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TEMPERATURE IN SEMI-INFINITE AND FINITE CYLINDER WITH MOVING HEATING OVER THE LATERAL SURFACE

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In this paper some relations describing a nonstationary temperature field in a semi-infinite and also a finite cylinder heated over a part of its lateral surface with a moving axial-symmetric thermal surface source and dependent on two cases of boundary conditions across the face plane; the temperature is equal to zero and there is perfectly thermal insulation, are derived. On the basis of the fundamental solution for a long cylinder in [1] a computer program has been made and the results have been widely analysed in [3]. The formulae of the temperature for various kinds of heating have been given. The temperature distribution within the cylinder of length L is presented as an infinite sum of elements being the fundamental solution alike. Finally, a transformation to Green's function as a limiting case of results has been proved. Numerical results are given.

Introduction

The present consideration deals with a nonstationary temperature field in a semi-infinite and also a finite cylinder heated suddenly on a part of its lateral surface $\zeta \geqslant \zeta_0$ to constant temperature θ_0 at the contact region by the effect of thermal surface sources which, at the same time, start to move with uniform velocity w in the positive axial direction ζ. The remaining part of the lateral surface is also held at a constant temperature but at quite a different one, say, at zero while the face plane $\zeta = 0$ or $\zeta = (0, L)$ may be kept either at zero or be perfectly thermally insulated.

The problem formulated above is equivalent to the case that at the moment when the surface thermal source is applied to a part of the lateral surface of the cylinder, it starts to move uniformly in the negative direction of the ζ — axis, whereas its free part of cylindrical surface is maintained at zero and the remainder of the boundary, i.e. the face plane, is held at zero temperature or is thermally insulated.

1. Formulation of the problem. Solvent function

In the dimensionless system of the cylindrical coordinates and time (ϱ, ζ, τ) the heat conduction equation and appropriate initial and boundary conditions become, cf. [1, 2].

$$\left(\nabla^2 - \frac{\partial}{\partial t}\right)\bar{\theta} = 0, \quad 0 \leqslant \varrho < 1, \quad 0 < \tau < \infty \tag{1.1}$$

$$0 < \zeta < \begin{cases} \infty \text{ — semi-infinite cylinder} \\ L \text{ — finite cylinder,} \end{cases}$$

$$\bar{\theta}(\varrho, \zeta; 0) = 0, \quad 0 \leqslant \varrho < 1, \quad 0 < \zeta < \{ {}^{\infty}_{L}, \tag{1.2}$$

$$\bar{\theta}(1, \zeta; \tau) = \theta_0\, \eta(\zeta - \zeta_0 - w\tau), \quad \text{at the lateral surface,} \tag{1.3}$$

and on the face plane, respectively:

I) In the case of a semi-infinite cylinder
if temperature equal to zero is required

$$\bar{\theta}(\varrho, 0; \tau) = 0, \tag{1.4$_1$}$$

or if the face plane is perfectly thermally insulated

$$\left.\frac{\partial \bar{\theta}(\varrho, \zeta; \tau)}{\partial \zeta}\right|_{\zeta=0} = 0. \tag{1.4$_2$}$$

II) In the case of a finite cylinder with length L:
if the temperature of both face planes equal to zero is required

$$\bar{\theta}(\varrho, 0; \tau) = \bar{\theta}(\varrho, L; \tau) = 0, \tag{1.4'$_1$}$$

or if both face planes are perfectly thermally insulated

$$\left.\frac{\partial \bar{\theta}(\varrho, \zeta; \tau)}{\partial \zeta}\right|_{\zeta=0} = \left.\frac{\partial \bar{\theta}(\varrho, \zeta; \tau)}{\partial \zeta}\right|_{\zeta=L} = 0. \tag{1.4'$_2$}$$

$\bar{\theta} = \bar{\theta}(\varrho, \zeta; \tau)$ denotes the dimensionless temperature field in the domain, $\eta(x)$ — Heaviside's function, $\theta = T - T_0$ — increment of absolute temperature, T_0 — temperature of natural state. We recall the restriction that $\theta/T_0 \ll 1$. The other notations referred to [1] and [2]; $\bar{\theta} = \theta/\theta_0$ and θ_0 is the source temperature on the lateral surface of the cylinder.

We shall confine ourselves primarily to the consideration of a temperature field in an infinite cylinder heated suddenly on a semi-infinite part of its lateral surface, $\zeta \geqslant 0$, with thermal sources moving uniformly in the positive direction of the ζ — axis. The formulation of the problem is as follows, cf. [1, 2]:

$$(1.5)\qquad \left(\nabla^2-\frac{\partial}{\partial\tau}\right)\bar{\theta}=0,\quad 0\leqslant\varrho<1,\quad |\zeta|<\infty,\quad 0<\tau<\infty,$$

$$(1.6)\qquad \bar{\theta}(\varrho,\zeta;0)=0,\quad 0\leqslant\varrho<1,\quad |\zeta|<\infty,$$

$$(1.7)\qquad \bar{\theta}(1,\zeta;\tau)=\eta(\zeta-w\tau),\quad 0<\tau<\infty,\quad |\zeta|<\infty.$$

The solution of the initial boundary value problem (1.5)—(1.7) determining temperature distribution $\bar{\theta}(\varrho,\zeta;\tau)$ in the entire cylinder has been expressed by Eqs. (12), (17.1), (17.2) and (21) in [1] and we call it to our further purpose the fundamental solution. The function $\bar{\theta}(\varrho,\zeta;\tau)$ was investigated numerically in [3] and its variation calculated in three aspects of functional dependence, namely as $\bar{\theta}=\bar{\theta}(\tau)$, $\bar{\theta}=\bar{\theta}(\varrho)$, $\bar{\theta}=\bar{\theta}(\zeta)$. Diagrams were plotted and the most interesting fragments of the curves from the practical points of view were discussed.

We introduce now a function of four variables $\mathscr{F}=\mathscr{F}(\varrho,\lambda_1,\lambda_2,\tau)$ and define it as follows:

$$(1.8)\qquad \mathscr{F}(\varrho,\lambda_1,\lambda_2,\tau)=\frac{1}{2}-\sum_{n=1}^{\infty}\frac{\mathscr{J}_0(\omega_n\varrho)}{\omega_n\mathscr{J}_1(\omega_n)}\exp[-\omega_n^2\tau]+ \\ +\mathscr{F}_1(\varrho,\lambda_1;\tau)+\mathscr{F}_2(\varrho,\lambda_2;\tau),$$

where

$$(1.9)\qquad \mathscr{F}_1(\varrho,\lambda_1;\tau)=\begin{cases}-\dfrac{1}{2}+\displaystyle\sum_{n=1}^{\infty}\frac{\mathscr{J}_0(\omega_n\varrho)}{\mathscr{J}_1(\omega_n)}\frac{\omega_n\exp[a_n\lambda_1]}{a_n\left(\omega_n^2+\dfrac{w^2}{4}\right)^{1/2}}, & \lambda_1<0,\\[2ex] \dfrac{1}{2}+\displaystyle\sum_{n=1}^{\infty}\frac{\mathscr{J}_0(\omega_n\varrho)}{\mathscr{J}_1(\omega_n)}\frac{\omega_n\exp[a_{-n}\lambda_1]}{a_{-n}\left(\omega_n^2+\dfrac{w^2}{4}\right)^{1/2}}, & \lambda_1>0,\end{cases}$$

