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THE TORSION OF A NONHOMOGENEOUS VISCO-ELASTIC BAR OF ELLIPTIC CROSS-SECTION

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An exact solution for a torsion problem of a nonhomogeneous visco-elastic bar of elliptic cross-section is presented. A method based on elastic-viscoelastic analogy is used to obtain the stress distribution in the cross-section of the bar. The numerical results show that the nonhomogeneity reduces the stress as compared to the homogeneous one.

INTRODUCTION

The influence of nonhomogeneity is of great importance in the study of visco-elastic materials such as composite materials, fibre reinforced plastics, fibre glasses and glass epoxy which are widely used in engineering design and technology to increase the strength of the construction. In such materials the material properties vary with position in random manner. So far only a few papers have been devoted to the investigation of the effect of nonhomogeneity on the stresses of torsional problems. The torsional problem of a circular bar and a hollow sphere of nonhomogeneous visco-elastic material have been studied KARAMANY [1, 2]. In the present work, an analysis of a torsional problem of a nonhomogeneous visco-elastic bar of elliptic cross-section is presented. The creep function is assumed here as a function of time and coordinates. The method of solution as discussed by ILIOUSHIN [3] for the homogeneous visco-elastic problem is applied here to the nonhomogeneous visco-elastic problem and the numerical results for the stress distribution in the elliptic cross-section of the bar are obtained and compared with those of the homogeneous one.

1. FORMULATION OF THE NONHOMOGENEOUS VISCO-ELASTIC PROBLEM

Consider the boundary value problem

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \varrho F_i = 0, \quad \epsilon_{ijk}\epsilon_{lmn}\frac{\partial^2 \epsilon_{km}}{\partial x_j \partial x_n} = 0; \tag{1.1}$$

$$S_{ij}=\int_0^t R(t-\tau,x,y,z)\,de_{ij}(\tau),\qquad \sigma(t)=K\theta(t),$$

$$e_{ij}=\int_0^t \Pi(t-\tau,x,y,z)\,dS_{ij}(\tau),\qquad \theta(t)=\frac{\sigma(t)}{K},\tag{1.2}$$

$$\sigma_{ij}n_j|_{S_\sigma}=q_i(x_s,y_s,z_s,t);\qquad u_i|_{S_u}=\varphi_i(x_s,y_s,z_s,t),\tag{1.3}$$

where

$$\sigma=\frac{\sigma_{kk}}{3},\qquad \theta=\varepsilon_{kk},\qquad S_{ij}=\sigma_{ij}-\sigma\delta_{ij},\qquad e_{ij}=\varepsilon_{ij}-\frac{\theta}{3}\delta_{ij}.$$

σ_{ij}, ε_{ij} respectively, denote the stress and strain tensors; $\Pi(t,x,y,z)$ is the creep function; K — the bulk modulus; (x,y,z) — the coordinates of an arbitrary point M situated inside the body; t is the time; ϵ_{ijk} — the permutation symbol; u_i — the displacement; S_σ — the boundary surface where surface tractions are specified. S_u — the boundary surface where displacements are specified, n_j — the unit vector along the outer normal to the surface, (x_s,y_s,z_s) — the coordinates of an arbitrary point on the surface S_σ or S_u, and q_i, φ_i are given functions $(i,j=1,2,3)$.

It is assumed that the loading is quasi-static and the relaxation effects of the volume properties of the material are ignored [4].

