufficiently small a ($a \leqslant 0.2$) $0.02 \leqslant \mathrm{Pr} \leqslant 6$, $1.2 \leqslant R \leqslant 2.8$ and $\mathrm{Ra} \leqslant 3000$. With increasing value of a, the multicellar flow pattern is observed (see Fig. 4b). In the left half of Figs. 4a and 4b, the isothermal patterns are shown. The isothermal started from the point (R, π) separates the cavity into two regions (heavy dashed line in Figs. 4a). The inner region includes the concentric isothermals and the outer one includes isothermals starting from some point of the outer cylinder. The outer region increases with increasing Rayleigh number or the parameter a. It enables to observe that the heat transfer in some way depends on the area of this region. It should be noted that by letting a tend to zero and keeping R fixed, one can obtain results approaching the results of MACK and BISHOP [6].

The local heat transfer rates at the inner and outer cylinders can be expressed in terms of the corresponding local Nusselt numbers Nu_i and Nu_0 defined by

$$\mathrm{Nu}_i = \ln R \left[r \frac{\partial T}{\partial r} \right], \qquad r = 1,$$

$$\mathrm{Nu}_0 = \ln R \left[r \frac{\partial T}{\partial r} \right], \qquad r = R.$$

For $\Theta = \cos \varphi$ both local Nusselt numbers Nu_i and Nu_0 are decreasing functions of $\varphi\left(0 \leqslant \varphi \leqslant \frac{\pi}{2}\right)$ and $Nu_i\left(\frac{\pi}{2}\right) = Nu_0\left(\frac{\pi}{2}\right) = 0$ (see Fig. 5a, heavy dashed line). In the case when $\Theta = 1 + a \cos \varphi$, the behaviour of Nusselt

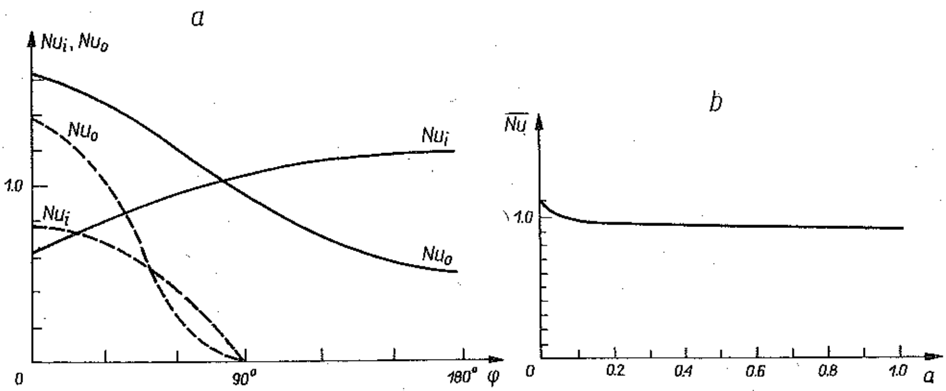

FIG. 5. a) Local Nusselt numbers Nu_i and Nu_0
——— $\Theta = 1 + 0.1 \cos \varphi$, $R = 1.85$, $Pr = 0.7$, $R_a = 3000$,
– – – $\Theta = \cos \varphi$, $R = 1.85$, $Pr = 0.7$, $R_a = 1500$;
b) $\overline{Nu}$ as a function of a for $R_a = 1500$, $R = 1.85$.

numbers depends significantly on a. Of course, for all values of R, Ra, and Pr the maximum of the value of Nu_0 is located in $\varphi = 0$. Such regularity is not observed for Nu_i, whose behaviour depends on a. For $a = 0.1$, Nu_i is an increasing function of φ (see Fig. 5a) but with a increasing from 0.1 to 1, the location of the maximum value of Nu_i changes from the top of the cylinder to the bottom. For $a = 1$, Nu_i is decreasing function of φ.