$$(1.10)\qquad \mathscr{F}_2(\varrho,\lambda_2;\tau)=\sum_{n=1}^{\infty}\frac{\omega_n\mathscr{J}_0(\omega_n\varrho)}{\mathscr{J}_1(\omega_n)}\left\{-\frac{1}{\omega_n^2+a_n^2}\exp[(-\omega_n^2+a_n^2)\tau+ \right. \\ +a_n\lambda_2]\operatorname{erfc}\left(a_n\sqrt{\tau}\mp\frac{\lambda_2}{2\sqrt{\tau}}\right)+ \\ +\frac{1}{\omega_n^2+a_{-n}^2}\exp[(-\omega_n^2+a_{-n}^2)\tau+a_{-n}\lambda_2]\operatorname{erfc}\left(-a_{-n}\sqrt{\tau}-\frac{\lambda_2}{2\sqrt{\tau}}\right)- \\ \left.-\frac{1}{\omega_n^2}\exp[-\omega_n^2\tau]\operatorname{erf}\left(\frac{\lambda_2}{2\sqrt{\tau}}\right)\right\},\quad |\lambda_2|<\infty,$$

by notice

$$a_n = \left[\left(\omega_n^2 + \frac{w^2}{4}\right)^{1/2} - \frac{w}{2}\right] > 0,$$

$$a_{-n} = -\left[\left(\omega_n^2 + \frac{w^2}{4}\right)^{1/2} + \frac{w}{2}\right] < 0,$$

$\mathscr{J}_0(x)$, $\mathscr{J}_1(x)$ indicates the Bessel function on the first kind of order zero and one, ω_n are roots of the equation $\mathscr{J}_0(x) = 0$, $n = 1, 2, \ldots$.

We state that the function

$$\bar{\theta}(\varrho, \zeta; \tau) \equiv \mathscr{F}(\varrho, \zeta - w\tau, \zeta, \tau), \tag{1.11}$$

is the solution of the problem (1.5)—(1.7), exactly. The proof can be achieved by resorting to the method applied to the succeeding Theorem in Paragraph 4 or directly to [1].

Conclusion. The function

$$\bar{\theta}_{\zeta_0}(\varrho, \zeta; \tau) = \bar{\theta}(\varrho, \zeta - \zeta_0, \tau) = \mathscr{F}(\varrho, \zeta - \zeta_0 - w\tau, \zeta - \zeta_0, \tau), \tag{1.12}$$

in the solution of Eq. (1.5) with the conditions (1.6) and (1.3) where $\theta_0 = 1$.

In this way the suitable formulae are made ready to define the temperature field in the cylindrical space bounded one-sided or two-sided by a face plane perpendicular to the generator.

2. Nonstationary temperature field in semi-infinite and finite cylinder

2.1. *Semi-infinite cylinder*

We shall now look for a solution of the problem formulated by Eqs. (1.1)—(1.3) and (1.4). We pay our attention to Eq. (1.12) and note that the condition $(1.4)_1$ will be satisfied in the time interval $0 < \tau < \infty$ if a hypothetic temperature field is maintained to the infinite cylinder symmetric to Eq. (1.12) regard to the face plane $\zeta = 0$ but with a sign opposite to Eq. (1.12) to reduce the residuary thermal state at section $\zeta = 0$.

By virtue of the formulae (1.12) with (1.8) the desired solution $\hat{\theta}$ may be expressed as follows:

$$\hat{\theta}_T(\varrho, \zeta; \tau) = \mathscr{F}(\varrho, \zeta - \zeta_0 - w\tau, \zeta - \zeta_0, \tau) - \mathscr{F}(\varrho, -(\zeta + \zeta_0 + w\tau), -(\zeta + \zeta_0), \tau). \tag{2.1}$$

Continuing the above idea, we define the temperature field in a semi--infinite cylinder with the boundary condition $(1.4)_2$ by adding to the solution (1.12) the hypothetic temperature field to assure the face plane $\zeta=0$ against loss of heat flux. Thus we have

$$(2.2) \qquad \hat{\theta}_{\mathrm{I}}(\varrho, \zeta; \tau) = \mathscr{F}(\varrho, \zeta-\zeta_0-w\tau, \zeta-\zeta_0, \tau) + \\ + \mathscr{F}\big(\varrho, -(\zeta+\zeta_0+w\tau), -(\zeta+\zeta_0), \tau\big).$$

If the delay is $\zeta_0 \to 0$, then a qualitative and quantitative analysis of such thermal performances may be carried out obviously in some limit on the data published in [3] combining the proper temperature values in agreement with the formulae (2.1) and (2.2) putting in both $\zeta=0$. It is possible to deduce thermal distributions from the given temperature plots if the delay ζ_0 does not vanish.

2.2. *Cylinder of a finite length L*

We extend our previous considerations of the temperature distribution problem to a cylindrical medium bounded by two parallel planes $\zeta=0$ and $\zeta=L$ initially at zero temperature when a part of the lateral surface from $\zeta=\zeta_0$ to $\zeta=L$ is suddenly heated at time $\tau=0$ by the affect of thermal surface sources at the contact region to constant temperature θ_0 and at the same time the thermal sources start to shrink with uniform velocity w towards $+\zeta$. A free part of the cylindrical surface is maintained at zero temperature and the face planes remain either at zero or are perfectly thermally insulated during the whole process.

This problem is formulated by Eqs. (1.1)—(1.3) and (1.4′) and is more complicated than for a semi-infinite cylinder. The method of solution will be based on the fundamental solution (infinite cylinder) and the hypothetic temperature field; the last one will be called later an image, see for instance [4, 7].

At first we locate a sink (negative source) as the image of a given surface source symmetric to the plane $\zeta=0$ of an infinite system beginning from $\zeta=-\zeta_0$ towards $-\infty$ to reduce in this way the temperature at the bounding face plane $\zeta=0$ so to be zero if we mean the condition $(1.4')_1$. Similarly we put a sink as an image symmetrically with respect to the plane $\zeta=L$, that is from $\zeta=2L-\zeta_0$ in the direction ∞ to reduce the temperature at $\zeta=L$. We perceive, however, that new sources must now be introduced as the reflection of the image sinks to maintain the bounding planes $\zeta=(0, L)$ at zero temperature and so on. Then it follows that an infinite succession of alternating sources and sinks are required and they must be taken into account to satisfy the condition on the face planes.

Finally, the temperature function in the cylinder of length L is written in the form

$$(2.3)\qquad \overset{\nabla}{\theta}_{T,\mathrm{I}}(\varrho,\zeta;\tau)=\sum_{m=-\infty}^{\infty}\left[\mathscr{F}(\varrho,\zeta-\zeta_0-w\tau-2mL,\zeta-\zeta_0-2mL,\tau)\pm\right.$$
$$\left.\pm\mathscr{F}(\varrho,-(\zeta+\zeta_0+w\tau-2mL),-(\zeta+\zeta_0-2mL),\tau)\right],$$

valid for $0<\tau\leqslant(L-\zeta_0)/w$. The magnitude $\mathscr{F}(\varrho,\lambda_1,\lambda_2,\tau)$ is defined by Eqs. (1.8)—(1.10). For numerical purposes the next formula may be useful:

$$(2.3)'\qquad \overset{\nabla}{\theta}_{T,\mathrm{I}}(\varrho,\zeta;\tau)=\sum_{m=-\infty}^{\infty}\{\mathscr{F}(\varrho,\zeta-\zeta_0-w\tau-2mL,\zeta-\zeta_0-2mL,\tau)-$$
$$-\mathscr{F}(\varrho,\zeta-(2m-1)L,\zeta-(2m-1)L,\tau)]\pm$$
$$\pm[\mathscr{F}(\varrho,-(\zeta+\zeta_0+w\tau-2mL),-(\zeta+\zeta_0-2mL),\tau)-$$
$$-\mathscr{F}(\varrho,-(\zeta-(2m-1)L),-(\zeta-(2m-1)L),\tau)]\}.$$

The foregoing expressions (2.3), (2.3′) are met if the face planes are kept at zero temperature (the condition $(1.4')_1$) and if they are insulated (the condition $(1.4')_2$) where the minus sign refers to the planes at temperature to be held at zero, $\overset{\nabla}{\theta}_T$, while the plus sign refers to the insulated planes, $\overset{\nabla}{\theta}_{\mathrm{I}}$.