We shall use the Laplace–Carson transform with the real parameter p. The image of $f(t)$ is $\bar f(p)$ defined by [5].

$$\bar f(p)=p\int_0^\infty e^{-pt}f(t)\,dt.\tag{1.4}$$

Taking the Laplace-Carson transform of the problem (1.1)—(1.3), we obtain the following boundary value problem in terms of images:

$$\frac{\partial\bar\sigma_{ij}}{\partial x_j}+\varrho\bar F_i=0,\qquad \epsilon_{ijk}\epsilon_{lmn}\frac{\partial^2\bar\varepsilon_{km}}{\partial x_j\,\partial x_n}=0,\qquad \bar e_{ij}=\bar\Pi\bar S_{ij},$$

$$\bar\theta=\frac{\bar\sigma}{K},\qquad \bar\sigma_{ij}n_j|_{S_\sigma}=\bar q_i,\qquad \bar u_i|_{S_u}=\bar\varphi_i.\tag{1.5}$$

2. The method of solution

We shall consider the materials for which the creep function Π can be written as

$$\Pi(t,x,y,z)=\Pi_0(t)\,g(x,y,z),\tag{2.1}$$

where $g(x, y, z) \neq 0$ everywhere inside or at the boundary of the body.

Let the function $\Pi_0(t)$ be given by [6]:

$$\Pi_0(t) = \frac{1}{2G_0}\left[1 + \int_0^t L(t)\,dt\right], \tag{2.2}$$

where

$$L(t) = \frac{\bar{e}^{\beta t}}{t} \sum_{n=1}^{\infty} \frac{[A\Gamma(\alpha)]^n}{\Gamma(\alpha n)}\, t\alpha^n; \tag{2.3}$$

A, β, α are empirical constants, $\Gamma(\alpha)$ is the gamma function, G_0 is the shear modulus which is constant for the homogeneous body and $\Pi_0(0) = 1/2G_0$.

Using Eq. (2.1) in Eq. (1.2), we get

$$e_{ij} = g(x, y, z) \int_0^t \Pi_0(t-\tau)\, dS_{ij}(\tau). \tag{2.4}$$

From Eq. (1.4) and (2.4) one obtains

$$\bar{e}_{ij} = g\bar{\Pi}_0 \bar{S}_{ij}. \tag{2.5}$$

In addition to the problem described by Eqs. (1.5) and (2.5), let us consider the problem of the nonhomogeneous theory of elasticity with the following nohomogeneity law:

$$G = G_0/g(x, y, z), \qquad K = \text{constant}, \tag{2.6}$$

where G is the shear modulus. In this case we have the following boundary value problem [7]:

$$\begin{gathered} \frac{\partial \sigma_{ij}}{\partial x_j} + \varrho F_i = 0, \qquad \epsilon_{ijk}\epsilon_{lmn} \frac{\partial^2 \varepsilon_{km}}{\partial x_j \partial x_n} = 0, \\ \theta = \frac{\sigma}{K}, \qquad \sigma_{ij} n_j|_{S_\sigma} = q_i, \qquad u_i|_{S_u} = \varphi_i. \end{gathered} \tag{2.7}$$

According to the elastic-viscoelastic correspondence principle [8], and from Eqs. (1.5) and (2.7), it follows that the present nonhomogeneous visco-elastic problem in terms of the images is identical to the corresponding nonhomogeneous elastic problem with the condition (2.6) and the following substitutions:

$$\begin{gathered} 1/2G_0 \to \bar{\Pi}_0, \qquad S_{ij} \to \bar{S}_{ij}, \qquad \sigma \to \bar{\sigma}, \qquad \sigma_{ij} \to \bar{\sigma}_{ij}, \\ e_{ij} \to \bar{e}_{ij}, \qquad \varepsilon_{ij} \to \bar{\varepsilon}_{ij}, \qquad \theta \to \bar{\theta}, \qquad F_i \to \bar{F}_i, \qquad q_i \to \bar{q}_i, \\ \varphi_i \to \bar{\varphi}_i, \qquad u_i \to \bar{u}_i. \end{gathered} \tag{2.8}$$

3. The torsion of a nonhomogeneous bar of elliptic cross-section

Consider a long bar of elliptic cross-section. It is assumed that during torsion, the cross-section of the bar remains plane and rotates without any distortion. The non-vanishing components of stress of the Saint–Venant's torsional elastic problem are then defined as [7]

$$\sigma_{xz} = \eta G \frac{F}{y}, \quad \sigma_{yz} = -\eta G \frac{F}{x}, \tag{3.1}$$

where $F = F(x, y)$ is the stress function, $G = G(x, y) = G_0/g$ is the shear modulus and η is the twist in radians per unit length of the bar for the non-homogeneous elastic problem, $g = g(x, y, z)$.