While the local Nusselt numbers indicate the distribution of the heat flow across a given surface, the total heat flow across that surface is given by the average Nusselt number as defined by

$$\overline{Nu} = \frac{1}{\pi} \int_0^\pi Nu_i \, d\varphi \quad \text{or} \quad \overline{Nu} = \frac{1}{\pi} \int_0^\pi Nu_0 \, d\varphi,$$

(where, by virtue of the balance of energy, the integration over the inner and outer surfaces must be the same).

In both cases $\Theta = \cos \varphi$ and $\Theta = 1 + a \cos \varphi$, for our approximation, $\overline{Nu}$ does not depend on Pr. It follows from the fact that the integrals of the terms depending on Pr vanish. This indicates that the influence of Pr on $\overline{Nu}$ is the higher order effect. In the case when $\Theta = 1 + a \cos \varphi$, $\overline{Nu}$ decreases with increasing a $(a \leqslant 1)$, (see Fig. 5b). It is interesting to note that the effect of convection causes a decrease of the heat transfer.

6. Final remark

The method of the solution presented in this paper can be applied to a limited range of Rayleigh numbers, and for some class, quite wide, but also a limited range of boundary conditions. This method gives approximate solutions and requires computer calculations. In spite of this, in many cases it is more convenient to use it than to perform direct computer calculations. First of all, when this method is applied, the numerical calculations are very simple and they are limited only to the determination of some constant coefficients, from the boundary conditions. When these coefficients are known, the solutions can be expressed by analytical formulas. Having such expressions is very useful, especially for the investigation of the problem of stability of the obtained solution.

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Streszczenie

NATURALNA KONWEKCJA MIĘDZY DWOMA NIEJEDNOSTAJNIE NAGRZANYMI, WSPÓŁOSIOWYMI CYLINDRAMI

W pracy badany jest problem naturalnej konwekcji między dwoma współosiowymi, poziomymi cylindrami. Temperatura jednego cylindra jest stała, a temperatura drugiego zależy od współrzędnej kątowej. W celu znalezienia parametrów przepływu podana została przybliżona metoda. Jest ona uogólnieniem metody zaproponowanej przez Macka i Bishopa dla przypadku stałych temperatur na cylindrach. Podana metoda może być stosowana dla niezbyt dużych liczb Rayleigha i przy symetrycznym, względem pionowej płaszczyzny symetrii, rozkładzie temperatur na cylindrach. Dokładnie przeanalizowano dwa przypadki warunków na temperaturę. Pierwszy dotyczy rozkładu temperatury na zewnętrznym cylindrze danego wzorem $\Theta = \cos\varphi$ (φ — kątowa współrzędna), podczas gdy temperatura wewnętrznego cylindra jest stała. W drugim przypadku $\Theta = \cos\varphi$ zastąpiono przez $\Theta = 1+a\cos\varphi$, a — const.

Резюме

НАТУРАЛЬНАЯ КОНВЕКЦИЯ МЕЖДУ ДВУМЯ НЕРАВНОМЕРНО НАГРЕТЫМИ СООСНЫМИ ЦИЛИНДРАМИ

В работе исследуется проблема натуральной конвекции между двумя соосными, горизонтальными цилиндрами. Температура одного цилиндра постоянная, а температура второго зависит от угловой координаты. С целью нахождения параметров течения приведен приближенный метод. Является он обобщением метода предложенного Маком и Бишопом для случая постоянных температур на цилиндрах. Приведенный метод может применяться для не слишком больших чисел Рэлея и при симметричном, по отношению к вертикальной плоскости симметрии, распределении температур на цилиндрах. Детально проанализированы два случая условий для температуры. Первый касается распределения температуры на внешнем цилиндре данного формулой $\Theta = \cos\varphi$ (φ— угловая координата), в то время как температура внутреннего цилиндра является постоянной. Во втором случае $\Theta = \cos\varphi$, заменен $\Theta = 1+a\cos\varphi$, a—const.

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