Further, we observe that Eq. (2.3) is true when $w\to 0$. Then

$$(2.4)\qquad \overset{\nabla}{\theta}_{T,\mathrm{I}}(\varrho,\zeta;\tau)\big|_{w=0}=\lim_{w\to 0}\sum_{m=-\infty}^{\infty}\left[\mathscr{F}(\varrho,\zeta-\zeta_0-w\tau-2mL,\zeta-\zeta_0-\right.$$
$$\left.-2mL,\tau)\pm\mathscr{F}(\varrho,-(\zeta+\zeta_0+w\tau-2mL),-(\zeta+\zeta_0-2mL),\tau)\right],$$

and at the limit Eqs. (22) in [1] are to our disposal.

As an example of the concept we bring the solution of the one-dimensional question of heat conduction in an infinite domain initially at zero temperature subject to the thermal instantaneous point source at x_0, in the form, cf. [4], pp. 167—177, Eqs. (6.2.5) and (6.5.12):

$$(2.5)\qquad \theta(x,t)=\frac{Q}{2\sqrt{\pi\varkappa t}}\exp\left[-(x-x_0)^2/4\varkappa t\right].$$

The solution for a one-dimensional stick of length L heated with a point source at x_0 and faces at zero temperature or thermally insulated may be written immediately from the expression (2.4), (2.5) as

$$\theta(x,t)=\frac{Q}{2\sqrt{\pi\varkappa t}}\sum_{m=-\infty}^{\infty}\{\exp[-(x-x_0-2mL)^2/4\varkappa t]\pm \pm\exp[-(x+x_0-2mL)^2/4\varkappa t]\}, \tag{2.6}$$

where Q — strength of point source at x_0, t — time.

The solution (2.3) in a meaning of the formula (2.3′) was tabulated numerically and the results are illustrated at Fig. 1 where the variation

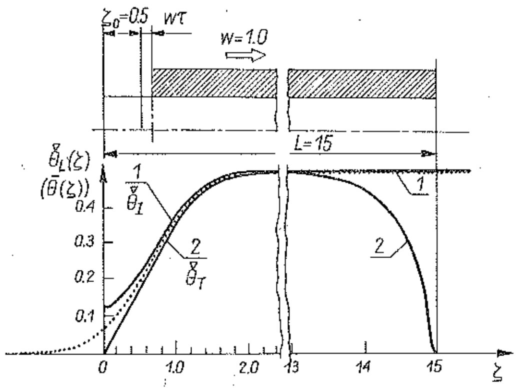

FIG. 1. Variation of temperature $\theta_L(\zeta)$ on the ζ-axis of the cylinder of length $L=15$ at time $\tau=0.2$ heated over lateral surface, $\zeta\geqslant\zeta_0$ with moving sources shrinking to zero with velocity $w=1$. Face planes are insulated perfectly, curve *1*, or held at temperature zero, curve *2*. Dotted curve referred to the infinite cylinder, $\bar{\theta}(\zeta)$.

of temperature into a cylinder of length $L=15$ at time $\tau=0.2$ along the ζ — axis is given. It is visible that an influence of the boundary condition at face planes is important in a relative short distance from the end-points.

3. CYLINDER WITH RING-TYPED HEATING ON A PART OF ITS LATERAL SURFACE

In more practical problems moving heating over a lateral cylindrical surface of bodies maintained to ring-typed thermal sources, say, of width 2β, is met quite often. We point out the procedure in an attempt to construct a solution in this case using the fundamental solution (1.11) and the results obtained in foregoing sections.

We start with the formulation of the problem for an infinite cylinder signing with $\bar{\theta}(\varrho,\zeta,\tau;\beta)$ the temperature inside the cylinder caused by thermal ring-typed sources moving with uniform velocity,

$$(3.1)\qquad \left(\nabla^2-\frac{\partial}{\partial\tau}\right)\bar{\theta}=0,\quad 0\leqslant\varrho<1,\quad |\zeta|<\infty,\quad 0<\tau<\infty,$$

$$(3.2)\qquad \bar{\theta}(\varrho,\zeta,0;\beta)=0,\quad 0\leqslant\varrho<1,\quad |\zeta|<\infty,$$

$$(3.3)\qquad \bar{\theta}(1,\zeta,\tau;\beta)=\eta(\lambda^+)-\eta(\lambda^-),$$

where

$$\lambda^+=\zeta+\beta-w\tau,\quad \lambda^-=\zeta-\beta-w\tau,\quad \beta>0.$$

The solution of Eq. (3.1) with the conditions (3.2) and (3.3) can be expressed by the function $\mathscr{F}$ well defined in Sect. 1 by Eq. (1.8)—(1.10).

$$(3.4)\qquad \bar{\theta}(\varrho,\zeta,\tau;\beta)=\mathscr{F}(\varrho,\lambda^+,\zeta+\beta,\tau)-\mathscr{F}(\varrho,\lambda^-,\zeta-\beta,\tau).$$

Proof may be performed in the way shown in the paragraph 4 or directly as in [2].

If a delay of the argument ζ in the boundary condition over the lateral surface is taken into account, say ζ_0, Eq. (3.3) requires to be modified:

$$(1.3')\qquad \bar{\theta}(1,\zeta,\tau;\beta)=\eta(\lambda_1^+)-\eta(\lambda_1^-),$$

where

$$\lambda_1^+=\zeta-\zeta_0+\beta-w\tau,\quad \lambda_1^-=\zeta-\zeta_0-\beta-w\tau.$$

We now have our problem described by Eqs. (1.1), (1,2), (1.3′) and (1.4). Seeking for a solution we assume at first multivariable function $\mathscr{G}=\mathscr{G}(\varrho,\hat{\lambda}^+,\hat{\lambda}^-,\overset{*}{\lambda}{}^+,\overset{*}{\lambda}{}^-,\tau)$ defined as

$$(3.5)\qquad \mathscr{G}(\varrho,\hat{\lambda}^+;\hat{\lambda}^-,\overset{*}{\lambda}{}^+,\overset{*}{\lambda}{}^-,\tau)=\mathscr{G}_1(\varrho,\hat{\lambda}^+,\hat{\lambda}^-;\tau)-\mathscr{G}_2(\varrho,\overset{*}{\lambda}{}^+,\overset{*}{\lambda}{}^-;\tau),$$

where

$$(3.6)\qquad \mathscr{G}_1(\varrho,\hat{\lambda}^+,\hat{\lambda}^-;\tau)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}\frac{\omega_n\mathscr{J}_0(\omega_n\varrho)}{\mathscr{J}_1(\omega_n)a_n\left(\omega_n^2+\frac{w^2}{4}\right)^{1/2}}\left(e^{a_n\hat{\lambda}^+}-e^{a_n\hat{\lambda}^-}\right), & \hat{\lambda}^+<0,\\[2ex] \displaystyle 1-\sum_{n=1}^{\infty}\frac{\omega_n\mathscr{J}_0(\omega_n\varrho)}{\mathscr{J}_1(\omega_n)\left(\omega_n^2+\frac{w^2}{4}\right)^{1/2}}\left(\frac{1}{a_n}e^{a_n\hat{\lambda}^-}-\frac{1}{a_{-n}}e^{a_{-n}\hat{\lambda}^+}\right), & \hat{\lambda}^+>0,\quad \hat{\lambda}^-<0,\\[2ex] \displaystyle\sum_{n=1}^{\infty}\frac{\omega_n\mathscr{J}_0(\omega_n\varrho)}{\mathscr{J}_1(\omega_n)a_{-n}\left(\omega_n^2+\frac{w^2}{4}\right)^{1/2}}\left(e^{a_{-n}\hat{\lambda}^+}-e^{a_{-n}\hat{\lambda}^-}\right), & \hat{\lambda}^->0;\end{cases}$$