The stress components (3.1) satisfy the equilibrium equation

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0. \tag{3.2}$$

Substitution of Eq. (3.1) in Eq. (3.2) gives

$$G = G(F). \tag{3.3}$$

Now, considering the expression (2.6), we write the shear modulus (3.3) in the form

$$G = \frac{G_0}{g(F)}, \quad G_0 = \text{constant}. \tag{3.4}$$

Further, the stress components (3.1) also satisfy the compatibility equation

$$\frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{yz}}{\partial x} = -\eta, \tag{3.5}$$

where

$$\varepsilon_{ij} = \frac{1}{2G} \sigma_{ij},$$

Substitution of Eq. (3.1) in Eq. (3.5) gives

$$\nabla^2 F = -2, \tag{3.6}$$

where ∇^2 is the Laplace operator.

The boundary condition here is

$$F|_{\mathscr{L}} = 0, \tag{3.7}$$

where $\mathscr{L}$ is the contour of elliptic cross-section.

η can be determined from the relation

$$M = \iint (x\sigma_{yz} - y\sigma_{xz})\, ds,$$

where M is the twisting moment.

The solution of Eq. (3.6) with Eq. (3.7) is

$$F = -B\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right), \tag{3.9}$$

where

$$B = \frac{a^2 b^2}{a^2+b^2}.$$

Thus an exact solution is obtained [7] of the torsional nonhomogeneous elastic problem of an elliptic bar in stresses as

$$\sigma_{xz} = \frac{\eta G_0}{g(F)} \frac{\partial F}{\partial y}, \quad \sigma_{yz} = -\frac{\eta G_0}{g(F)} \frac{\partial F}{\partial x}. \tag{3.10}$$

If the nonhomomeneity of the material is taken in view of Eq. (2.6) as

$$G = G_0 [1-(F/B)^n], \quad n>0 \quad \text{(integer or fraction)}. \tag{3.11}$$

One gets from Eq. (3.8), using Eqs. (3.10) and (3.11),

$$\eta = \frac{(2+n)(1+n)}{n(3+n)} \cdot \frac{2}{(\pi ab)} \cdot \frac{M}{2G_0 B}. \tag{3.12}$$

The components of stress and strain in the form independent of η would then be

$$\begin{aligned}
\sigma_{xz} &= -\frac{2Ny}{\pi ab^3}[1-(F/B)^n]\, M, \\
\sigma_{yz} &= \frac{2Nx}{\pi a^3 b}[1-(F/B)^n]\, M, \\
\varepsilon_{xz} &= -\frac{2Ny}{\pi ab^3} \cdot \frac{M}{2G_0}, \\
\varepsilon_{yz} &= \frac{2Nx}{\pi a^3 b} \cdot \frac{M}{2G_0}.
\end{aligned} \tag{3.13}$$

where

$$N = \frac{(2+n)(1+n)}{n(n+3)}.$$

Using Eqs. (2.8) and applying the inverse Laplace–Carson transform, one gets the components of stress and strain in the nonhomogeneous visco-elastic case as

(3.14)
$$\sigma_{xz} = -\frac{2Ny}{\pi ab^3}[1-(F/B)^n]\,M,$$

$$\sigma_{yz} = \frac{2Nx}{\pi a^3 b}[1-(F/B)^n]\,M,$$

$$\varepsilon_{xz} = -\frac{2Ny}{\pi ab^3}\int_0^t \Pi_0(t-\tau)\,dM(\tau),$$

$$\varepsilon_{yz} = \frac{2Nx}{\pi a^3 b}\int_0^t \Pi_0(t-\tau)\,dM(\tau\tau).$$