(3.6) [cont.]

$$\mathscr{G}_2(\varrho, \overset{*}{\lambda}{}^+, \overset{*}{\lambda}{}^-; \tau) = \sum_{n=1}^{\infty} \frac{\omega_n \mathscr{J}_0(\omega_n \varrho)}{\mathscr{J}_1(\omega_n)} \left\{ \frac{1}{\omega_n^2} e^{-\omega_n^2 \tau} \left[\operatorname{erf}\left(\frac{\overset{*}{\lambda}{}^+}{2\sqrt{\tau}} \right) - \operatorname{erf}\left(\frac{\overset{*}{\lambda}{}^-}{2\sqrt{\tau}} \right) \right] + \frac{1}{a_n - a_{-n}} \left(\frac{1}{a_n} [K(-\overset{*}{\lambda}{}^+, a_n, \omega_n; \tau) - K(-\overset{*}{\lambda}{}^-, a_n, \omega_n; \tau)] - \frac{1}{a_{-n}} [H(-\overset{*}{\lambda}{}^+, -a_{-n}, \omega_n; \tau) - H(-\overset{*}{\lambda}{}^-, -a_{-n}, \omega_n; \tau)] \right) \right\}, \quad \overset{*}{\lambda}{}^+ < 0,$$

$$\mathscr{G}_2(\varrho, \overset{*}{\lambda}{}^+, \overset{*}{\lambda}{}^-; \tau) = \sum_{n=1}^{\infty} \frac{\omega_n \mathscr{J}_0(\omega_n \varrho)}{\mathscr{J}_1(\omega_n)} \left\{ \frac{1}{\omega_n^2} e^{-\omega_n^2 \tau} \left[\operatorname{erf}\left(\frac{\overset{*}{\lambda}{}^+}{2\sqrt{\tau}} \right) - \operatorname{erf}\left(\frac{\overset{*}{\lambda}{}^-}{2\sqrt{\tau}} \right) \right] - \frac{1}{a_n - a_{-n}} \left(\frac{1}{a_n} [H(\overset{*}{\lambda}{}^+, a_n, \omega_n; \tau) + K(-\overset{*}{\lambda}{}^-, a_n, \omega_n; \tau)] - \frac{1}{a_{-n}} [K(\overset{*}{\lambda}{}^+, -a_{-n}, \omega_n; \tau) + H(-\overset{*}{\lambda}{}^-, -a_{-n}, \omega_n; \tau)] \right) \right\}, \quad \overset{*}{\lambda}{}^+ > 0, \quad \overset{*}{\lambda}{}^- < 0;$$

$$\mathscr{G}_2(\varrho, \overset{*}{\lambda}{}^+, \overset{*}{\lambda}{}^-, \tau) = \sum_{n=1}^{\infty} \frac{\omega_n \mathscr{J}_0(\omega_n \varrho)}{\mathscr{J}_0(\omega_n)} \left\{ \frac{1}{\omega_n^2} e^{-\omega_n^2 \tau} \left[\operatorname{erf}\left(\frac{\overset{*}{\lambda}{}^+}{2\sqrt{\tau}} \right) - \operatorname{erf}\left(\frac{\overset{*}{\lambda}{}^-}{2\sqrt{\tau}} \right) \right] - \frac{1}{a_n - a_{-n}} \left(\frac{1}{a_n} [H(\overset{*}{\lambda}{}^+, a_n, \omega_n; \tau) - H(\overset{*}{\lambda}{}^-, a_n, \omega_n; \tau)] - \frac{1}{a_{-n}} [K(\overset{*}{\lambda}{}^+, -a_n, \omega_n; \tau) - K(\overset{*}{\lambda}{}^-, -a_{-n}, \omega_n; \tau)] \right) \right\}, \quad \overset{*}{\lambda}{}^- > 0.$$

and further

$$H(\lambda, a, \omega; \tau) = -\exp[(a^2 - \omega^2)\tau + a\lambda] \operatorname{erfc}\left(a\sqrt{\tau} + \frac{\lambda}{2\sqrt{\tau}} \right),$$

$$K(\lambda, a, \omega; \tau) = \exp[(a^2 - \omega^2)\tau - a\lambda] \operatorname{erfc}\left(a\sqrt{\tau} - \frac{\lambda}{2\sqrt{\tau}} \right),$$

true if $\operatorname{Re}[a] > 0$, $\lambda > 0$.

Studying Eqs. (3.6), we discuss some properties of the function $\mathscr{F}$ and $\mathscr{G}$.

i) The function $\mathscr{G}(\varrho, \lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^-, \tau)$ is continuous at the cross-sections

$$\lambda_1^+ = 0, \quad \lambda_1^- = 0,$$
$$\lambda_2^+ = 0, \quad \lambda_2^- = 0, \quad 0 \leqslant \varrho < 1,$$

where

$$\lambda_2^+ = \zeta - \zeta_0 + \beta, \quad \lambda_2^- = \zeta - \zeta_0 - \beta.$$

ii) It can be proved that

$$(3.7) \qquad \begin{aligned} &\mathscr{F}_1(\varrho, \lambda_1^+; \tau) - \mathscr{F}_1(\varrho, \lambda_1^-; \tau) = \mathscr{G}_1(\varrho, \lambda_1^+, \lambda_1^-; \tau), \\ &\mathscr{F}_2(\varrho, \lambda_2^+; \tau) - \mathscr{F}_2(\varrho, \lambda_2^-; \tau) = -\mathscr{G}_2(\varrho, \lambda_2^+, \lambda_2^-; \tau). \end{aligned}$$

Hence we have the following implication:

$$(3.8) \qquad \mathscr{F}(\varrho, \lambda_1^+, \lambda_2^+, \tau) - \mathscr{F}(\varrho, \lambda_1^-, \lambda_2^-, \tau) = \mathscr{G}_1(\varrho, \lambda_1^+, \lambda_1^-; \tau) - \mathscr{G}_2(\varrho, \lambda_2^+, \lambda_2^-; \tau) = \mathscr{G}(\varrho, \lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^-, \tau).$$

It may be verified that Eq. (3.8) is the solution of Eq. (3.1) with the conditions (3.2) and (1.3′).

iii) It is obvious that

$$(3.9) \qquad \bar{\theta}(\varrho, \zeta, \tau; \beta) = \mathscr{G}_1(\varrho, \lambda^+, \lambda^-; \tau) - \mathscr{G}_2(\varrho, \zeta+\beta, \zeta-\beta; \tau) = \mathscr{G}(\varrho, \lambda^+, \lambda^-, \zeta+\beta, \zeta-\beta, \tau)$$

being the solution of Eqs. (3.1)—(3.3).

We specify the symmetric function to $\mathscr{G}(\varrho, \lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^-, \tau)$, Eq. (3.8), with regard to the plane $\zeta = 0$.

$$(3.10) \qquad \mathscr{G}(\varrho, -\tilde{\lambda}_1^-, -\tilde{\lambda}_1^+, -\tilde{\lambda}_2^-, -\tilde{\lambda}_2^-, \tau) = -\mathscr{G}_1(\varrho, -\tilde{\lambda}_1^-, -\tilde{\lambda}_1^+; \tau) - \mathscr{G}_2(\varrho, -\tilde{\lambda}_2^-, -\tilde{\lambda}_2^+; \tau),$$

where

$$\tilde{\lambda}_1^+ = \zeta + \zeta_0 + \beta + w\tau, \quad \tilde{\lambda}_1^- = \zeta + \zeta_0 - \beta + w\tau,$$
$$\tilde{\lambda}_2^+ = \zeta + \zeta_0 + \beta, \quad \tilde{\lambda}_2^- = \zeta + \zeta_0 - \beta,$$

and recall

$$\zeta_0 > 0, \quad \beta > 0, \quad w > 0.$$

In accordance to Sect. 2.1, we can present the solution of a problem described by Eqs. (1.1), (1.2), (1.3′) and (1.4). This means we give an expression for the temperature field in a semi-infinite cylinder heated by a ring-typed thermal source at a part of the lateral surface making use of Eqs. (3.8) and (3.10); thus we have

$$(3.11)\qquad \overset{\vdash}{\theta}(\varrho, \zeta, \tau; \beta) = \mathcal{G}(\varrho, \lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^-, \tau) \pm \\ \pm \mathcal{G}(\varrho, -\tilde{\lambda}_1^-, -\tilde{\lambda}_1^+, \tilde{\lambda}_2^-, -\tilde{\lambda}_2^+, \tau),$$

where the sign — at the right side in the last equality is referred to the requirement $(1.4)_1$, i.e. to the temperature zero at the face $\zeta = 0$ whereas the sign + to the data $(1.4)_2$ i.e. to zero heat flux across this face plane.

On the other hand, the result of Sect. 2.2 allows to obtain the temperature distribution in a cylinder of finite length L. This is the solution of Eq. (1.1) with the conditions (1.2), (1.3′) and (1.4′) in the next form valid for $0 < \tau \leqslant [L-(\zeta_0+\beta)]/w$.

$$(3.12)\qquad \overset{\vdash\!\dashv}{\theta}(\varrho, \zeta, \tau; \beta) = \sum_{m=-\infty}^{\infty} [\mathcal{G}(\varrho, \lambda_1^+ - 2mL, \lambda_1^- - 2mL, \lambda_2^+ - 2mL, \lambda_2^- - \\ -2mL, \tau) \pm \mathcal{G}(\varrho, -(\tilde{\lambda}_1^- - 2mL), -(\tilde{\lambda}_1^+ - 2mL), -(\tilde{\lambda}_2^- - \\ -2mL), -(\tilde{\lambda}_2^+ - 2mL), \tau)],$$

where the minus sign refers to the face planes at temperature zero and the plus sign to the face planes perfectly thermally insulated. We underline that the functions at the right side of Eqs. (3.11) and (3.12) are defined by Eqs. (3.5) and (3.6) and it can be justified that they hold at the limit if $w \to 0$, too. In the case of Eq. (3.11), we have the expression

$$(3.13)\qquad \overset{\vdash}{\theta}(\varrho, \zeta, \tau; \beta)\big|_{w=0} = \lim_{w\to 0} [\mathcal{G}(\varrho, \lambda_1^+, \lambda_1^-, \lambda_2^+, \lambda_2^-, \tau) \pm \\ \pm \mathcal{G}(\varrho, -\tilde{\lambda}_1^-, -\tilde{\lambda}_1^+, -\tilde{\lambda}_2^-, -\tilde{\lambda}_2^+, \tau)],$$

which is accurately connected with Eqs. (3.9) and (3.10) in [5], as a consequence of passing to the limit.

It may be verified that the formula (3.12) pertaining to the finite cylinder $0 \leqslant \zeta \leqslant L$ reduces to the corresponding Eq. (3.11) for the semi-infinite body when we put $L \to \infty$ since in this case only one term of indicated series remains.

A finite cylinder with one bounding face plane at zero temperature and the other face plane thermally insulated may also be studied by a similar procedure. In fact any arbitrary boundary condition across the face planes may be imposed and just a different one at either of these planes since a combination of Eq. $(1.4')_1$ and Eq. $(1.4')_2$ with proper values of the delay ζ_0 allows to meet practically every set of data gained by the experiment.. Hence a computer technique and experimental data give rise to one research system of investigated problems. Some conclusions from the foregoing considerations are expressed in Figs. 2, 3 and 4 as a result of numerical calculation

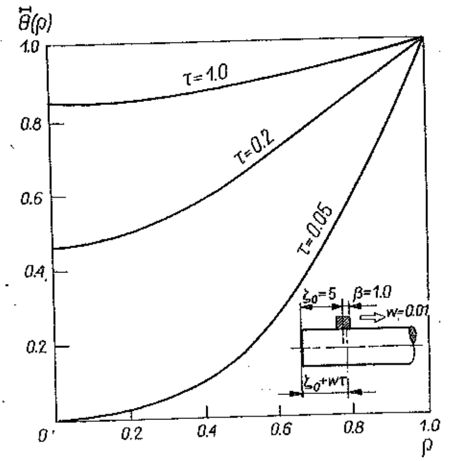

FIG. 2. Variation of temperature $\overset{\vdash\!\dashv}{\theta} = \overset{\vdash\!\dashv}{\theta}(\varrho)$ in sections $\zeta_* = \zeta_0 + w\tau$ across the cylinder, $L = 15$, $\beta = 1.0$, $\zeta_0 = 5$, $w = 0.01$; at time $\tau = 0.05$; 0.2; 1.0; under ring-typed heating.

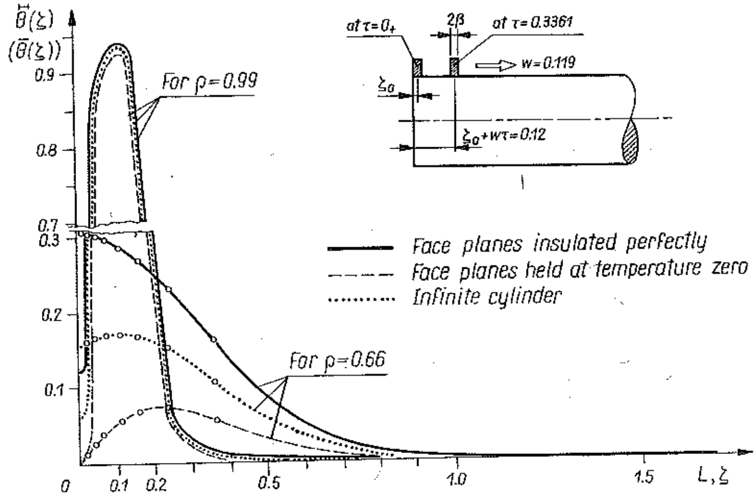

FIG. 3. Variation of temperature $\overset{\vdash\!\dashv}{\theta}(\zeta)$ into finite cylinder of length $L = 8.0$ for $\varrho = 0.66$, 0.99 heated with ring-typed sources which symmetry plane is located at $\zeta_* = \zeta_0 + w\tau = 0.12$ ($\tau = 0.3361$). Accordingly $\beta = 0.08$; $w = 0.119$, $\zeta_0 = 0.08$. For a comparison the suitable data of infinite cylinder are included, $\bar{\theta}(\zeta)$.