4. The torsion of a homogeneous visco-elastic bar of elliptic cross-section

The components of stress and strain for the case of the homogeneous visco-elastic bar may be obtained straight away as a particular case by taking the shear modulus $G = G_0$ (a constant) in the preceding nonhomogeneous problem. Equation (3.11) then implies that n should tend to ∞ as $F/B < 1$. The components of stress and strain given in Eq. (3.14) then reduce to give the components of the homogeneous visco-elastic problem as

(4.1)
$$\sigma_{xz} = -\frac{2y}{\pi ab^3}\cdot M,$$

$$\sigma_{yz} = \frac{2x}{\pi a^3 b}\cdot M,$$

$$\varepsilon_{xz} = -\frac{2y}{\pi ab^3}\int_0^t \Pi_0(t-\tau)\,dM(\tau),$$

$$\varepsilon_{yz} = \frac{2x}{\pi a^3 b}\int_0^t \Pi_0(t-\tau)\,dM(\tau).$$

5. Numerical solution

In order to compute the components of strain for the homogeneous or nonhomogeneous visco-elastic problem, one has to find the value of the integral

$$\int_0^t \Pi'_0 (t-\tau)\, dM(\tau). \tag{5.1}$$

For that, let $M(\tau) = M_0\, h(\tau)$, where M_0 is a constant and $h(\tau)$ is the Heaviside's Unit Step Function.

Substitution in the integral (5.1) gives

$$\int_0^t \Pi_0 (t-\tau)\, dM(\tau) = \int_0^t \Pi_0 (t-\tau)\, M_0\, h'(\tau)\, d\tau = \tag{5.2}$$

$$= M_0 \int_0^t \Pi_0 (t-\tau)\, \delta(\tau)\, d\tau = M_0\, \Pi_0 (t), \quad (t \geqslant 0),$$

where, $\delta(\tau)$ is the Dirac delta function.

Now, from Eqs. (2.2) and (2.3) we have

$$\Pi_0 (t) = \frac{1}{2G_0} \left[1 + \int_0^t L(t)\, dt \right], \tag{5.3}$$

where

$$\int_0^t L(t)\, dt = \int_0^t \frac{\bar{e}^{\beta t}}{t} \sum_{n=1}^{\infty} \frac{[A\Gamma(\alpha)]^n}{\Gamma(\alpha n)}\, t^{\alpha n}\, dt$$

can be evaluated for different values of t (in minutes corresponding to some special values of the empirical parameters α, A, β).

Introducing the dimensionless components of stress and strain as

$$\begin{aligned} \sigma^*_{xz} &= \frac{\pi a b^2}{M_0}\, \sigma_{xz}, \\ \sigma^*_{yz} &= \frac{\pi a b^2}{M_0}\, \sigma_{yz}, \\ \varepsilon^*_{xz} &= \pi a b^2 \left(\frac{2G_0}{M_0} \right) \varepsilon_{xz}, \\ \varepsilon^*_{yz} &= \pi a b^2 \left(\frac{2G_0}{M_0} \right) \varepsilon_{yz}, \qquad \gamma = a/b \end{aligned} \tag{5.4}$$

and using Eqs. (5.2) and (5.3) in Eqs. (5.1) and (4.1) respectively, one obtains the components of the stress and strain tensors in the homogeneous visco-elastic case as

$$\sigma^*_{xz} = -\frac{2y}{b}, \quad \sigma^*_{yz} = \frac{2x}{a\gamma}, \tag{5.5}$$

$$\varepsilon_{xz}^{*} = -\frac{2y}{b}\left[1+\int_{0}^{t} L(t)\,dt\right], \quad \varepsilon_{yz}^{*} = \frac{2x}{a\gamma}\left[1+\int_{0}^{t} L(t)\,dt\right].$$

and the components of the stress and strain tensors in the nonhomogeneous visco-elastic case as

$$\text{(5.6)} \quad \begin{aligned} \sigma_{xz}^{*} &= -\frac{2Ny}{b}\,[1-(F/B)^{n}], \\ \sigma_{yz}^{*} &= \frac{2xN}{a\gamma}\,[1-(F/B)^{n}], \\ \varepsilon_{xz}^{*} &= -\frac{2Ny}{b}\left[1+\int_{0}^{t} L(t)\,dt\right], \\ \varepsilon_{yz}^{*} &= \frac{2Nx}{a\gamma}\left[1+\int_{0}^{t} L(t)\,dt\right]. \end{aligned}$$

CASE I. Stress and strain components at the point ($a/2$, $b/2$) under the condition $n = 2$.

At the point $x = a/2$, $y = b/2$, on the elliptic cross-section of the rod under torsion for $n = 2$, one has $F/B = 1/2$ and consequently the components of stress and strain for various values of γ are otained from Eqs. (5.5) and (5.6) for visco-elastic homogeneous and non-homogeneous materials, respectively. The results are given in Tables 1, 2 and Fig. 1.

CASE II. Stress components along the axis under the condition $n = 1$

For any point $(x, 0)$ on the axis (i.e. x-axis) of the elliptic cross--section of the rod under torsion for $n = 1$, one has $F/B = 1 - x^2/a^2$ and the stress components $\sigma_{xz}^{*} = 0$. The other stress component σ_{yz}^{*} for various

Table 1. Showing stress components δ_{xz}^{*} and δ_{yz}^{*}

γ	σ_{xz}^{*}		σ_{yz}^{*}	
	Homogeneous visco-elastic	Nonhomogeneous visco-elastic	Homogeneous visco-elastic	Nonhomogeneous visco-elastic
1.0	-1	-0.9	1	0.9
1.5	-1	-0.9	0.667	0.6
2.0	-1	-0.9	0.5	0.45
2.5	-1	-0.9	0.4	0.36

Table 2. Showing strain component ε^*_{yz}.

t	$L(t)$ for $\alpha = 0.25$ $A = 0.106$ $\beta = 0.05$	ε^*_{yz}			
		$\gamma = 1$		$\gamma = 2$	
		Homogeneous visco-elastic	Nonhomogeneous visco-elastic	Homogeneous visco-elastic	Nonhomogeneous visco-elastic
1	0.6758	1.6758	2.0110	0.8376	1.0040
2	0.8900	1.8900	2.2680	0.9450	1.1340
3	1.0556	2.4667	1.0278	1.0294	1.2334
5	1.3216	2.3216	2.7859	1.1608	1.3930
7	1.5407	2.5407	3.0488	1.2704	1.5245
8	1.6388	2.6388	3.1666	1.3194	1.5833
9	1.7309	2.7309	3.2771	1.3655	1.6386
10	1.8180	2.8180	3.3816	1.4090	1.6908
12	1.9734	2.9734	3.5681	1.4867	1.7840
14	2.1263	3.1263	3.7516	1.5632	1.8758
16	2.2613	3.2613	3.9136	1.6307	1.9568