of the temperature function $\overset{\vdash\!\dashv}{\theta}(\varrho, \zeta, \tau; \beta)$ according to Eq. (3.12). Figure 2 shows the temperature distribution $\overset{\vdash\!\dashv}{\theta}(\varrho)$ in a cross-section of the cylinder, $L = 15$, under ring-typed heat occupancy at time $\tau = 0.05$; 0.2; 1.0. The variation of temperature $\overset{\vdash\!\dashv}{\theta}(\zeta)$ in a cylinder of length $L = 8.0$ for a short

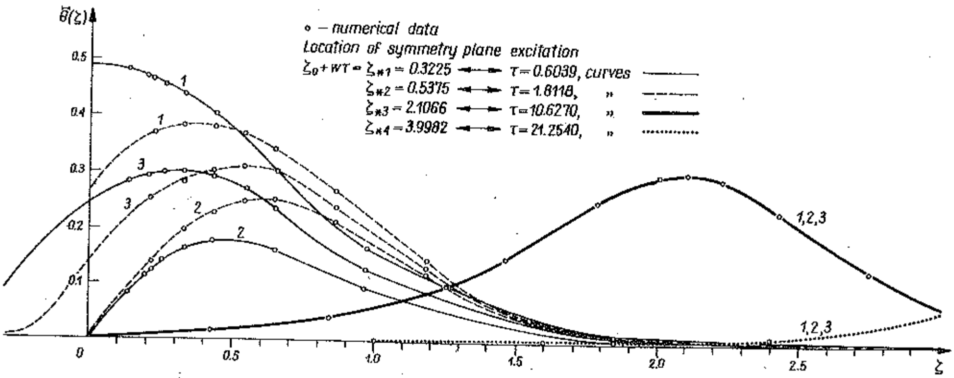

FIG. 4. Temperature field along ζ-axis at $\varrho = 0.33$ into cylinder of length $L = 8.0$ heated suddenly over its lateral surface with moving ring-typed thermal source by assumption that the face planes are insulated perfectly, curves *1*, or are held at zero, curves *2*, at the moments $\tau = 0.6039$; 1.8118; 10.6270; 21.2540. Accordingly $\beta = 0.2150$, $w = 0.178$, $\zeta_0 = 0.2150$.
For comparison the temperature variation into infinite cylinder is included, curves *3*.

Table 1.

$\varrho = RO_1 = 0.6600$
$\varrho = RO_2 = 0.9900$
$\tau = TAU = 0.3361$

zeta 0+w. TAU = 0.1200
ζ_0 τ

ζ = zeta	TETA (FI. IN.) for RO_1	RO_2	TETA (INF) RO_1	RO_2	TETA (FIN. CO) RO_1	RO_2
0.0000	0.311	0.125	0.155	0.063	0.000	0.000
0.0240	0.310	0.196	0.162	0.161	0.013	0.126
0.0480	0.308	0.721	0.167	0.698	0.026	0.675
0.0720	0.303	0.900	0.170	0.884	0.038	0.867
0.0800	0.302	0.915	0.171	0.900	0.041	0.885
0.0960	0.297	0.931	0.173	0.918	0.048	0.905
0.1200	0.290	0.935	0.173	0.926	0.057	0.916
0.1600	0.274	0.907	0.171	0.900	0.067	0.893
0.2400	0.234	0.066	0.153	0.062	0.073	0.058
0.3600	0.167	0.012	0.113	0.010	0.059	0.008
1.6960	0.002	0.000	0.001	0.000	0.001	0.000
3.2720	0.000	0.000	0.000	0.000	0.000	0.000
4.8480	0.000	0.000	0.000	0.000	0.000	0.000
6.4240						

FI.IN — endsinsulated, FIN.CO — ends at constant (zero) temperature

time, $\tau = 0.3361$, at $\varrho = 0.66$ and $\varrho = 0.99$ is given in Fig. 3 and for different time in Fig. 4 at $\varrho = 0.33$. The influence of requirements on the face planes is compared with data for the infinite cylinder, Tables 1 and 2.

Thus it was shown that the solutions of the problem formulated above have been obtained in the form of Eqs. (3.11) and (3.12) definite by a combi-

Table 2.

ϱ = RO = 0.33
τ = TAU_1 = 0.6039
τ = TAU_2 = 1.8118

ζ = zeta	TETA (FI. IN.) for TAU_1	TETA (FI. IN.) TAU_2	TETA (INF) TAU_1	TETA (INF) TAU_2	TETA (FIN. CO.) TAU_1	TETA (FIN. CO.) TAU_2
0.0000	0.490		0.245		0.000	
0.0645	0.488		0.264		0.040	
0.1290	0.482		0.280		0.077	
0.1935	0.473		0.292		0.110	
0.2150	0.469	0.369	0.294	0.251	0.120	0.132
0.2580	0.459		0.298		0.137	
0.3225	0.442	0.38	0.300	0.28	0.158	0.19
0.4300	0.403	0.375	0.289	0.300	0.176	0.225
0.6450	0.302	0.341	0.232	0.295	0.162	0.249
0.9675	0.160	0.200	0.127	0.182	0.095	0.165
1.8580	0.017	0.032	0.014	0.028	0.010	0.024
3.3935	0.000	0.000	0.000	0.000	0.000	0.000
	curve 1		curve 3		curve 2	

nation of the function $\mathscr{G} = \mathscr{G}(\varrho, \hat{\lambda}^+, \hat{\lambda}^-, \overset{*}{\lambda}{}^+, \overset{*}{\lambda}{}^-, \tau)$, where the arguments $\hat{\lambda}^+, \hat{\lambda}^-, \overset{*}{\lambda}{}^+, \overset{*}{\lambda}{}^-$ are taken with different delay. The properties of the function $\mathscr{G}$ carried in points ii) and iii), Eqs. (3.7)—(3.9), allow us to state that the formulae (3.11) and (3.12) are expressed by the fundamental solution (1.11) of the system (1.5)—(1.7) indeed, what was the main task in this work, keep Eqs. (1.12) and (1.11) in mind.

4. Cylinder subject to moving axi-symmetric point heat overall lateral surface

Let us now try to enlarge our considerations concerned with the relations (3.1), (3.2), (1.3′) and (3.8) defining the temperature in a long cylinder subject to the thermal ring-typed sources to some limit case when a surface source approaches a point source distribution at circumference of the lateral boundary. In this case the problems described by Eq. (1.1) and the conditions (1.2)—(1.4) should be changed, with regard to the data given by Eq. (1.3) only. Thus we have

$$(1.3'') \qquad \bar{\theta}(1, \zeta; \tau) = \bar{\theta}^{\delta}(1, \zeta; \tau) = \theta_0' \, \delta(\mathring{\lambda}_1), \qquad \mathring{\lambda}_1 = \zeta - \zeta_0 - w\tau,$$

where $\delta(x)$ denotes the Dirac delta function.

It can be readily seen that if a limit is taken in the formula (1.3′) Sect. 3, when $\beta \to 0$ an identical result is obtained as before. To show this, there is enough to write a suitable quotient and to tend with β to zero bearing in mind that $\lambda_1^+ = \lambda_1^+ (\zeta, \tau)$, $\lambda_1^- = \lambda_1^- (\zeta, \tau)$. Hence

$$(4.1) \qquad \lim_{\beta \to 0} \frac{\bar{\theta}(1, \zeta, \tau; \beta)}{2\beta} = \theta_0 \lim_{\beta \to 0} \frac{\eta(\lambda_1^+) - \eta(\lambda_1^-)}{2\beta} = \theta_0 \frac{\partial}{\partial \zeta} \eta(\mathring{\lambda}_1) = \theta_0\, \delta(\mathring{\lambda}_1).$$

Conclusions connected with the relation (4.1) will be presented later but now we wish to prove the following theorem.

THEOREM. *Let B be the infinite domain bounded by the cylindrical surface ∂B with radius $\varrho = 1$. The solution $\theta(\varrho, \zeta; \tau)$ of the heat conduction equation* (1.5) *satisfying the initial condition* (1.6) *and the boundary condition* (1.3″) *may be introduced by the relation*

$$(4.2) \qquad \bar{\theta}(\varrho, \zeta; \tau) = \bar{\theta}_\delta(\varrho, \mathring{\lambda}_1, \mathring{\lambda}_2; \tau) = \bar{\theta}_1^\delta(\varrho, \mathring{\lambda}_1; \tau) + \bar{\theta}_2^\delta(\varrho, \mathring{\lambda}_2; \tau),$$

where $\bar{\theta}_1^\delta$ and $\bar{\theta}_2^\delta$ are given in the form:

I. Integral expression in the Fourier transform space

$$(4.3) \qquad \begin{aligned} \bar{\theta}_1^\delta(\varrho, \mathring{\lambda}_1; \tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{I_0[(\alpha^2 + i\alpha w)^{1/2} \varrho]}{I_0[(\alpha^2 + i\alpha w)^{1/2}]} e^{-i\alpha\lambda_1} d\alpha, \\ \bar{\theta}_2^\delta(\varrho, \mathring{\lambda}_2; \tau) &= -\frac{2}{2\pi} \sum_{n=1}^{\infty} \omega_n \frac{\mathscr{J}_0(\omega_n \varrho)}{\mathscr{J}_1(\omega_n)} \int_{-\infty}^{\infty} \frac{e^{-(\omega_n^2 + \alpha^2)\tau - i\alpha\mathring{\lambda}_2}}{\omega_n^2 + \alpha^2 + i\alpha w} d\alpha, \end{aligned}$$

or

II. Equivalent series representation

$$(4.4) \qquad \bar{\theta}_1^\delta(\varrho, \mathring{\lambda}_1; \tau) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{\mathscr{J}_0(\omega_n \varrho)\, \omega_n \exp[a_n \mathring{\lambda}_1]}{\mathscr{J}_1(\omega_n) \left(\omega_n^2 + \dfrac{w^2}{4}\right)^{1/2}}, & \mathring{\lambda}_1 < 0, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\mathscr{J}_0(\omega_n \varrho)\, \omega_n \exp[a_{-n} \mathring{\lambda}_1]}{\mathscr{J}_1(\omega_n) \left(\omega_n^2 + \dfrac{w^2}{4}\right)^{1/2}}, & \mathring{\lambda}_1 > 0, \end{cases}$$

$$(4.5) \qquad \bar{\theta}_2^\delta(\varrho, \mathring{\lambda}_2; \tau) = \sum \frac{\omega_n \mathscr{J}_0(\omega_n \varrho)}{\mathscr{J}_1(\omega_n)} \left\{ -\frac{1}{\omega_n^2 + a_n^2} \exp[(-\omega_n^2 + a_n^2)\tau + a_n \mathring{\lambda}_2] \times \right.$$

$$\times\left(a_n \operatorname{erfc}\left(a_n\sqrt{\tau+\frac{\mathring{\lambda}_2}{2\sqrt{\tau}}}\right)-\frac{1}{\sqrt{\pi\tau}}\exp\left[-\left(a_n\sqrt{\tau}+\frac{\mathring{\lambda}_2}{2\sqrt{\tau}}\right)^2\right]\right)+$$

$$+\frac{1}{\omega_n^2+a_{-n}^2}\exp\left[(-\omega_n^2+a_{-n}^2)\tau+a_{-n}\mathring{\lambda}_2\right]\times$$

$$\times\left(a_{-n}\operatorname{erfc}\left(-a_{-n}\sqrt{\tau}-\frac{\mathring{\lambda}_2}{2\sqrt{\tau}}\right)+\frac{1}{\sqrt{\pi\tau}}\exp\left[-\left(a_{-n}\sqrt{\tau}+\frac{\mathring{\lambda}_2}{2\sqrt{\tau}}\right)^2\right]\right)-$$

$$-\frac{1}{\omega_n\sqrt{\pi\tau}}\exp\left[-\left(\omega_n^2\tau+\frac{(\mathring{\lambda}_2)^2}{4\tau}\right)\right]\Bigg\},\quad |\mathring{\lambda}_2|<\infty.$$

Designations are adopted according to Sect. 1. Here $\mathring{\lambda}_2=\zeta-\zeta_0$. It is obvious that the functions $\bar{\theta}_1^\delta$ and $\bar{\theta}_2^\delta$ are continuous in B and regular if $\zeta\to\infty$, $\theta_0=1$.

Proof. The thesis will be accomplished if the desirable function (4.2) satisfies: i) the equation in space B, ii) the initial data in B, iii) the boundary conditions on ∂B. We note that $\left(\nabla^2-\frac{\tau}{\partial\tau}\right)(\bar{\theta}_1^\delta+\bar{\theta}_2^\delta)=\left(\nabla^2-\frac{\partial}{\partial\tau}\right)\bar{\theta}_1^\delta+\left(\nabla^2-\frac{\partial}{\partial\tau}\right)\bar{\theta}_2^\delta$ since the operator $\nabla^2-\frac{\partial}{\partial t}$ is linear.

Ad. I. Inspection of the relations (4.3).

i) Carrying out the indicated operations under the sign of the integral and the sum and using the next formulae

$$\frac{d}{dz}I_0(z)=I_1(z),\qquad z\frac{d}{dz}I_1(z)=-I_1(z)+zI_0(z),$$

$$\frac{d}{dz}\mathcal{J}_0(z)=-\mathcal{J}_1(z),\ z\frac{d}{dz}\mathcal{J}_1(z)=-\mathcal{J}_1(z)+z\mathcal{J}_0(z),$$

we proceed to the relations

$$\nabla^2\bar{\theta}_1^\delta=\left(\frac{\partial^2}{\partial\varrho^2}+\frac{1}{\varrho}\frac{\partial}{\partial\varrho}+\frac{\partial^2}{\partial\zeta^2}\right)\bar{\theta}_1^\delta=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{I_0[(\alpha^2+i\alpha w)^{1/2}\varrho]}{I_0[(\alpha^2+i\alpha w)^{1/2}]}\,i\alpha w\exp[-i\alpha\mathring{\lambda}_1]\,d\alpha,$$

$$\nabla^2\bar{\theta}_2^\delta=\frac{2}{2\pi}\sum_{n=1}^{\infty}\frac{\mathcal{J}_0(\omega_n\varrho)}{\mathcal{J}_1(\omega_n)}\int_{-\infty}^{\infty}\frac{\omega_n^2+\alpha^2}{\omega_n^2+\alpha^2+i\alpha w}\times$$

$$\times\exp[-(\omega_n^2+\alpha^2)\tau-i\alpha\mathring{\lambda}_2]\,d\alpha,$$

which, in an evident manner, are reduced with proper terms of an operation $\left(-\frac{\partial}{\partial \tau}\right)$ acting on the functions (4.3).