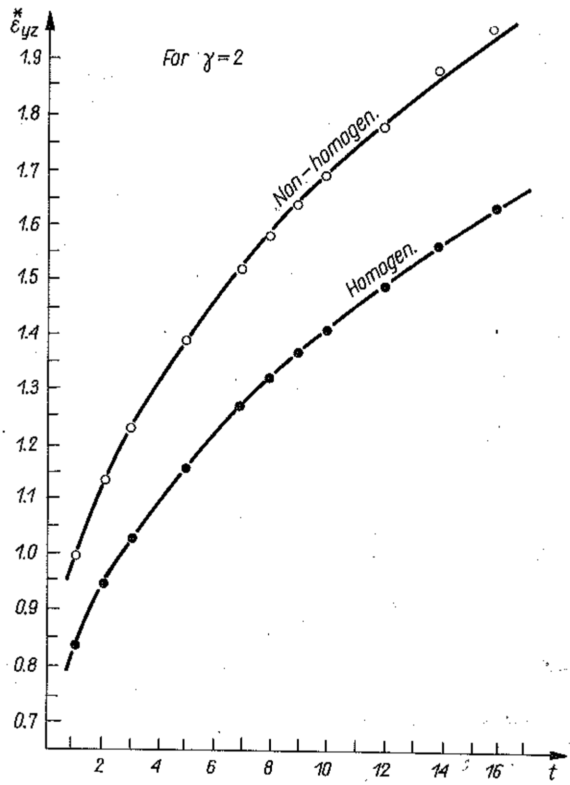

FIG. 1. Relation between t and ε_{yz} of the homogeneous and nonhomogeneous visco-elastic cases.

values of x is calculated from Eqs. (5.5) and (5.6) for visco-elastic homogeneous and nonhomogeneous cases. The results are given in Table 3 and Fig. 2.

In both cases the creep function $\Pi_0(t)$ is taken in the form (2.2) and $L(t)$ in the form (2.3) when

$$\alpha = 0.25, \quad A = 0.106, \quad \beta = 0.05.$$

Table 3. Showing stress component σ^*_{yz}.

Points along X-axis	$\gamma\,\sigma^*_{yz}$	
	visco-elastic homogeneous	visco-elastic nonhomogeneous
a/8,0	0.25	0.0059
a/4,0	0.50	0.0469
$3a/8$,0	0.75	0.1582
a/2,0	1.00	0.3750
$5a/8$,0	1.25	0.7324
$3a/4$,0	1.50	1.2656
$0.8a$,0	1.60	1.5360
$0.85a$,0	1.70	1.8424
$0.90a$,0	1.80	2.1870
$0.95a$,0	1.90	2.5721
0,0	2.00	3.0000

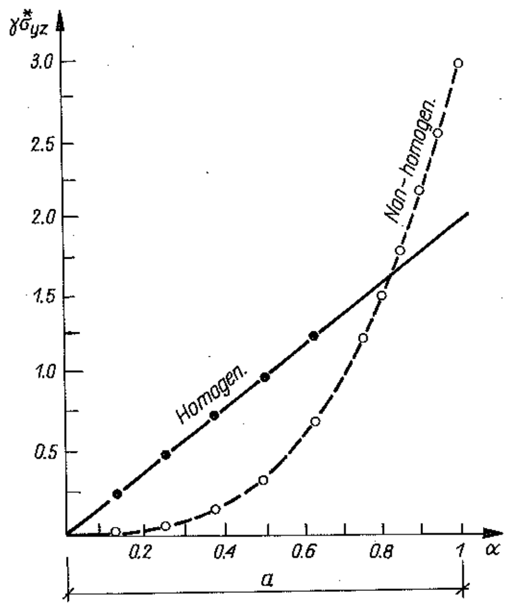

FIG. 2. Stress distribution along x-axis of the cross-section for the homogeneous and non-homogeneous visco-elastic cases.

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Streszczenie

SKRĘCANIE NIEJEDNORODNEGO PRĘTA LEPKOSPRĘŻYSTEGO O PRZEKROJU ELIPTYCZNYM

Przedstawiono ścisłe rozwiązanie zagadnienia skręcania niejednorodnego pręta lepkosprężystego o przekroju eliptycznym. Zastosowano metodę analogii sprężysto-lepkosprężystej do wyznaczenia rozkładu naprężeń w przekoju poprzecznym pręta. Wyniki numeryczne wskazują, że niejednorodność prowadzi do zmniejszenia tych naprężeń.

Резюме

СКРУЧИВАНИЕ НЕОДНОРОДНОГО ВЯЗКОУПРУГОГО СТЕРЖНЯ С ЭЛЛИПТИЧЕСКИМ СЕЧЕНИЕМ

Представлено точное решение задачи окручивания неоднородного вязкоупругого стержня с эллиптическим сечением. Применен метод упруго-вязкоупругой аналогии для определения распределения напряжений в поперечном сечении стержня. Численные результаты указывают, что неоднородность приводит к уменьшению этих напряжений.

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