ii) We take a limit of the relations (4.3) at $\tau \to 0$ and apply the identity (3.4.2) in [5] to get

$$\bar{\theta}_1^\delta(\varrho, \mathring{\lambda}_1; 0) = \bar{\theta}_1^\delta(\varrho, \mathring{\lambda}_2; 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{I_0[(\alpha^2+i\alpha w)^{1/2}\varrho]}{I_0[(\alpha^2+i\alpha w)^{1/2}]} \times$$

$$\times \exp[-i\alpha\mathring{\lambda}_2]\, d\alpha,$$

$$\bar{\theta}_2^\delta(\varrho, \mathring{\lambda}_2; 0) = -\frac{2}{2\pi} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\omega_n \mathscr{J}_0(\omega_n \varrho)}{\mathscr{J}_1(\omega_n)(\omega_n^2+\alpha^2+i\alpha w)} \exp[-i\alpha\mathring{\lambda}_2]\, d\alpha =$$

$$= -\bar{\theta}_1^\delta(\varrho, \mathbf{k}\mathring{\lambda}_2; 0).$$

Thus we have, according to the relation (4.2),

$$\bar{\theta}_\delta(\varrho, \mathring{\lambda}_1, \mathring{\lambda}_2, 0) = \bar{\theta}_1^\delta(\varrho, \mathring{\lambda}_1; 0) + \bar{\theta}_2^\delta(\varrho, \mathring{\lambda}_2; 0) = 0.$$

iii) Passing to the limit under the sign of the integral in the expression $(4.3)_1$ and the sum in Eq. $(4.3)_2$ at $\varrho \to 1$ we obtain, in agreement with Eq. (4.2),

$$(4.6)\qquad \bar{\theta}_\delta(1, \mathring{\lambda}_1, \mathring{\lambda}_2, \tau) = \bar{\theta}_1^\delta(1, \mathring{\lambda}_1; \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-i\alpha\mathring{\lambda}_1]\, d\alpha \Leftrightarrow \delta(\mathring{\lambda}_1),$$

since $\bar{\theta}_2^\delta(1, \mathring{\lambda}_2; \tau) \to 0$ because of $\lim\limits_{\varrho \to 1} \mathscr{J}_0(\omega_n \varrho) = 0$.

We see that the expression (4.6) exhibits a formal description of Dirac's delta function, compare with [6], p. 35, (86f).

Ad II. Inspection of the relations (4.4) and (4.5).

We return to the function (4.4) and (4.5) evaluating the denoted operations and arranging appropriate terms of the obtained results in groups similarly as was done above at point I in agrrement with i), ii) and iii), respectively. Finally we state out that the sequence solution is quite right.

For instance, the requirement ii) may be justified as follows. There are two cases of interest to be argued: for $\mathring{\lambda}_1 < 0$ and $\mathring{\lambda}_1 > 0$ both at $\tau \to 0$. Analysing the function (4.4) within the interval $\mathring{\lambda}_1 < 0$ and remembering the limit values of $\operatorname{erfc}(\infty) = 0$, $\operatorname{erfc}(-\infty) = 2$, $\operatorname{erf}(\infty) = 1$, $\operatorname{erf}(-\infty) = -1$, and the definition of the magnitude a_n whose particular consequence is that $a_n + \frac{w}{2} = \left(\omega_n^2 + \frac{w^2}{4}\right)^{1/2}$, we proceed to the simple dependence

$$a_n^2 + wa_n - \omega_n^2 = 0,$$

satisfied from the assumption.

The case when $\mathring{\lambda}_1 > 0$ is taken into account can be interpreted similarly.

At the end some remarks are given. It is clear that the relation (4.1). suggests to write the following conclusions but it is the Theorem that permits us to do so.

CONCLUSION 1. The solutions (4.4) and (4.5) of the conduction of the heat problem in an infinite circular cylinder subject to the moving boundary condition on a lateral surface described by the Dirac delta function is the derivative of Eq. (1.12) with respect to the axi-coordinate ζ with the fundamental solution involved.

CONCLUSION 2. The solutions of Eq. (1.1) in a semi-infinite and finite cylindric domain initially at zero temperature and the boundary conditions (1.3''), (1.4) and (1.4') are the derivatives with respect to the axi-coordinate ζ of the solutions (2.1), (2.2) and (2.3) adequate to appropriate boundary data across the face planes.

REFERENCES

1. T. ROŻNOWSKI, *Non-steady state of temperature in a long cylinder with moving boundary condition*, Bull. Acad. Polon Sci., **13**, 1, 45—51, 1965.
2. T. ROŻNOWSKI, *Non-steady state temperature in a long circular cylinder with heat moving over the lateral surface*, Bull. Acad. Polon. Sci., **12**, 5, 333—340, 1964.
3. T. ROŻNOWSKI, *Quantitative analysis of a non-steady-state temperature field in a long cylinder*, Arch. Mech. Stos., **20**, 5, 565—574, 1968.
4. B. A. BOLEY, J. H. WEINER, *Theory of thermal stresses*, New York-London John Wiley, 1962.
5. T. ROŻNOWSKI, *An axially symmetric thermoelastic problem with a moving boundary condition*, Arch. Mech. Stos. **17**, 5, 693—710, 1965.
6. I. N. SNEDDON, *Fourier transforms*, McGraw Hill Book Comp., Inc., New York 1951.
7. H. S. CARSLAW, J. C. JAEGER, *Conduction of heat in solids*, Oxford at the Clarendon Press, 2 edit. 1959.

STRESZCZENIE

POLE TEMPERATURY W PÓŁNIESKOŃCZONYM I SKOŃCZONYM CYLINDRZE PRZY RUCHOMYM OGRZANIU POBOCZNICY

W pracy wyprowadzono związki opisujące niestacjonarne pole temperatury w półnieskończonym i skończonym cylindrze ogrzanym na części pobocznicy ruchomymi, osiowo-symetrycznymi, powierzchniowymi źródłami ciepła. Założono dwa przypadki warunków brzegowych na powierzchni czołowej: temperaturę równą zero i doskonałą izolację cieplną. Na podstawie rozwiązania fundamentalnego dla długiego cylindra w [1], które zostało oprogramowane i przeanalizowane wyczerpując numerycznie w [3], wyprowadzono wzory na temperaturę przy różnych sposobach ogrzewania pobocznicy. Rozkład temperatury wewnątrz cylindra o długości L jest przedstawiony jako nieskończona suma elementów podobnych do rozwiązania fundamentalnego. W zakończeniu udowodniono istnienie transformacji otrzymanych wyników do funkcji Greena w sensie pewnego przejścia granicznego. Zauważono, że możliwe jest założenie bardziej złożonych warunków na powierzchni czołowej systemu i prowadzenie obliczeń numerycznych na podstawie rozwiązania fundamentalnego w sensie analizy jakościowej i ilościowej pod kątem zastosowań. Wyniki numeryczne przedstawiono na wykresach.

Резюме

ПОЛЕ ТЕМПЕРАТУРЫ В ПОЛУБЕСКОНЕЧНОМ И КОНЕЧНОМ ЦИЛИНДРЕ ПРИ ПОДВИЖНОМ НАГРЕВЕ БОКОВОЙ ПОВЕРХНОСТИ

В работе выведены соотношения описывающие нестационарное поле температуры в полубесконечном и конечном цилиндре, нагреваемом в части боковой поверхности подвижными, осесимметричными, поверхностными источниками тепла, предполагая два случая граничных условий на лобовой поверхности: температуру равную нулю и идеальную тепловую изоляцию. Опираясь на фундаментальное решение для длинного цилиндра в [1], которое опрограммировано и проанализировано исчерпывающим образом численно в [3], выведены формулы для температуры, при разных способах нагрева боковой поверхности. Распределение температуры внутри цилиндра с длиной L представлено как бесконечная сумма элементов аналогичных фундаментальному решению. В заключении доказано существование преобразования полученных результатов к функцию Грина в смысле некоторого предельного перехода. Замечается, что возможно предположение более сложных условий на лобовой поверхности системы и ведение численных расчетов, опираясь на фундаментальное решение в смысле качественного и количественного анализа с точки зрения применений. Приведены результаты численных расчетов в виде диаграмм.

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